A Tableau-Based Proof System for Concurrent Actions and Events

Jussi Rintanen*
University of Texas at Austin
Department of Computer Sciences
Taylor Hall 2.124
Austin, Texas 78712

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Abstract

A proof system for a class of problems in temporal reasoning is presented. It is an extension of the method of analytic tableau for the classical propositional logic. The class of of problems addressed is determined by the requirement that transitions of the world are unique: state constraints and minimal change are not supported, effects of actions may not be disjunctive, and the world is deterministic. The class allows the expression of concurrent actions and change that is not caused by actions.

1 Introduction

Recent research on actions focuses on worlds where all changes at a certain moment of time are the effects of a single action. Hence concurrent actions are not allowed and all change is caused (directly or indirectly) by actions. Neither of these restrictions is motivated by the real world. Reasoning about actions in a setting where the world may change by itself without any actions taking place is a subject that has received little attention [Lifschitz and Rabinov, 1989; Cordier and Siegel, 1992; Thielscher, 1995]. We continue this research by giving a proof system for a class of scenarios where concurrency and change that is not directly caused by actions is allowed.

The proof system extends the method of analytic tableaux for classical propositional logic [Smullyan, 1968].

*On leave from Helsinki University of Technology.

2 Semantics

The symbol $\mathcal{L}$ represents the language of classical propositional logic consisting of propositional variables P and all compound formulae constructed from propositional variables and the usual unary and binary connectives $\neg, \vee, \wedge, \rightarrow$, and $\leftrightarrow$. The superset of $\mathcal{L}$ that includes also connectives $\boxplus$ and $[i]$ for $i \in \mathcal{Z}$ and is closed under all connectives, is $\mathcal{L}_t$. A formula $[i]\phi$ expresses that ϕ is true at time point i, and $\boxplus\phi$ expresses that ϕ is true at the next time point. The symbol $\mathcal{Z}$ refers to the set of integers. The subset $P \cup \{\neg p | p \in P\}$ of $\mathcal{L}$ is the set of literals $\mathcal{L}_l$. If the literal l is a propositional variable $p \in P$, then $\overline{l} = \neg p$. If l is $\neg p$, then $\overline{l}$ is p. Occasionally, we view sets of literals $\{l_1, \ldots, l_n\} \subseteq \mathcal{L}_l$ as conjunctions $l_1 \wedge \cdots \wedge l_n$. The following definition is like the ordinary definition of valuations except that there is an additional parameter for time.

Definition 2.1 *A* valuation *is a function* $v : \mathcal{Z} \times P \rightarrow \{true, false\}$.

Valuations are extended to arbitrary formulae of $\mathcal{L}_t$ as follows. The extension follows the standard truth-definition of the connectives, and is unique for a given valuation of propositional variables. We use the notation $v \models_i \phi$ for $v^*(i, \phi) = \mathrm{true}$, where v^* is a function $v^* : \mathcal{Z} \times \mathcal{L}_t \rightarrow \{\mathrm{true}, \mathrm{false}\}$ such that the following holds.

1. $v \models_i p$ iff $v(i, p) = \mathrm{true}$, for propositional variables $p \in P$.
2. $v \models_i \neg\phi$ iff not $v \models_i \phi$.
3. $v \models_i \phi \wedge \phi'$ iff $v \models_i \phi$ and $v \models_i \phi'$.
4. $v \models_i \phi \vee \phi'$ iff $v \models_i \phi$ or $v \models_i \phi'$.
5. $v \models_i \phi \rightarrow \phi'$ iff not $v \models_i \phi$ or $v \models_i \phi'$.
6. $v \models_i \phi \leftrightarrow \phi'$ iff $v \models_i \phi \rightarrow \phi'$ and $v \models_i \phi' \rightarrow \phi$.
7. $v \models_i [j]\phi$ iff $v \models_j \phi$.
8. $v \models_i \boxplus\phi$ iff $v \models_{i+1} \phi$.

In the proof system given later each formula has an integer prefix. For prefixed formulae we define $v \models i : \phi$ as $v \models_i \phi$.

Changes in the world are formalized as *transition rules* $\langle \phi, L \rangle \in \mathcal{L} \times 2^{\mathcal{L}_l}$: the literals L will be true at time i, if the formula ϕ is true at time $i-1$. We do not have a category of actions, but we formalize effects of actions as special transition rules that are applicable only at some points of time. Transition rules that express spontaneous change are formalized as sets $T \in 2^{\mathcal{L} \times 2^{\mathcal{L}_l}}$. Transition rules that express effects of actions and are therefore applicable only at some points of time are formalized as functions $A : \mathcal{Z} \rightarrow 2^{\mathcal{L} \times 2^{\mathcal{L}_l}}$. Here A is called an *action schedule.*

We make the assumption that only those changes take place that are expressed as transition rules and the action schedule. This is the principle of inertia: a state of affairs persists if there is no reason not to. The change of world is formalized as follows.

$$\frac{i : \phi \vee \phi'}{i : \phi \;\big|\; i : \phi'} \qquad \frac{i : \neg(\phi \vee \phi')}{\begin{array}{c} i : \neg\phi \\ i : \neg\phi' \end{array}}$$

$$\frac{i : \phi \wedge \phi'}{\begin{array}{c} i : \phi \\ i : \phi' \end{array}} \qquad \frac{i : \neg(\phi \wedge \phi')}{i : \neg\phi \;\big|\; i : \neg\phi'}$$

$$\frac{i : \phi \leftrightarrow \phi'}{i : \phi \wedge \phi' \;\big|\; i : \neg\phi \wedge \neg\phi'} \qquad \frac{i : \neg(\phi \leftrightarrow \phi')}{i : \phi \wedge \neg\phi' \;\big|\; i : \neg\phi \wedge \phi'}$$

$$\frac{i : \phi \rightarrow \phi'}{i : \neg\phi \;\big|\; i : \phi'} \qquad \frac{i : \neg(\phi \rightarrow \phi')}{\begin{array}{c} i : \phi \\ i : \neg\phi' \end{array}}$$

$$\frac{i : \neg\neg\phi}{i : \phi}$$

Figure 1: Tableau rules for propositional logic

$$\frac{i : \boxplus\phi}{i+1 : \phi} \qquad \frac{i : \neg\boxplus\phi}{i+1 : \neg\phi} \qquad \frac{i : [j]\phi}{j : \phi} \qquad \frac{i : \neg[j]\phi}{j : \neg\phi}$$

Figure 2: Tableau rules for temporal operators

Definition 2.2 *Let $T \in 2^{\mathcal{L} \times 2^{\mathcal{L}_l}}$ be a set of transition rules and $A : \mathcal{Z} \rightarrow 2^{\mathcal{L} \times 2^{\mathcal{L}_l}}$ an action schedule. Define A, T-*valuations *as valuations v that satisfy the following.*

1. *If $v \models_i \phi$ and $\langle \phi, L \rangle \in T \cup A(i)$, then $v \models_{i+1} l$ for all $l \in L$.*
2. *If $v \models_i l$ for $l \in \mathcal{L}_l$ and $v \models_i \phi$ for no $\langle \phi, L \rangle \in T \cup A(i)$ such that $\bar{l} \in L$, then $v \models_{i+1} l$.*

3 The Tableau Method

The tableau method is an extension of the analytic tableau as presented by Smullyan [1968]. The extensions concern the temporal operators and the principle of inertia. The standard tableau rules for the connectives of classical propositional logic are given in Figure 1, and the rules for temporal operators $\boxplus$ and $[i]$ in Figure 2. The first tableau rule in Figure 3 is for inferring the changes expressed as transition rules and action schedules. The second tableau rule is for handling inertia. It is used for case-analysis concerning whether the value of a certain propositional variable p persists. Either it persists at moment i, in which case it has the same value both at i and $i+1$, or it does not persist. Due to the inertia assumption, the non-persistence of a propositional variable p may only be due to a transition rule that changes the value of p. In the inertia rule, this is reflected as the

$$\frac{}{i : \phi \to \boxplus L} \quad \text{if } \langle \phi, L \rangle \in T \cup A(i)$$

$$\frac{}{i : p \leftrightarrow \boxplus p \mid \chi} \quad \text{where } \chi = i : \bigvee \{\phi | \langle \phi, L \rangle \in T \cup A(i), \{p, \neg p\} \cap L \neq \emptyset\}$$

Figure 3: Tableau rules for actions and transition rules, and inertia

branch with $i : p \leftrightarrow \boxplus p$ for the case with no change, and the branch with a formula that is true if at least one of the rules that may change the propositional variable is applicable.

3.1 The Uniform Notation

To follow Smullyan [1968], we classify the prefixed formulae to α and β formulae. The classification stems from the two kinds of rules. The tableau rules with a vertical bar | in them are β-rules (with the exception of the inertia rule). If the formula above the line is β, then the two formulae below the line are β_1 and β_2. The rest of the tableau rules (excluding the tableau rules for inertia and transition rules) are α-rules. An α-rule does not have a vertical bar in it. The formula above the line in an α-rule is α and the formulae below the line are α_1 and α_2. In case there is only one formula below the line, take $\alpha_2 = \alpha_1$. Now for each prefixed formula ϕ (excluding prefixed literals) there is a unique rule with the antecedent of the same form as ϕ. If that rule is an α-rule, the formula is accordingly an α formula, otherwise it is a β formula. The uniform notation simplifies many proofs because of the following useful property.

Lemma 3.1 *For each α-formula α and valuation v, $v \models \alpha$ iff $v \models \alpha_1$ and $v \models \alpha_2$. For each β-formula β and valuation v, $v \models \beta$ iff $v \models \beta_1$ or $v \models \beta_2$.*

4 How a Tableau is Constructed

The tableau rules given in the previous section can be used for demonstrating that for a given set of transition rules T, an action schedule A, and a set of prefixed formulae O, there are no A, T-valuations that satisfy O. The β-rules perform case-analysis: if β has to be satisfied, one of β_1 and β_2 must be satisfied. The α-rules are about formulae α that uniquely determine the truth-values of α_1 and α_2: they both have to be satisfied to satisfy α.

A tableau constructed using the rules is an exhaustive case-analysis of possible ways to satisfy all formulae. If in all cases the satisfaction of both $i : p$ and $i : \neg p$ is required for some $i \in \mathcal{Z}$ and $p \in P$, we are allowed to conclude that no satisfying valuations exist.

Let A be an action schedule and T a set of transition rules. an A, T-tableau for a set of prefixed formulae O is constructed as follows. A tableau is constructed step by step beginning from the empty tableau. In the following, a *branch* is a path from the root of the tableau to a leaf.

- Any prefixed formula from O may be added as a new leaf to any of the branches in the tableau.
- An α-rule can be applied if there is a branch in the tableau that has a formula of the form α in it. As a result of the application, the formulae α_1 and α_2 are added to the tip of the branch. Now α_2, that is written below α_1, becomes the new leaf.
- A β-rule can be applied if there is a branch in the tableau that has a formula of the form β in it. As a result of the application, the formulae β_1 and β_2 are added to the tableau below the leaf node of the old branch, splitting the old branch to two branches, one having β_1 as leaf and the other having β_2.
- The rule for actions and transitions can be applied to any branch in the tableau. For any $i \in \mathcal{Z}$, if $\langle \phi, L \rangle \in T \cup A(i)$, then $i : \phi \rightarrow \boxplus L$ may be added to the branch as a new leaf.
- The inertia rule can be applied to any branch in the tableau. For any propositional variable $p \in P$ and time point $i \in \mathcal{Z}$, the formulae $i : p \leftrightarrow \boxplus p$ and $i : \bigvee \{\phi | \langle \phi, L \rangle \in T \cup A(i), \{p, \neg p\} \cap L \neq \emptyset\}$ are added to the tableau below the leaf node of the old branch, essentially splitting the old branch to two branches, one having $i : p \leftrightarrow \boxplus p$ as the leaf and the other having $i : \bigvee \{\phi | \langle \phi, L \rangle \in T \cup A(i), \{p, \neg p\} \cap L \neq \emptyset\}$. If the disjunction is empty, the rightmost branch is left out.

Branches that contain $i : p$ and $i : \neg p$ for some $i \in \mathcal{Z}$ and $p \in P$, are *closed*. If all branches of the tableau are closed, the tableau is closed. If a branch or a tableau is not closed, it is *open*. A closed A, T-tableau for O indicates that there are no A, T-valuations v such that $v \models \phi$ for all $\phi \in O$. If a tableau is open, in general no conclusions about the existence of A, T-valuations satisfying O can be made.

The tableau method can be used for consequence testing in the standard way. A formula ϕ is true in all A, T-valuations that satisfy O if and only if no A, T-valuation satisfies $O \cup \{\neg \phi\}$ if and only if there is a closed A, T-tableau for $O \cup \{\neg \phi\}$. Tableau methods are sound, and completeness is achieved by using systematic methods of constructing tableaux. Soundness, completeness and decidability are discussed in Section 6.

5 Examples

The Yale Shooting Problem consists of two actions, loading a gun, and firing it. The firing causes death if the gun was loaded. The problem is to show that after loading, doing nothing, and firing, death arrives.

$$\begin{array}{rcl@{\qquad}rcl} A(0) & = & \{\langle \top, \{\text{Loaded}\} \rangle\} & T & = & \emptyset \\ A(1) & = & \emptyset & & & \\ A(2) & = & \{\langle \text{Loaded}, \{\text{Dead}\} \rangle\} & O & = & \{3 : \neg \text{Dead}\} \end{array}$$

For brevity, we use $\top$ as an abbreviation for $p \vee \neg p$ and consider a branch containing $\neg \top$ closed. The symbol $\times$ is signifies a closed branch. The tableau in Figure 4 was

$$
\begin{array}{l}
3 : \neg \mathrm{Dead} \\
0 : \top \rightarrow \boxplus \mathrm{Loaded} \\
2 : \mathrm{Loaded} \rightarrow \boxplus \mathrm{Dead} \\
\begin{array}{l|c}
\begin{array}{l} 0 : \neg \top \\ \times \end{array} &
\begin{array}{c}
0 : \boxplus \mathrm{Loaded} \\
1 : \mathrm{Loaded} \\
\begin{array}{c|l}
\begin{array}{c}
2 : \neg \mathrm{Loaded} \\
1 : \mathrm{Loaded} \leftrightarrow \boxplus \mathrm{Loaded} \\
\begin{array}{l|l}
\begin{array}{l} 1 : \mathrm{Loaded} \\ 1 : \boxplus \mathrm{Loaded} \\ 2 : \mathrm{Loaded} \\ \times \end{array} &
\begin{array}{l} 1 : \neg \mathrm{Loaded} \\ 1 : \neg \boxplus \mathrm{Loaded} \\ \times \\ \ \end{array}
\end{array}
\end{array} &
\begin{array}{l} 2 : \boxplus \mathrm{Dead} \\ 3 : \mathrm{Dead} \\ \times \end{array}
\end{array}
\end{array}
\end{array}
\end{array}
$$

Figure 4: The Yale Shooting Problem

constructed by delaying the application of β rules last because they split branches, and otherwise always applying the highest unused formula.

The next example contains concurrency and spontaneous change. The scenario is about a employee and an elevator. The employee can be waiting on the first floor outside the elevator or be in the elevator when it is on the first or on the second floor (propositions W, D, U, respectively). The elevator is the kind that has several cabins, there are no doors in the cabins, and the cabins continuously move up in one shaft and then come down in another. The employee can be ordered to enter the elevator or to exit it at the first floor (propositions E, X). The employee has no memory, so an order is in effect only one point of time.

$$
\begin{array}{rcll}
T & = & \{\langle E \wedge W, \{\neg W, D\}\rangle, & \langle D \wedge \neg X, \{\neg D, U\}\rangle, \\
 & & \langle D \wedge X, \{\neg D, W\}\rangle, & \langle U, \{\neg U, D\}\rangle, \\
 & & \langle X, \{\neg X\}\rangle, & \langle E, \{\neg E\}\rangle\}
\end{array}
$$

In our scenario, there are two actions. The first action is giving an order to enter the elevator (between time points 0 and 1), and the second action is an order to exit the elevator (between time points 3 and 4).

$$
A(0) \;=\; \{\langle \top, \{E\}\rangle\} \quad A(3) \;=\; \{\langle \top, \{X\}\rangle\}
$$

The observations are that the employee is on the first floor and there are no orders given at time point 0. We want to infer that the employee is back to the first floor at time point 5, so we include the negation of a corresponding proposition in the set of observations.

$$
O \;=\; \{0 : W, 0 : \neg E, 0 : \neg X, 5 : \neg W\}
$$

We give a closed A, T-tableau for O in Figure 5 that indicates the inexistence of A, T-valuations for O, and hence shows that the presence of the employee on the first floor at time point 5 is a consequence of our scenario description. The model of the scenario we are interested in is the following.

$0 : W$
$0 : \neg E$
$0 : \neg X$
$5 : \neg W$
$0 : \top \to \boxplus E$ from A(0)

$0 : \neg\top$
$\times$

$0 : W \leftrightarrow \boxplus W$ inertia

(I)

$0 : \neg W \wedge \neg \boxplus W$
$0 : \neg W$
$0 : \neg \boxplus W$
$\times$

$0 : \boxplus E$
$1 : E$
$0 : (W \wedge E) \vee (D \wedge X)$ inertia

$0 : W \wedge E$
$0 : W$
$0 : E$
$\times$

$0 : D \wedge X$
$0 : D$
$0 : X$
$\times$

I :

$0 : W \wedge \boxplus W$
$0 : W$
$0 : \boxplus W$
$1 : W$
$1 : W \wedge E \to \boxplus(\neg W \wedge D)$ from T

$1 : \neg(W \wedge E)$

$1 : \neg W$
$\times$

$1 : \neg E$
$\times$

$1 : \boxplus(\neg W \wedge D)$
$2 : \neg W \wedge D$
$2 : \neg W$
$2 : D$
$2 : D \wedge \neg X \to \boxplus(\neg D \wedge U)$ from T

(II)

$2 : \boxplus(\neg D \wedge U)$
$3 : \neg D \wedge U$
$3 : \neg D$
$3 : U$
$3 : U \to \boxplus(\neg U \wedge D)$ from T

$3 : \neg U$
$\times$

(III)

II :

$2 : \neg(D \wedge \neg X)$

$2 : \neg D$
$\times$

$2 : \neg\neg X$
$2 : X$
$1 : X \leftrightarrow \boxplus X$
$0 : X \leftrightarrow \boxplus X$

$0 : X \wedge \boxplus X$
$0 : X$
$0 : \boxplus X$
$\times$

$0 : \neg X \wedge \neg \boxplus X$
$0 : \neg X$
$0 : \neg \boxplus X$
$0 : \boxplus \neg X$
$1 : \neg X$

$1 : X \wedge \boxplus X$
$1 : X$
$1 : \boxplus X$
$\times$

$1 : \neg X \wedge \neg \boxplus X$
$1 : \neg X$
$1 : \neg \boxplus X$
$1 : \boxplus \neg X$
$2 : \neg X$
$\times$

III :

$3 : \boxplus(\neg U \wedge D)$
$4 : \neg U \wedge D$
$4 : \neg U$
$4 : D$
$4 : D \wedge X \to \boxplus(\neg D \wedge W)$ from T

$4 : \neg(D \wedge X)$

$4 : \neg D$
$\times$

$4 : \neg X$
$3 : \top \to \boxplus X$ from $A(3)$

$\neg\top$
$\times$

$3 : \boxplus X$
$4 : X$
$\times$

$4 : \boxplus(\neg D \wedge W)$
$5 : \neg D \wedge W$
$5 : \neg D$
$5 : W$
$\times$

Figure 5: A tableau proof concerning an elevator

proposition/time	0	1	2	3	4	5
U	f	f	f	**t**	f	f
D	f	f	**t**	f	**t**	f
W	**t**	**t**	f	f	f	**t**
E	f	**t**	f	f	f	f
X	f	f	f	f	**t**	f

6 Soundness and Completeness

Theorem 6.1 (Soundness) *Let T be a set of transition rules, A an action schedule, and O a set of prefixed formulae such that $v \models O$ for some A, T-valuation v. Then there is no closed A, T-tableau for O.*

Proof: The proof proceeds in the standard way [Smullyan, 1968; Fitting, 1983]. Assume there is an A, T-valuation v such that $v \models O$. Initially, the tableau consists of a single branch that contains the formulae in O. We show that each rule application preserves the property that there is a branch B in the tableau and an A, T-valuation such that $v \models B$. Now the branch cannot be closed, because there is no valuation v such that $v \models i : p$ and $v \models i : \neg p$ for some $p \in P$ and $i \in \mathcal{Z}$.

If a rule is applied to a branch that does not have a satisfying A, T-valuation, the property is trivially preserved. So assume a rule is applied to a branch B such that $v \models B$ for some A, T-valuation v. We analyze by cases all tableau rules.

1. The application of an α-rule for a formula α extends the branch with α_1 and α_2. By Lemma 3.1, if $v \models \alpha$, then $v \models \alpha_1$ and $v \models \alpha_2$. Therefore $v \models B \cup \{\alpha_1, \alpha_2\}$.

2. The application of a rule for a formula β splits the branch B to two new branches $B \cup \{\beta_1\}$ and $B \cup \{\beta_2\}$. By Lemma 3.1, if $v \models \beta$, then $v \models \beta_1$ or $v \models \beta_2$. Therefore at least one of $v \models B \cup \{\beta_1\}$ and $v \models B \cup \{\beta_2\}$.

3. The application of a rule $\overline{i : \phi \rightarrow \boxplus L}$ for $\langle \phi, L \rangle \in T \cup A(i)$ preserves the property: by the definition of A, T-valuations, if $v \models_i \phi$, then $v \models_{i+1} L$. Therefore $v \models B \cup \{i : \phi \rightarrow \boxplus L\}$.

4. The application of the inertia rule for some $p \in P$ and $i \in \mathcal{Z}$ splits the branch B to branches $B \cup \{i : p \leftrightarrow \boxplus p\}$ and $B \cup \{i : \bigvee\{\phi | \langle \phi, L \rangle \in T \cup A(i), \{p, \neg p\} \cap L \neq \emptyset\}\}$.

   If $v \models_i p \leftrightarrow \boxplus p$, the property is preserved as $v \models B \cup \{i : p \leftrightarrow \boxplus p\}$. So assume this is not the case. Let l be the literal p or $\neg p$ whichever fulfills $v \models_i l$. By the second item in the definition of A, T-valuations, now $v \models_i \phi$ for some $\langle \phi, L \rangle \in T \cup A(i)$ such that $\bar{l} \in L$. Therefore $v \models_i \delta$ for one of the disjuncts δ in $i : \bigvee\{\phi | \langle \phi, L \rangle \in T \cup A(i), \{p, \neg p\} \cap L \neq \emptyset\}$. Hence $v \models B \cup \{i : \bigvee\{\phi | \langle \phi, L \rangle \in T \cup A(i), \{p, \neg p\} \cap L \neq \emptyset\}\}$, and the property is preserved in this case as well.

$\square$

Not all constructions of an A, T-tableau for O result in a closed tableau, even if there is a closed A, T-tableau for O. There are, however, systematic procedures for constructing a closed tableau whenever such a tableau exists. A simple systematic procedure is given next.

In the following discussion we restrict to finite sets T of transitions, finite action schedules A (meaning that the set $\{\langle i, R\rangle | i \in \mathcal{Z}, R \in A(i)\}$ is finite), and finite sets of observations O.

Definition 6.2 (Systematic tableaux) *Let A be an action schedule, T a finite set of transition rules, and O a finite set of prefixed formulae. The* systematic tableau for A, T, O *is constructed as follows. Initially the tableau consists of the branch O. Then extend the tableau in stages as follows (starting from $0, 1, -1, 2, -2, \ldots$). At stage i, extend all open branches by first applying the inertia rule for i and all $p \in P$ occurring in A, T, O to each open branch, and the action/transition rule for all $\langle\phi, L\rangle \in T \cup A(i)$ to each open branch, and finally applying α- and β-rules to all branches and all formulae that have not yet been used in extending the tableau.*

This process stops at some stage $i \geq 0$ at a closed tableau, or otherwise continues forever and we get an infinite tableau. By König's lemma we can now infer – because the tableau is finitely branching – that there is at least one infinite open branch in the tableau [Smullyan, 1968; Fitting, 1983]. Open branches of infinite systematic tableaux fulfill the conditions in the following definition[1].

Definition 6.3 (Downward saturated sets) *A set of prefixed formulae Φ is* downward saturated *if it fulfills the following conditions.*

1. *For no $p \in P$ and $i \in \mathcal{Z}$, $\{i : p, i : \neg p\} \subseteq \Phi$.*
2. *For each $\alpha \in \Phi$, both $\alpha_1 \in \Phi$ and $\alpha_2 \in \Phi$.*
3. *For each $\beta \in \Phi$, either $\beta_1 \in \Phi$ or $\beta_2 \in \Phi$.*

Lemma 6.4 *Let Φ be a downward saturated set. Then there is a valuation v such that $v \models \Phi$.*

Proof: Let $v(i, p) =$ true whenever $i : p \in \Phi$, and $v(i, p) =$ false otherwise. Proof that $v \models \phi$ for all $\phi \in \Phi$ is an easy induction on the structure of the formulae in Φ. □

Theorem 6.5 (Completeness) *Let A be an action schedule, T a set of transition rules, and O a set of formulae, and let there be no closed A, T-tableaux for O. Then there is an A, T-valuation v such that $v \models O$.*

[1] The term "downward saturated set" is from Fitting [1983]. Smullyan [1968] uses "Hintikka set".

Proof: The proof proceeds in the standard way [Smullyan, 1968; Fitting, 1983]: given a systematic tableau with an open branch, note that the branch is downward saturated, and construct an A, T-valuation v that satisfies O.

So let there be a systematic A, T-tableau for O that has an open branch B. Because B is downward saturated, by Lemma 6.4 there is a valuation v such that $v \models B$. Because $O \subseteq B$, $v \models O$. It remains to verify that v is an A, T-valuation.

1. Let $\langle \phi, L \rangle$ be a member of $T \cup A(i)$ for some $i \in \mathcal{Z}$.

   The branch B contains $i : \phi \rightarrow \boxplus L$, and hence $v \models_i \phi \rightarrow \boxplus L$. By the truth-definition of the connectives, if $v \models_i \phi$, then $v \models_{i+1} L$.

2. Let $v \models_i l$ for $l \in \mathcal{L}_l$ and let there be no $\langle \phi, L \rangle \in T \cup A(i)$ such that $\bar{l} \in L$ and $v \models_i \phi$.

   Let $p \in P$ be such that $l = p$ or $\bar{l} = p$. The branch B contains at least one of $\iota = i : p \leftrightarrow \boxplus p$ and $\nu = i : \bigvee\{\phi | \langle \phi, L \rangle \in T \cup A(i), \{p, \neg p\} \cap L \neq \emptyset\}$.

   Assume the branch contains ι. Because $v \models_i p \leftrightarrow \boxplus p$ and $v \models_i l$, also $v \models_{i+1} l$. Assume the branch contains ν. Because there is no $\langle \phi, L \rangle \in T \cup A(i)$ such that $\bar{l} \in L$ and $v \models_i \phi$, the disjunct of ν that is satisfied must be ϕ for an L such that $l \in L$ and $\langle \phi, L \rangle \in T \cup A(i)$. Now because $i : \phi \rightarrow \boxplus L$ is in B, $v \models_{i+1} l$.

□

6.1 Decidability

The systematic tableau construction provides only a semidecision procedure as it terminates only when there are no A, T-valuations that satisfy O. This is because the systematic tableau construction applies inertia and action rules for time points arbitrarily far in the past and in the future. It is not sufficient to restrict the rule application for example to time points i such that $[i]$ occurs in i or $A(i)$ is non-empty, because the transition rules T may run into inconsistencies after the last actions and observations. There is, however, a wide class of transition rules for which this is impossible. First we give an example of an inconsistency that does not involve actions or observations.

Example 6.1 Take any set of transition rules $T \supseteq \{\langle p, \neg q \rangle, \langle p, q \rangle\}$. If p is true at i, then both q and $\neg q$ should be true at $i + 1$. Therefore for all $\emptyset, T$-valuations v and all $i \in \mathcal{Z}$, $v \models_i \neg p$. ■

By restricting the class of transition rules, all inconsistencies obtained in the tableau construction involve at least one of the observations or an action occurrence, and hence looking past the last observation and action is unnecessary. Intuitively, whenever a pair of action or transition rules have conclusions that contradict each other, it must not be possible that the rules are applicable at the same point of time.

Definition 6.6 (Non-conflicting) *Let T be a set of transition rules. Then T is* non-conflicting, *if for all $\{\langle\phi, L\rangle, \langle\phi', L'\rangle\} \subseteq T$, if $L \cap \{\bar{l} | l \in L\} \neq \emptyset$, then $\models \neg(\phi \wedge \phi')$.*

Claim 6.7 *For action schedules A, non-conflicting sets of transition rules T, and observations O, there is a complete systematic tableau construction that applies action and inertia rules for integers $i, b \leq i \leq s$ such that $A(j) = \emptyset$ for all $j \in \mathcal{Z}$ such that $j < b$ or $j > s$, and $[j]$ does not occur in O and $j : \phi \notin O, \phi \in \mathcal{L}_t$ for any $j \in \mathcal{Z}$ such that $j < b$ or $j > s$.*

Actually, the systematic tableau can be made finite even in cases where the transition rules are not non-conflicting. With a finite number n of distinct propositional variables occurring in O, A and T, it suffices to look at 2^n time points past the last action or observation and 2^n time points before the first action or observation. This is because the world is deterministic, there are only 2^n different states of the world, and the history starts repeating itself at the 2^n:th time point at the latest.

7 Conclusions and Future Work

We have given a tableau-based proof system for a class of action scenarios. The proof system supports reasoning about concurrent actions, spontaneous change not caused by actions, and observations that may contain disjunctions and references to any points of time. All the laws that govern world change are strict in our system: there are no rules with a default behavior in the sense that the rules could be violated. This differs from some earlier work on dynamic worlds [Thielscher, 1995]. Nonmonotonicity in the proof system manifests itself in the tableau rule for inertia. Any extension of the set of transition rules or the action schedule may weaken the inertia rule.

Extending the proof system to support ramifications induced by state constraints does not seem easy. After the world changes, the state constraints are satisfied by making a minimal number of modifications to the world. These modifications are the indirect effects or ramifications of the change. To ensure that the change is minimal, we conceptually need to compare the world history to another world history that satisfies the state constraints. This comparison, in general, involves looking at the whole state of the world at two consecutive time points, and there does not seem to be an easy way to do this with the usual kind of tableau rules. Similarly, extending the proof system with causal rules [Lin, 1995; McCain and Turner, 1995] that model immediate indirect effects seems to require a test for the minimality of change to prevent unmotivated changes.

Our proof system can be easily extended with a primitive form of nondeterminism in which for example coin tossing can be represented. This extension is achieved by exempting the coin from inertia at the moment when it is tossed.

The possibility of supporting predicate symbols, terms, a restricted form of quantification, and the unique-names assumption, all without sacrificing decidability, is an important subject for future research because of their usefulness in practical applications.

References

[Cordier and Siegel, 1992] M. O. Cordier and P. Siegel. A temporal revision model for reasoning about world change. In B. Nebel, C. Rich, and W. Swartout, editors, *Proceedings of the 3rd International Conference on Principles of Knowledge Representation and Reasoning*, Cambridge, MA, October 1992. Morgan Kaufmann Publishers.

[Fitting, 1983] M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. Reidel, Dordrecht, Netherlands, 1983.

[Hanks and McDermott, 1987] S. Hanks and D. McDermott. Nonmonotonic logic and temporal projection. *Artificial Intelligence*, 33:379–412, 1987.

[Lifschitz and Rabinov, 1989] V. Lifschitz and A. Rabinov. Things that change by themselves. In *Proceedings of the 11th International Joint Conference on Artificial Intelligence*, pages 864–867, Detroit, MI, 1989.

[Lin, 1995] Fangzhen Lin. Embracing causality in formal theories of action. In *Proceedings of the 14th International Joint Conference on Artificial Intelligence*, pages 1985–1991, Montréal, Canada, 1995.

[McCain and Turner, 1995] Norman McCain and Hudson Turner. A causal theory of ramifications and qualifications. In *Proceedings of the 14th International Joint Conference on Artificial Intelligence*, pages 1978–1984, Montréal, Canada, 1995.

[Smullyan, 1968] R.M. Smullyan. *First-Order Logic*. Springer-Verlag, Berlin, 1968.

[Thielscher, 1995] Michael Thielscher. The logic of dynamic systems. In *Proceedings of the 14th International Joint Conference on Artificial Intelligence*, pages 1956–1962, Montréal, Canada, 1995. Morgan Kaufmann Publishers.