A Solution Algorithm for a System of Interval Linear Equations Based on the Constraint Interval Point of View*

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Abstract

This paper focuses on solving systems of interval linear equations in a computationally efficient way. The absence of additive inverse in the interval computations motivated the authors to consider this subject. To his end, Lodwick ("Interval linear systems as a necessary step in fuzzy linear systems," 2015) proposed a technical approach, and we extend it to a practical one. Since the computational complexity of most interval enclosure numerical methods is often prohibitive, a procedure to obtain a relaxation of the interval enclosure solution that is computationally tractable is proposed. We show that our approach unifies the four standard interval solutions – the weak, strong, control and tolerance solutions. Numerical examples illustrate advantages of our approach.

Keywords: interval linear system, constraint intervals, extended interval enclosure solutions
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1 Introduction to Interval Analysis and the Lack of Additive Inverse

Systems of linear equations are applicable in various fields of science such as engineering, physics, computer science, technology, business, and economics. However, since many applications deal with data that are not deterministic, the parameters of corresponding systems are often non-deterministic and assumed to vary within prescribed intervals. To model applications, it is necessary to consider uncertainty. One approach for uncertainty quantification is to consider an interval as an encoding of uncertainty. Thus, interval linear systems are frequently used to model linear problems subject to interval uncertainty.

On the other hand, interval analysis is not only useful but necessary to the understanding of fuzzy interval analysis, especially in the context of linear systems [6]. The lack of understanding of interval linear equations and the mathematical space of interval entities has led to various anomalies in fuzzy linear system research [6].

We apply constraint intervals in our analysis (see [4], [6]) instead of standard interval analysis [1] to overcome the lack of additive inverses. Lack of inverses is a characteristic of classical interval spaces. Constraint intervals represent an interval as a parametrization, a linear function with nonnegative slope. That is, an interval, as $[\underline{x}, \overline{x}]$ in which $\underline{x} \leq \overline{x}$ and $\underline{x}, \overline{x} \in \mathbb{R}$ is represented as a function of a single parameter (variable), $f(\lambda_x) = \underline{x} + (\overline{x} - \underline{x})\lambda_x$. For example, expression $[3, 4] - [1, 2]$ is converted to $(3+(4-3)\lambda_1) - (1+(2-1)\lambda_2) = 2 - \lambda_1 - \lambda_2; 0 \leq \lambda_i \leq 1, i = 1, 2$. Interval expressions using this transformation have their resulting calculations in the space of real numbers, a parameter space, instead of the space of intervals. To obtain an interval solution, a global minimum and maximum with respect to all the associated parameters (there is one parameter per distinct interval variable) is required over the unit hypercube, the minimum being the left endpoint and the maximum yielding the right endpoint. If the expression is well-posed, for example, it is continuous, an interval solution is always theoretically possible to obtain.

This study uses the constraint interval approach to solve interval systems of equations and interval linear programming problems. This method is different from the usual way interval systems are typically solved, and it unifies the four interval linear system solution types. Furthermore, we propose two nonlinear minimizing and maximizing problems to obtain extended interval enclosure solutions in a computationally tractable manner, which is not possible using any other interval approach. That is, our main contribution is two-fold. We present a method to solve the interval linear system problem that is tractable, which is used in interval linear programming problems, and we unify the four interval linear system solutions (strong, control, tolerant, weak) associated with interval linear programming problems. In addition, this technique can be used for the rectangular matrix problems typical of linear programming. To the best of our knowledge, this computationally tractable method for rectangular systems is new.

Research papers akin to our study of interval spaces have the aim of solving interval problems. This paper adopts and extends what is proposed in [5] to obtain an interval enclosure (IE) (see [8]) we call the extended IE(EIE). Some well-known methods for solving interval linear equation are Oettli [9], Jansson [2] and Shary [12]. Oettli [9] proves that the intersection of the solution set of a real linear interval system with each orthant is a convex polyhedron. This fact was used in linear programming to compute exact lower and upper bounds of the solution set in each orthant. Since there are 2^n orthants for $x \in \mathbb{R}^n$, the method proposed in [9] can be used only for small dimensions. Jansson [2] proposed some topological and graph theoretical properties of the solution set of linear algebraic systems with interval coefficients. In [2] a method is described in which, in a finite number of steps, either the method calculates the exact bounds for each component of the solution set, or it finds a singular matrix within the given interval coefficients. The calculation of exact bounds of the solution set is known to be NP-hard [10]. The method proposed in [2] needs p calls of a polynomial-time algorithm, where p is the number of nonempty intersections of the solution set in each orthant. Moreover, Jansson [2] also mentions that due to physical or economical requirements, many variables do not change in sign. This eliminates the number of orthants that need to be consulted. In these cases p is small, and our extended IE method works efficiently. It is in this sense that our methods are tractable. Shary [12] investigated basic properties of the algebraic (classic) solutions to the interval linear systems and proposed a number of numerical methods to compute them. While we use Shary's [12] taxonomy and definition of solution of interval linear systems, we depart from his solution methods.

Remark 1 *In this paper, we demonstrate that constraint intervals have theoretical and computational advantages. This paper contributes to the theoretical and computational understanding in which solutions to interval linear systems are formulated and solved in a unique, powerful and unified way. There are many research results associated with interval linear systems [12], the most important difference and advantage of our approach is to be independent of calculating the inverse and to apply to non-square matrices which appear in linear programming problems.*

Remark 2 *We do not claim that solving interval linear systems is new nor necessarily computationally more efficient. Certainly interval linear systems have been efficiently solved for 50 years. We claim that our approach unifies the theory and that linear systems can be solved from this point of view.*

Remark 3 *There are many interval arithmetic methods (see [5] pages 22-33). There, in particular, affine arithmetic of [13] is discussed. In [5] many types of interval arithmetics are discussed, not just affine arithmetic. Affine arithmetic deals with dependencies via a series of parameters that are known a-priori. A constraint interval is explicitly given as a* ***function*** *of a single parameter. This is not how affine intervals are defined in [13]. Constraint intervals are elements in a space of functions (see [7]). Constraint intervals encode both dependencies and symbolically represent the whole interval by one parameter that is constrained to lie between 0 and 1. The constraint interval point of view leads to global optimization algorithms, whereas the affine point of view leads to an arithmetic.*

2 Preliminaries

Let a system of linear equations $[A]x = [b]$, for which $[A]$ and $[b]$ are an $n \times n-$matrix and an $n-$vector right-hand side, respectively, with interval-valued elements. The aim is to obtain its interval solution set. Strong solution (pessimistic, classic), control solution, tolerance solution, algebraic solution, and weak solution (optimistic, united set solution) are some definitions of solution sets associated with systems of constraints, which is our interest (see [6] for our point of view). We begin with the four already known solution types proposed by interval analysis researchers (see Lodwick [3], Rohn

[10] and Chapter 2 [1], and Shary [12]) for a real-valued interval linear system $[A]x = [b]$.

Definition 1 ***United solution set*** *[6] A vector $x \in \mathbb{R}^n$ is called a united solution of the interval linear system $[A]x = [b]$, if $Ax = b$ is satisfied for some $A \in [A]$ and for some $b \in [b]$. We denote the set of united solutions by*

$$\Omega_{\exists\exists} = \{x | \exists A \in [A], \exists b \in [b] \text{ such that } Ax = b\}.$$

Definition 2 ***Tolerance solution set*** *[6] A vector $x \in \mathbb{R}^n$ is called a tolerance solution of the interval linear system $[A]x = [b]$, if $Ax = b$ is satisfied for all $A \in [A]$ and for some $b \in [b]$. We denote the set of tolerance solutions by*

$$\Omega_{\forall\exists} = \{x | \forall A \in [A], \exists b \in [b] \text{ such that } Ax = b\}.$$

Definition 3 ***Control solution set*** *[6] A vector $x \in \mathbb{R}^n$ is called a control solution of the interval linear system $[A]x = [b]$, if $Ax = b$ is satisfied for some $A \in [A]$ and for all $b \in [b]$. We denote the set of control solutions by*

$$\Omega_{\exists\forall} = \{x | \forall b \in [b], \exists A \in [A] \text{ such that } Ax = b\}.$$

Definition 4 ***Classical solution set*** *[6]*

$$\Omega_{\forall\forall} = \Omega_{\forall\exists} \bigcap \Omega_{\exists\forall}.$$

In what follows, we use boxes for two-dimensional intervals, cubes for three-dimensional intervals, and hyper-boxes for intervals in dimensions higher than three. It can be verified that the solution sets $\Omega_{\exists\exists}, \Omega_{\forall\exists}, \Omega_{\exists\forall}$ and $\Omega_{\forall\forall}$ are not necessarily a box, cube or hyper-box. A new definition proposed by Lodwick and Jenkins [7] is: the smallest box that contains the weak solution (they called the united solution) that never loses a possible solution value, denoted by NLV. This is called the interval hull in the interval analysis literature. Here, we use the term, "interval enclosure" (IE), which is this also a standard term found in interval analysis. In what follows, we use constraint intervals, defined next, to construct the IE solution.

Definition 5 ***Constraint interval*** *(see [6]) A constraint interval is an interval that is defined as a linear function with non-negative slope over $[0, 1]$,*

$$[x] = x(\lambda_x) = \underline{x} + w_x.\lambda_x; w_x = \overline{x} - \underline{x}, 0 \leq \lambda_x \leq 1.$$

Interval matrices and vectors defined from the constraint interval point of view are:

$$[A] = A(\Lambda_A) = \underline{A} + W_A.\Lambda_A; W_A = \overline{A} - \underline{A}; \tag{1}$$

and

$$[b] = b(\lambda_b) = \underline{b} + w_b.\lambda_b; w_b = \overline{b} - \underline{b}; \tag{2}$$

where W_A and Λ_A are real valued matrices, w_b and λ_b are real valued vectors with

$$0 \leq (\Lambda_A)_{ij} \leq 1, 0 \leq (\lambda_b)_i \leq 1; i = 1, ..., m, j = 1, ..., n.$$

The product operator between W_A, Λ_A, w_b and λ_b in Equations (1) and (2) is the Hadamard product, denoted $W_A.\Lambda_A$ and $w_b.\lambda_b$. Using this transformation of an interval linear system into a constraint interval setting, the IE solution is obtained as follows (see [6]). For $[A]x = [b]$, the NLV solution is,

$$[x] = [\underline{x}, \overline{x}] = [\min_{0 \leq \Lambda_A, \lambda_b \leq 1} A(\Lambda_A)^{-1} b(\lambda_b), \max_{0 \leq \Lambda_A, \lambda_b \leq 1} A(\Lambda_A)^{-1} b(\lambda_b)]. \tag{3}$$

It can be proved that [6]

$$\Omega_{\forall\forall} \subseteq \Omega_{\forall\exists} \subseteq \Omega_{\exists\exists} \subseteq [x],$$

and

$$\Omega_{\forall\forall} \subseteq \Omega_{\exists\forall} \subseteq \Omega_{\exists\exists} \subseteq [x].$$

Note that (3) is a unified approach for the four solutions (see [7]), since we can consider all four cases by varying "for all" and "for each" with Λ_A and λ_b. This is what we do subsequently. Although the interval solution set of (3) is well-defined, this method is applicable for square systems with an invertible coefficient matrix, and thus not suitable for general linear programming problems. Moreover, (3) requires a constrained global optimization of a nonlinear non-convex problem. This means that numerical methods most often converge to a local optimal solution instead of a global optimal solution. This IE method needs first to calculate the inverse of the coefficient matrix, and then solve $2n$ nonlinear non-convex programming problems. Thus, (3) is a theoretical approach.

3 The Extended IE Solution

The theoretical IE solution proposed by Lodwick and Dubois [6] is tractable for small square matrices, and is a nonlinear programming problem. These two properties motivated the authors to extend the IE solution approach without the necessity to calculate the inverse, with the view to apply it to large scale problems and non-square problems. In addition, we propose an algorithm to relax the nonlinear non-convex problem to a linear convex one. Based on (3) consider the two following programming problems

$$\begin{array}{llcl} \mathrm{P1}: \min\limits_{\Lambda_A,\lambda_b} & e_j^T x & & \\ \text{s.t.} & A(\Lambda_A)x & \leq & b(\lambda_b) \\ & 0 \leq (\Lambda_A)_{ij} & \leq 1 & \\ & 0 \leq (\lambda_b)_i & \leq 1 & \end{array} \qquad \begin{array}{llcl} \mathrm{P2}: \max\limits_{\Lambda_A,\lambda_b} & e_j^T x & & \\ \text{s.t.} & A(\Lambda_A)x & \leq & b(\lambda_b) \\ & 0 \leq (\Lambda_A)_{ij} & \leq 1 & \\ & 0 \leq (\lambda_b)_i & \leq 1 & \end{array} \tag{4}$$

for $i = 1, ..., m$ and $j = 1, ..., n$. Let $\underline{x} = [\underline{x}_j]_{j=1,...,n}$ and $\overline{x} = [\overline{x}_j]_{j=1,...,n}$ be the symbolic representation of the optimal values for P1 and P2, respectively.

Clearly, problems P1 and P2 address the first of the two properties, but they are still nonlinear, non-convex and NP-hard. We next address the second of the two properties by transforming the nonlinear and non-convex problems via a relaxation. To this end, we change the nonlinear constraints of P1 and P2 to linear ones in such a way that the obtained solution contains all the solutions of P1 and P2. We recall that for the equality $A(\Lambda_A)x = b(\lambda_b)$ with

$$0 \leq (\Lambda_A)_{ij} \leq 1; 0 \leq (\lambda_b)_i \leq 1;$$

we have

$$A(\Lambda_A)x = (\underline{A} + W_A.\Lambda_A)x = \underline{b} + w_b.\lambda_b = b(\lambda_b); \tag{5}$$

or

$$\underline{A}x - \underline{b} - w_b.\lambda_b = -W_A.\Lambda_A x; \tag{6}$$

where $\leq (\Lambda_A)_{ij} \leq 1$ and $0 \leq (\lambda_b)_i \leq 1$. We know that Equation (6) can be rewritten as

$$\begin{cases} (\underline{A} + W_A.\Lambda_A)x \leq \underline{b} + w_b.\lambda_b \\ (\underline{A} + W_A.\Lambda_A)x \geq \underline{b} + w_b.\lambda_b. \end{cases} \tag{7}$$

Remark 4 *If we choose a particular λ, either 0 or 1 for all the parameters λ, we will obtain various kinds of solutions containing the united, control, tolerance and classical solutions. It is clear that (3) is the interval hull. Moreover, since it was shown in [6], that the constraint interval approach encompasses the four standard solution types (united extension, control, tolerance, classical-set theoretic), the methods applied to each of the four types contain as subsets each of the types.*

The right-hand side of Equation (6) is the only nonlinear part of the equality. Since $W_A \geq 0$, and $0 \leq \Lambda_A \leq 1$, depending on the sign of x, we can obtain a lower and an upper bound for the right-hand side. To determine the sign of x, we restrict the region to each orthant. Let e_j be the unique vector in $\mathbb{R}^n$ for which the j^{th} component is 1 and all of the other elements are zero. In an n-dimensional space, there are 2^n orthants. Corresponding to the q^{th} orthant ($q = 1, ..., 2^n$), we can construct matrix D^q such that $D^q = [d_1^q, d_2^q, ..., d_n^q]$, where d_n^q is either e_j or $-e_j$. We can also denote $D^q = \mathrm{Diag}^q(\alpha^q) = \mathrm{Diag}^q(\alpha_1^q, \alpha_2^q, ..., \alpha_n^q)$ where $\alpha_j^q = \{-1, 1\}; j = 1, ..., n, q = 1, ..., 2^n$. It can be shown that Diag^q is a permutation matrix. Let D^q be a permutation matrix. An element $z \in \mathbb{R}^n$ belongs to q^{th} orthant if and only if $D^q z \geq 0$. Two matrices $R^q = [R_j]$ and $S^q = [S_j]$ can be defined as bounds for $W_A \Lambda_A D^q$ as

$$R^q \leq W_A \Lambda_A D^q \leq S^q. \tag{8}$$

The relaxed problem is defined as follows for any $q = 1, .., 2^n$, and $j = 1, ..., n$;

$$\begin{array}{rlll} \mathrm{P1}_{relaxed} : (\underline{x}_{relaxed})_{jq} = \inf & e_j^T x & & \\ \text{s.t.} & (\underline{A} + L^q)x - w_b\lambda_b & \leq & \underline{b} \\ & (\underline{A} + U^q)x - w_b\lambda_b & \geq & \underline{b} \\ & 0 \leq & (\lambda_b)_i & \leq 1 \\ & D^q x \geq 0 & & \end{array} \tag{9}$$

$$\begin{array}{rlll} \mathrm{P2}_{relaxed} : (\overline{x}_{relaxed})_{jq} = \sup & e_j^T x & & \\ \text{s.t.} & (\underline{A} + L^q)x - w_b\lambda_b & \leq & \underline{b} \\ & (\underline{A} + U^q)x - w_b\lambda_b & \geq & \underline{b} \\ & 0 \leq & (\lambda_b)_i & \leq 1 \\ & D^q x \geq 0 & & \end{array} \tag{10}$$

Problems $\mathrm{P1}_{relaxed}$ and $\mathrm{P2}_{relaxed}$ are linear and convex. There are $n \times 2^n$ problems to be solved, and the algorithm is still NP-hard as the size of problem increases. Clearly in each orthant, all elements of the solution will be a minimum and maximum. Thus, we define the extended IE solution as $(\underline{x}_{relaxed})_j = \min\{(\underline{x}_{relaxed})_{jq}; q = 1, ..., 2^n\}$ and $(\overline{x}_{relaxed})_j = \max\{(\overline{x}_{relaxed})_{jq}; q = 1, ..., 2^n\}$

Theorem 3.1 *Let $\underline{x}$ and $\overline{x}$ be the optimal solutions of problems P1 and P2 respectively, and $\underline{x}_{relaxed}$ and $\overline{x}_{relaxed}$ be the optimal solutions of problems $P1_{relaxed}$ and $P2_{relaxed}$, respectively. Then,*

$$[\underline{x}, \overline{x}] \subseteq [\underline{x}_{relaxed}, \overline{x}_{relaxed}].$$

Proof: Suppose $\underline{x}$ and $\overline{x}$ are the optimal solutions of problems P1 and P2 . This means that $\underline{x}$ and $\overline{x}$ satisfy the constraints. Without loss of generality, we prove the theorem for either $\underline{x}$ or $\overline{x}$. Let us consider $\underline{x}$. From Equations (5) and (6), $A(\Lambda_x)\underline{x} = b(\lambda_b)$ implies that $(\underline{A} + W_A\Lambda_A)\underline{x} = (\underline{b} + w_b.\lambda_b)$ and $\underline{A}x - \underline{b} - w_b.\lambda_b = -W_A.\Lambda_A\underline{x}$. We have $W_A \geq 0$ and $0 \leq (\Lambda_A)_{ik} \leq 1; i = 1, ..., m; k = 1, ..., n$. On the other hand, since $D^q x \geq 0; q = 1, ..., 2^n$ multiplying relationship (8) by $D^q x$ leads to $R^q D^q x \leq W_A.\Lambda_A D^q D^q x \leq S^q D^q x$. Clearly, $D^q D^q = I$, and therefore we have $L^q x \leq W_A.\Lambda_A x \leq U^q x$, where $L^q = R^q D^q$ and $U^q = S^q D^q$. Consequently, Equation (6) implies that we have $-U^q x \leq \underline{A}x - \underline{b} - w_b.\lambda_b \leq -L^q x$. Hence $\underline{x}$ is a feasible point of the problem $\mathrm{P1}_{relaxed}$. On the other hand, since $\underline{x}$ is the optimal solution, for any x, we have $(e_j)^T \underline{x} \leq (e_j)^T \overline{x}$. $\square$

Theorem 3.2 *If A is invertible and all components of A^{-1} are negative, then* $[\underline{x}, \overline{x}] = [\underline{x}_{relaxed}, \overline{x}_{relaxed}]$.

Proof: Let $\underline{x}$ and $\overline{x}$ be the optimal solutions of the problems P1 and P2 respectively, $\underline{x}_{relaxed}$ and $\overline{x}_{relaxed}$ be the optimal solutions of problems $\mathrm{P1}_{relaxed}$ and $\mathrm{P2}_{relaxed}$ respectively. According to Theorem 3.1, it is sufficient to prove

$$[\underline{x}_{relaxed}, \overline{x}_{relaxed}] \subseteq [\underline{x}, \overline{x}].$$

That is, for any $x \in [\underline{x}_{relaxed}, \overline{x}_{relaxed}]$, we have $x \in [\underline{x}, \overline{x}]$. Let $x \in [\underline{x}_{relaxed}, \overline{x}_{relaxed}]$ be in the q^{th} orthant. Therefore, from the definition of $\mathrm{P1}_{relaxed}$, we have

$$\begin{array}{c} (A + L^q) - w_b.\lambda_b \leq \underline{b}, \text{ and} \\ (A + U^q) - w_b.\lambda_b \geq \underline{b} \end{array} \tag{11}$$

where $0 \leq \lambda_b \leq 1$. Since the matrix function $W_A.\Lambda_A D^q$ is continuous in terms of Λ_A's components and its domain is compact, the minimum and maximum of $W_A.\Lambda_A D^q$ exist, that is,

$$\exists \underline{\Lambda}, \overline{\Lambda}; R^q = W_A.\underline{\Lambda}_A D^q, S^q = W_A.\overline{\Lambda}_A D^q.$$

Therefore

$$\begin{array}{c} L^q = R^q D^q = W_A.\underline{\Lambda}_A D^q D^q = W_A.\underline{\Lambda}_A \\ U^q = S^q D^q = W_A.\overline{\Lambda}_A D^q D^q = W_A.\overline{\Lambda}_A. \end{array} \tag{12}$$

Inequalities (11) imply that

$$\begin{array}{c} (\underline{A} + W_A.\underline{\Lambda}_A)x \leq \underline{b} + w_b.\lambda_b \\ (\underline{A} + W_A.\overline{\Lambda}_A)x \geq \underline{b} + w_b.\lambda_b. \end{array} \tag{13}$$

Since $A(\Lambda_A)^{-1} \leq 0$, we have

$$\begin{cases} x \geq A(\underline{\Lambda}_A)^{-1} b(\lambda_b) \geq \min_{\Lambda_A, \lambda_b} A(\Lambda_A)^{-1} b(\lambda_b), \\ x \leq A(\overline{\Lambda}_A)^{-1} b(\lambda_b) \leq \max_{\Lambda_A, \lambda_b} A(\Lambda_A)^{-1} b(\lambda_b). \end{cases} \tag{14}$$

Thus,

$$\begin{cases} x \geq \min_{\Lambda_A, \lambda_b} A(\Lambda_A)^{-1} b(\lambda_b), \\ x \leq \max_{\Lambda_A, \lambda_b} A(\Lambda_A)^{-1} b(\lambda_b), \end{cases} \tag{15}$$

and the proof is completed. $\square$

Algorithm 1 To obtain the extended IE solution by $[A], [b], W_A, w_b$.

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for $q = 1, ..., 2^n$
construct $U^q = [u_1^q, u_2^q, ..., u_n^q]$ **and** $L^q = [l_1^q, l_2^q, ..., l_n^q]$ **where**

$$l_i^q = \begin{cases} (W_A)_i & if \quad d_i = e_i \\ 0 & if \quad d_i = -e_i \end{cases} \qquad u_i^q = \begin{cases} 0 & if \quad d_i = e_i \\ (W_A)_i & if \quad d_i = -e_i \end{cases}$$

in which d_i **is the** i^{th} **column of** $D^q = \text{Diag}^q(\alpha_1^q, ..., \alpha_n^q)$ **where**
$\alpha_i^q \in \{-1, 1\}$ **for** $j = 1, ..., n$ **compute**
$(\underline{x}_{relaxed})_{jq} = \inf e_j^T x$ subject to $(\overline{A} + L^q)x - w_b.\lambda_b \geq \underline{b}, (\underline{A} + U^q)x - w_b.\lambda_b \leq \underline{b}, 0 \leq \lambda_b \leq 1, D^q x \geq 0$
$(\underline{x}_{relaxed})_{jq} = \sup e_j^T x$ subject to $(\overline{A} + L^q)x - w_b.\lambda_b \geq \underline{b}, (\underline{A} + U^q)x - w_b.\lambda_b \leq \underline{b}, 0 \leq \lambda_b \leq 1, D^q x \geq 0$
end
end
$(\underline{x}_{relaxed})_j = \min\{(\underline{x}_{relaxed})_{jq}, q = 1, ..., 2^n\}$
$(\overline{x}_{relaxed})_j = \max\{(\underline{x}_{relaxed})_{jq}, q = 1, ..., 2^n\}$

We compare IE and the extended IE,

	Size	Inverse	Problem	Optimal Solution	Number of problems to be solved
IE	Square	need	NLP	Local	$2n$
EIE	Free	No need	LP	Global	$n2^n$

Example 1 *Consider the system of linear interval equations (see [6])*

$[2, 4]x_1 + [-2, 1]x_2 = [-2, 2]$
$[-1, 2]x_1 + [2, 4]x_2 = [-2, 2]$.

According to the IE solution and using INTLAB (see [11]), its solution is $-4 \leq x_1 \leq 4; -4 \leq x_2 \leq 4$. *To obtain the extended IE solution*

$$\underline{A} = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}; \; W_A = \begin{pmatrix} 4 & 1 \\ 2 & 4 \end{pmatrix}; \; \underline{b} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}; \; w_b = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

Orthant 1: $x_1 \geq 0$; $x_2 \geq 0$ *then* $D^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $L^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; $U^1 = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$; *and* $0 \leq x_1 \leq 4 \Rightarrow \underline{x}_1 = 0;\; \overline{x}_1 = 4;\; 0 \leq x_2 \leq 3 \Rightarrow \underline{x}_1 = 0;\; \overline{x}_1 = 3.$

Orthant 2: $x_1 \leq 0$; $x_2 \geq 0$ *then* $D^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; $L^2 = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}$; $U^2 = \begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix}$; *and* $-3 \leq x_1 \leq 0 \Rightarrow \underline{x}_1 = -3;\; \overline{x}_1 = 0;\; 0 \leq x_2 \leq 4 \Rightarrow \underline{x}_1 = 0;\; \overline{x}_1 = 4.$

Orthant 3: $x_1 \leq 0$; $x_2 \leq 0$ *then* $D^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$; $L^3 = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$; $U^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; *and* $-4 \leq x_1 \leq 0 \Rightarrow \underline{x}_1 = -4;\; \overline{x}_1 = 0;\; -3 \leq x_2 \leq 0 \Rightarrow \underline{x}_1 = -3;\; \overline{x}_1 = 0.$

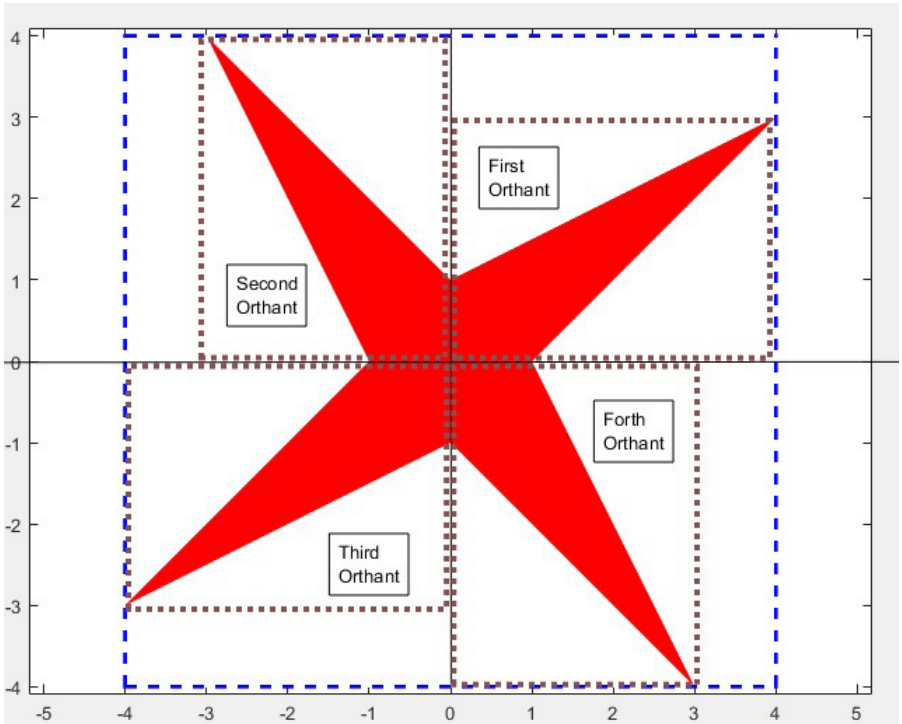

Figure 1: The red filled star, the blue dash box and the brown dot box in each orthant show the weak solution, IE solution, and our extended IE solution, respectively.

Orthant 4: $x_1 \leq 0$; $x_2 \leq 0$ *then* $D^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; $L^4 = \begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix}$; $U^4 = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}$; *and* $0 \leq x_1 \leq 4 \Rightarrow \underline{x}_1 = 0$; $\overline{x}_1 = 4$; $-4 \leq x_2 \leq 0 \Rightarrow \underline{x}_1 = -4$; $\overline{x}_1 = 0$.
Therefore,

$$\underline{x}_1 = \min -4, -3, 0 = -4$$
$$\overline{x}_1 = \max 0, 3, 4 = 4$$
$$\underline{x}_2 = \min -4, -3, 0 = -4$$
$$\overline{x}_2 = \max 0, 3, 4 = 4.$$

Figure 1 depicts the solution using INTLAB [11] with both the IE and Extended IE solutions.

Table 1 of the appendix shows the numerical results obtained by IE and the extended IE in detail, where OS stands for Obtained Solution and RT stands for Running Time. Clearly, the extended IE solution is either equal to the IE solution or includes it. On the other hand, for both algorithms, we deal with a programming problem. In this case, the IE is considered as a nonlinear programming problem. Therefore, as the size of the system increases, the number of orthants increases, which means that the run time of the extended IE also increases. On the other hand, IE obtains the optimal solution independent of orthants. Therefore, we expect that for large scale systems, the run time of the Extended IE to be more than that of IE, and this fact is confirmed by our preliminary numerical results.

4 Conclusion

The IE solution for a system of linear equations in the presence of interval parameters involves two nonlinear programming problems. For IE solutions, the coefficient matrix needs to be square and invertible. Moreover, the optimal solution is a local optimum. We proposed two optimization problems without having to invert coefficient matrices. Then a relaxation is proposed to transform the nonlinear problem to a linear one. Next, we obtained convexity by analyzing the solution set in each orthant. This extended IE solution contains the weak, strong, control, tolerance and IE solutions, and it is equal to the IE solution for square coefficient matrices.

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5 Appendix

size	sample	OS(IE)	OS(EIE)	RT(IE)	RT(EIE)
$n=2$	$A=\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$ $W_A=\begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$ $b^T=\begin{pmatrix} -2 & -2 \end{pmatrix}$ $w_b^T=\begin{pmatrix} 4 & 3 \end{pmatrix}$	$\begin{pmatrix} [-1.0789, 4] \\ [-5, 3.5] \end{pmatrix}$	$\begin{pmatrix} [-3, 4] \\ [-5, 3.5] \end{pmatrix}$	1.167084	0.287524
$n=2$	$A=\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$ $W_A=\begin{pmatrix} 3 & 4 \\ 3 & 3 \end{pmatrix}$ $b^T=\begin{pmatrix} -2 & -2 \end{pmatrix}$ $w_b^T=\begin{pmatrix} 4 & 3 \end{pmatrix}$	$\begin{pmatrix} [-2, 4] \\ [-5, 1.6585] \end{pmatrix}$	$\begin{pmatrix} [-3, 4] \\ [-5, 3.5] \end{pmatrix}$	1.268958	0.289187
$n=2$	$A=\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$ $W_A=\begin{pmatrix} 3 & 4 \\ 3 & 3 \end{pmatrix}$ $b^T=\begin{pmatrix} -3 & -2 \end{pmatrix}$ $w_b^T=\begin{pmatrix} 4 & 5 \end{pmatrix}$	$\begin{pmatrix} [-6, 0.6695] \\ [-2.25, 7.5] \end{pmatrix}$	$\begin{pmatrix} [-6, 3] \\ [-4, 7.5] \end{pmatrix}$	1.125899	0.294201
$n=2$	$A=\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$ $W_A=\begin{matrix} 3 & 4 \\ 2 & 3 \end{matrix}$ $b^T=\begin{pmatrix} -3 & -3 \end{pmatrix}$ $w_b^T=\begin{pmatrix} 4 & 4 \end{pmatrix}$	$\begin{pmatrix} [-21, 1.4045] \\ [-14.9998, 4.9998] \end{pmatrix}$	$\begin{pmatrix} [-21, 7] \\ [-15, 5] \end{pmatrix}$	0.943582	0.301753
$n=3$	$A=\begin{pmatrix} 3 & -4 & 4 \\ -2 & 3 & 5 \\ 3 & 4 & 3 \end{pmatrix}$ $W_A=\begin{pmatrix} 3 & 4 & 2 \\ 2 & 3 & 5 \\ 3 & 4 & 2 \end{pmatrix}$ $b^T=\begin{pmatrix} -3 & -3 & -3 \end{pmatrix}$ $w_b^T=\begin{pmatrix} 4 & 4 & 4 \end{pmatrix}$	$\begin{pmatrix} [-1.3153, 1.4496] \\ [-10.250, 10.750] \\ [-13.500, 12.500] \end{pmatrix}$	$\begin{pmatrix} [-8.1667, 8.500] \\ [-10.250, 10.75] \\ [-13.500, 12.50] \end{pmatrix}$	0.943582	0.301753
$n=4$	Random	$\begin{pmatrix} [-0.0010, 0.1949] \\ [-5.9060, 0.1943] \\ [-0.0046, 0.0493] \\ [-0.1204, 0.0648] \end{pmatrix}$	$\begin{pmatrix} [-24.4672, 6.5633] \\ [-12.7777, 4.9699] \\ [-1.0965, 1.0965] \\ [-1.2771, 1.2771] \end{pmatrix}$	4.237973	1.946557
$n=5$	Random	$\begin{pmatrix} [-4.4259, 0.0136] \\ [-0.1223, 0.0899] \\ [-0.0095, 2.0353] \\ [-0.0113, 0.0060] \\ [-0.0790, 0.0132] \end{pmatrix}$	$\begin{pmatrix} [-36.5793, 10.9590] \\ [-0.0000, 0.0000] \\ [-2.0581, 1.1558] \\ [-14.8230, 5.0579] \\ [-47.1432, 21.8656] \end{pmatrix}$	5.772440	4.662293
$m=3$ $n=2$	$A=\begin{pmatrix} 1 & 0.5 \\ 0.5 & 2 \\ 1.5 & 2 \end{pmatrix}$ $W_A=\begin{pmatrix} 2 & 0.5 \\ 0.5 & 1 \\ 1.5 & 2 \end{pmatrix}$ $b^T=\begin{pmatrix} 1.25 & 3.25 & 3.75 \end{pmatrix}$ $w_b^T=\begin{pmatrix} 5.75 & 5.75 & 10.75 \end{pmatrix}$	–	$\begin{pmatrix} [-6.5, 10.75] \\ [-3.75, 7.75] \end{pmatrix}$	–	0.233373

Table 1: Some numerical examples that compare the IE and EIE solutions