New Higher Order Root Finding Algorithm Using Interval Analysis*

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Abstract

We propose an implicit interval Newton algorithm for finding a root of a nonlinear equation. Our implicit process avoids the burden of selecting an initial point very close to the root as required by the classical Newton method. The conventional interval Newton method converges quadratically, while our modified algorithm converges cubically under some conditions and bi-quadratically under other conditions. These convergence results are proved using an image extension of a Taylor expansion. The algorithm is illustrated by numerical examples, and we provide a comparison between the existing interval Newton method and our proposed method.

Keywords: interval analysis, image extension, inclusion isotonic extension, Lipschitz continuity, interval Newton method
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1 Introduction

Interval computation plays an important role in many mathematical models involving quantities that are not exactly representable. In this situation, lower and upper bound of these parameters can describe a solution rigorously. Interval analysis has been used for this purpose by many researchers for root finding (Moore [19], Neumaier [21]), for solving a system of equations (Krawczyk [16]), for optimization (Hansen [14]), and for many real applications. In most of these algorithms, a solution of a real problem is determined by considering an initial interval and reducing it in subsequent

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steps to an narrow interval containing the solution. A widely used algorithm in this category is the interval Newton method (Moore in 1966 [19]) for finding an isolated root of an equation and for solving a system of equations. The classical floating-point Newton method requires the selection of the initial guess close to the root/solution, but the interval Newton method is free from this burden. The interval Newton method for finding simple root of a real-valued single variable function converges to the root quadratically if the image extension of derivative of this function does not contain 0. This algorithm was modified by Alefeld [1] in 1968 and Hansen [10] in 1978 (independently) to include the case where image extension of derivative of the function may contain 0. The modified interval Newton method developed in [10] finds the bounds of all real roots.

The interval Newton method for finding root of a system of equations introduced by Moore [19] was extended by several researchers. The methods due to Hansen [8], Alefeld and Herzberger [2], and Madsen [17] avoid inverting an interval matrix, and the methods of Hansen and Sengupta [13] and Hansen and Greenberg [12] focus on greater efficiency of the algorithm. Further developments due to Petković [26, 27] and Carstensen and Petković [5] focus on higher order root finding algorithms. Solution of the equations with interval coefficients is studied by several researchers (see [11, 23, 30, 31]). Readers may see [4, 7, 18, 25, 29, 32] for some algorithms to find simple and multiple complex polynomial zeros in the complex plane using circular complex arithmetic. These algorithms ([4, 7, 18, 25, 29, 32]) are applicable for finding root(s) of polynomials only; for transcendental equations, these methods fail. Developments using interval computations are summarized in [3, 14, 20, 28].

Recently, many researchers have extended the classical point Newton method (with no interval computations) for root finding with higher order convergence behavior using several techniques (see [6, 24, 33, 34]). These methods require an initial point be selected that is "sufficiently close" to the root being sought. In this paper, we develop a new algorithm using an implicit interval Newton method. Our algorithm has less restrictions on initial point selection, and it has higher than second order convergence, converging cubically under certain conditions and bi-quadratically under stronger conditions. Another advantage of this algorithm is that selection of an initial point very close to the root is not essential; the user may select an initial interval containing the solution, with milder restrictions on the size. We will apply such a higher order algorithm for a system of equations in the future.

The paper is organized as follows. Some prerequisites on interval analysis and the classical second order interval Newton method are discussed in Section 2. A new algorithm is proposed in Section 3, and convergence of the algorithm is proved. The usual interval Newton algorithm and our algorithm are compared in Section 4 with some numerical examples.

2 Prerequisites

We follow the interval notations of [15]. Let $\mathbb{IR}$ be the set of all closed intervals. Bold letters denote intervals. $\boldsymbol{x} \in \mathbb{IR}$ is the set $\boldsymbol{x} = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} | \underline{x} \leq x \leq \overline{x}\}$. For $\boldsymbol{x} \in \mathbb{IR}$, $m(\boldsymbol{x}) = \frac{\overline{x}+\underline{x}}{2}$ is the midpoint of $\boldsymbol{x}$, and $w(\boldsymbol{x}) = \overline{x} - \underline{x}$ is the width of $\boldsymbol{x}$.

For $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathbb{IR}$, $\boldsymbol{x} = [\underline{x}, \overline{x}]$, $\boldsymbol{y} = [\underline{y}, \overline{y}]$,

- $\boldsymbol{x} = \boldsymbol{y}$ iff $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$.
- $\boldsymbol{x}$ is degenerate if $\underline{x} = \overline{x}$. (Every real number can be expressed as a degenerate interval. A degenerate interval $[x, x]$ is denoted by $\widehat{x}$.)
- Absolute value of $\boldsymbol{x}$ is denoted by $|\boldsymbol{x}| = \max\{|\overline{x}|, |\underline{x}|\}$.

- An algebraic operation * in $I(\mathbb{R})$ is defined as $\boldsymbol{x}*\boldsymbol{y} = \{x * y, \quad x \in \boldsymbol{x}, \ y \in \boldsymbol{y}$, where $*$ is the algebraic operation$\{+, -, \cdot, /\}$ in the set of real numbers$\}$. Hence,
  $\boldsymbol{x}+\boldsymbol{y} = [\underline{x}+\underline{y}, \overline{x}+\overline{y}]$, $\boldsymbol{x}$-$\boldsymbol{y} = [\underline{x}-\overline{y}, \overline{x}-\underline{y}]$,
  $\boldsymbol{x}\cdot\boldsymbol{y} = [\min\{\overline{x}\cdot\overline{y}, \overline{x}\cdot\underline{y}, \underline{x}\cdot\underline{y}, \underline{x}\cdot\overline{y}\}, \max\{\underline{x}\cdot\underline{y}, \underline{x}\cdot\overline{y}, \overline{x}\cdot\underline{y}, \overline{x}\cdot\overline{y}\}]$,
  $\boldsymbol{x}/\boldsymbol{y} = [\min\{\overline{x}/\overline{y}, \overline{x}/\underline{y}, \underline{x}/\underline{y}, \underline{x}/\overline{y}\}, \max\{\underline{x}/\underline{y}, \underline{x}/\overline{y}, \overline{x}/\underline{y}, \overline{x}/\overline{y}\}]$, provided $0 \notin \boldsymbol{y}$.
- For any $c \in \mathbb{R}$, $c\boldsymbol{x} = \begin{cases} [c\underline{x}, \ c\overline{x}], & \text{if } c \geq 0; \\ [c\overline{x}, c\underline{x}], & \text{if } c < 0. \end{cases}$

Following results can be derived using the notation above:

$$\begin{aligned} w(\boldsymbol{x} \pm \boldsymbol{y}) &= w(\boldsymbol{x}) + w(\boldsymbol{y}), && (1) \\ w(c\boldsymbol{x}) &= |c| w(\boldsymbol{x}), && (2) \\ w(1/\boldsymbol{x}) &= \frac{1}{\underline{x}\overline{x}} w(\boldsymbol{x}), \text{ provided } 0 \notin \boldsymbol{x}, && (3) \\ w(\boldsymbol{x}\boldsymbol{y}) &\leq |\boldsymbol{x}| w(\boldsymbol{y}) + |\boldsymbol{y}| w(\boldsymbol{x}), && (4) \\ w(\boldsymbol{x}) &= 0 \text{ iff } \boldsymbol{x} \text{ is a degenerate interval}, && (5) \\ w(\boldsymbol{x} \cap \boldsymbol{y}) &\leq \min\{w(\boldsymbol{x}), w(\boldsymbol{y})\}, && (6) \\ \text{if } \boldsymbol{x} &\subseteq \boldsymbol{y} \text{ then } 1/\boldsymbol{x} \subseteq 1/\boldsymbol{y}, \text{ provided } 0 \notin \boldsymbol{y}. && (7) \end{aligned}$$

2.1 Set image and interval extension under a continuous function

The following results and definitions are from [3, 20].

Definition 1. *The set image of the interval $\boldsymbol{x}$ under a continuous function $f : \mathbb{R} \to \mathbb{R}$ is the set*

$$f(\boldsymbol{x}) = \{f(x) : x \in \boldsymbol{x}\} = \left[\min_{x \in \boldsymbol{x}} f(x), \max_{x \in \boldsymbol{x}} f(x)\right].$$

Definition 2. *F is said to be an interval extension of $f : \mathbb{R} \to \mathbb{R}$, if the degenerate interval arguments of F agrees with f. That is, $F([x, x]) = f(x)$.*

Definition 3. *An interval extension F of $f : \mathbb{R} \to \mathbb{R}$ is said to be an inclusion isotonic interval extension of f if $\boldsymbol{y} \subseteq \boldsymbol{x}$ implies $F(\boldsymbol{y}) \subseteq F(\boldsymbol{x})$ for any two intervals $\boldsymbol{x}$ and $\boldsymbol{y}$.*

Definition 4. *An interval extension F of $f : \mathbb{R} \to \mathbb{R}$ is said to be Lipschitz in $\boldsymbol{x}$ if $w(F(\boldsymbol{y})) \leq Lw(\boldsymbol{y})$ for every $\boldsymbol{y} \subseteq \boldsymbol{x}$ for some constant L.*

Note 1. *The set image of an interval under a continuous function is an inclusion isotonic interval extension.*

Note 2. *If f is a Lipschitz continuous function, then the set image of $\boldsymbol{x}$ under f is a Lipschitz extension in $\boldsymbol{x}$.*

Definition 5. *For $\boldsymbol{x} = [\underline{x}, \overline{x}]$ and $\boldsymbol{y} = \left[\underline{y}, \overline{y}\right]$, the distance between $\boldsymbol{x}$ and $\boldsymbol{y}$, denoted by $d(\boldsymbol{x}, \boldsymbol{y})$, is*

$$d(\boldsymbol{x}, \boldsymbol{y}) = \max\left\{|\underline{x} - \underline{y}|, |\overline{x} - \overline{y}|\right\}.$$

d is a metric.

Definition 6. *A sequence of intervals $\left\{\boldsymbol{x}^k\right\}$ is said to be convergent if there exists an interval $\boldsymbol{x}^*$ such that for every $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that $d(\boldsymbol{x}^k, \boldsymbol{x}^*) < \varepsilon$ whenever $k > N$. (In short, we may write $\lim\limits_{k\to\infty} \boldsymbol{x}^k = \boldsymbol{x}^*$.)*

For a sequence of intervals $\left\{\boldsymbol{x}^k\right\}$ which converges to $\boldsymbol{x}^*$ with $\boldsymbol{x}^* \subseteq \boldsymbol{x}^k$ for every k, there are two ways to measure the deviation of $\boldsymbol{x}^k$ from $\boldsymbol{x}^*$: calculating either $d(\boldsymbol{x}^k, \boldsymbol{x}^*)$ or $w(\boldsymbol{x}^k) - w(\boldsymbol{x}^*)$. Both are nonnegative, and

$$d(\boldsymbol{x}^k, \boldsymbol{x}^*) = 0 \text{ iff } \boldsymbol{x}^k = \boldsymbol{x}^* \text{ iff } w(\boldsymbol{x}^k) - w(\boldsymbol{x}^*) = 0.$$

Definition 7. *(See Appendix A of [3]) Let $\left\{\boldsymbol{x}^k\right\}$ be a nested sequence of intervals so that $\lim\limits_{k\to\infty} \boldsymbol{x}^k = \boldsymbol{x}^*$, and $\boldsymbol{x}^* \subseteq \boldsymbol{x}^k$ for every k. Then $p \geq 1$ is the order of convergence of $\left\{\boldsymbol{x}^k\right\}$ if there exists a nonnegative constant γ such that*

$$d\left(\boldsymbol{x}^{k+1}, \boldsymbol{x}^*\right) \leq \gamma\left(d(\boldsymbol{x}^k, \boldsymbol{x}^*)\right)^p \text{ or } w(\boldsymbol{x}^{k+1}) - w\left(\boldsymbol{x}^*\right) \leq \gamma\left(w(\boldsymbol{x}^k) - w\left(\boldsymbol{x}^*\right)\right)^p.$$

2.2 Existing classical interval Newton method (INM)

Before proposing a modification, we discuss the existing interval Newton method. For details, the reader may see [19] and [3].

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, ξ be a simple root of $f(x)$, $\boldsymbol{x}^0$ be an initial interval containing ξ, and $0 \notin F'(\boldsymbol{x}^0)$ for some inclusion isotonic extension F' of f' in $\boldsymbol{x}^0$. Then the sequence of nested intervals $\{\boldsymbol{x}^k\}$ generated by

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k \cap N(\boldsymbol{x}^k) \qquad k = 0, 1, 2, \ldots, \tag{8}$$

where $N(\boldsymbol{x}^k) = \widehat{m(\boldsymbol{x}^k)} - \frac{f(m(\boldsymbol{x}^k))}{F'(\boldsymbol{x}^k)}$, describes interval Newton iterates.

Theorem 1. *[Theorem 4, Chapter 7, [3]]. Let f be a continuously differentiable function in $\boldsymbol{x}^0 = [\underline{x^0}, \overline{x^0}]$, and for any $\boldsymbol{x} \subseteq \boldsymbol{x}^0$, f' satisfies $w(f'(\boldsymbol{x})) \leq Cw(\boldsymbol{x})$, $C \geq 0$. Furthermore, let $f(\underline{x^0}) < 0$ and $f(\overline{x^0}) > 0$. Then the sequence $\{\boldsymbol{x}^k\}$ generated by Equation (8) satisfies*

i. $\xi \in \boldsymbol{x}^k$, $k \geq 0$,

ii. $\boldsymbol{x}^0 \supset \boldsymbol{x}^1 \supset \boldsymbol{x}^2 \supset \ldots$, *and* $\lim\limits_{k\to\infty} \boldsymbol{x}^k = \widehat{\xi}$ *or the sequence comes to rest at* $\hat{\xi} = [\xi, \xi]$ *after a finite number of steps,*

iii. $w(\boldsymbol{x}^{k+1}) \leq \beta(w(\boldsymbol{x}^k))^2$, $\beta \geq 0$.

3 Modified Interval Newton Method (MINM)

In this section, we propose a new algorithm, which is a modified interval Newton algorithm with higher order convergence to find the simple root ξ of $f : \mathbb{R} \to \mathbb{R}$.

3.1 Algorithm:

Step 1: (Initialization) Choose an initial interval $\boldsymbol{x}^0$ which contains ξ and error of tolerance ε. Set $k := 0$.

Step 2: Compute set image $f'(\boldsymbol{x}^k)$ of $\boldsymbol{x}^k$ under f'.

Step 3: Select a point x^k in $\boldsymbol{x}^k$. In particular x^k may be selected as $m(\boldsymbol{x}^k)$. Compute $\widetilde{\boldsymbol{x}}^k = \boldsymbol{x}^k \bigcap \widetilde{N}(\boldsymbol{x}^k)$, where

$\widetilde{N}(\boldsymbol{x}^k) = \widehat{x^k} - \dfrac{f(x^k)}{f'(\boldsymbol{x}^k)}$, and $\widehat{x^k} = [x^k, x^k]$.

Step 4: Compute the set image $f'(\widetilde{\boldsymbol{x}}^k)$ of $\widetilde{\boldsymbol{x}}^k$ under f'.

Step 5: Select a point $\widetilde{x}^k$ in $\widetilde{\boldsymbol{x}}^k$. In particular, $\widetilde{x}^k$ may be selected as $m(\widetilde{\boldsymbol{x}}^k)$. Compute $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k \bigcap N(\boldsymbol{x}^k)$, where

$N(\boldsymbol{x}^k) = \widehat{\widetilde{x}^k} - \dfrac{2f(\widetilde{x}^k)}{f'(\boldsymbol{x}^k) + f'(\widetilde{\boldsymbol{x}}^k)}$, $\widetilde{x}^k \in \widetilde{\boldsymbol{x}}^k$, and $\widehat{\widetilde{x}^k} = [\widetilde{x}^k, \widetilde{x}^k]$.

Step 6: If $w(\boldsymbol{x}^{k+1}) < \varepsilon$, then stop. Else set $k := k + 1$ and go to Step 2.

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with a simple zero ξ in a closed interval $\boldsymbol{x}^0$, and $0 \notin f'(\boldsymbol{x}^0)$. Then the sequence of nested intervals $\{\boldsymbol{x}^k\}$ generated by Algorithm 3.1 converges to $\widehat{\xi}$.*

Proof. The statement $\hat{\xi} \subseteq \boldsymbol{x}^k$ for every k can be proved by induction. $\hat{\xi} \subseteq \boldsymbol{x}^{\mathbf{0}}$ is true from assumption. Let $\hat{\xi} \subseteq \boldsymbol{x}^k$ for some k. Denote $f'(\boldsymbol{x}^k) \triangleq [\underline{m}^k, \overline{m}^k]$ and $f'(\widetilde{\boldsymbol{x}}^k) \triangleq [\overline{M}^k, \underline{M}^k]$. Then

$$\underline{m}^k \le \frac{f(x) - f(\xi)}{x - \xi} = \frac{f(x)}{x - \xi} \le \overline{m}^k \text{ for all } x \in \boldsymbol{x}^k, \text{ and}$$
$$\overline{M}^k \le \frac{f(x) - f(\xi)}{x - \xi} = \frac{f(x)}{x - \xi} \le \underline{M}^k \text{ for all } x \in \widetilde{\boldsymbol{x}}^k.$$

Since $\widetilde{\boldsymbol{x}}^k \subset \boldsymbol{x}^k$ for all k, so from above inequalities, $\frac{2f(x)}{x-\xi} \in \left[\underline{m}^k + \overline{M}^k, \overline{m}^k + \underline{M}^k\right]$ for all $x \in \widetilde{\boldsymbol{x}}^k$. Hence using (7) for every $x \in \widetilde{\boldsymbol{x}}^k$, with $x \neq \xi$, we have

$$\xi = x - \frac{2f(x)}{\left(\frac{2f(x)}{x-\xi}\right)} \in \widehat{x} - 2f(x)\left[\frac{1}{\overline{m}^k + \underline{M}^k}, \frac{1}{\underline{m}^k + \overline{M}^k}\right] = \widehat{x} - \frac{2f(x)}{\left[\underline{m}^k + \overline{M}^k, \overline{m}^k + \underline{M}^k\right]}.$$

Hence, $\xi \in N(\boldsymbol{x}^k)$. So, $\xi \in \boldsymbol{x}^k \bigcap N(\boldsymbol{x}^k) = \boldsymbol{x}^{k+1}$. By induction, $\hat{\xi} \subseteq \boldsymbol{x}^k$ for every k.

Next, we show that $w(\boldsymbol{x}^k) \to 0$ as $k \to \infty$. Since $0 \notin f'(\boldsymbol{x}^k)$, without loss of generality, we may assume $f'(x) > 0$ for all $x \in \boldsymbol{x}^k$. From Step 5 of Algorithm 3.1, $\boldsymbol{x}^{k+1}$ can be computed as

$$\boldsymbol{x}^{k+1} = \begin{cases} [\max\{\underline{x}^k,\ \widetilde{x}^k - \frac{2f(\widetilde{x}^k)}{\underline{m}^k + \overline{M}^k}\},\ \widetilde{x}^k - \frac{2f(\widetilde{x}^k)}{\overline{m}^k + \underline{M}^k}], & f(\widetilde{x}^k) \ge 0; \\ [\widetilde{x}^k - \frac{2f(\widetilde{x}^k)}{\overline{m}^k + \underline{M}^k},\ \min\{\overline{x}^k,\ \widetilde{x}^k - \frac{2f(\widetilde{x}^k)}{\underline{m}^k + \overline{M}^k}\}], & f(\widetilde{x}^k) < 0. \end{cases}$$

Case 1. If $f(\widetilde{x}^k) = 0$, then $\mathbf{x}^{k+1} = \widehat{\widetilde{x}^k}$. Hence, the process terminates, and $\xi = \widetilde{x}^k$.
Case 2. If $f(\widetilde{x}^k) > 0$, $\mathbf{x}^{k+1} = \left[\max\left\{\underline{x}^k,\ \widetilde{x}^k - \frac{2f(\widetilde{x}^k)}{\underline{m}^k+\overline{M}^k}\right\},\ \widetilde{x}^k - \frac{2f(\widetilde{x}^k)}{\overline{m}^k+\underline{M}^k}\right]$. Two possibilities arise:
Sub-case 1. If $\underline{x}^k \geq \widetilde{x}^k - \frac{2f(\widetilde{x}^k)}{\underline{m}^k+\overline{M}^k}$, then

$$\begin{aligned}
w(\boldsymbol{x}^{k+1}) &= \widetilde{x}^k - \frac{2f(\widetilde{x}^k)}{\overline{m}^k + \underline{M}^k} - \underline{x}^k \\
&\leq \widetilde{x}^k - \underline{x}^k - (\widetilde{x}^k - \underline{x}^k)\frac{\underline{m}^k + \overline{M}^k}{\overline{m}^k + \underline{M}^k} \\
&= (\widetilde{x}^k - \underline{x}^k)\left(1 - \frac{\underline{m}^k + \overline{M}^k}{\overline{m}^k + \underline{M}^k}\right) \\
&= \left(1 - \frac{\underline{m}^k + \overline{M}^k}{\overline{m}^k + \underline{M}^k}\right) w(\boldsymbol{x}^k).
\end{aligned}$$

Sub-case 2. If $\underline{x}^k < \widetilde{x}^k - \frac{2f(\widetilde{x}^k)}{\underline{m}^k+\overline{M}^k}$, then

$$\begin{aligned}
w(\boldsymbol{x}^{k+1}) &= \widetilde{x}^k - \frac{2f(\widetilde{x}^k)}{\overline{m}^k + \underline{M}^k} - \widetilde{x}^k + \frac{2f(\widetilde{x}^k)}{\underline{m}^k + \overline{M}^k} \\
&= 2f(\widetilde{x}^k)\left(\frac{1}{\underline{m}^k + \overline{M}^k} - \frac{1}{\overline{m}^k + \underline{M}^k}\right) \\
&< (\widetilde{x}^k - \underline{x}^k)(1 - \frac{\underline{m}^k + \overline{M}^k}{\overline{m}^k + \underline{M}^k}) \\
&= \left(1 - \frac{\underline{m}^k + \overline{M}^k}{\overline{m}^k + \underline{M}^k}\right) w(\boldsymbol{x}^k).
\end{aligned}$$

Case 3. If $f(\widetilde{\boldsymbol{x}}^k) < 0$, then $\mathbf{x}^{k+1} = \left[\widetilde{x}^k - \frac{2f(\widetilde{x}^k)}{\overline{m}^k+\underline{M}^k},\ \min\{\overline{x}^k,\ \widetilde{x}^k - \frac{2f(\widetilde{x}^k)}{\underline{m}^k+\overline{M}^k}\}\right]$.
Proceeding as in Case 2, we can derive

$$w(\boldsymbol{x}^{k+1}) \leq \left(1 - \frac{\underline{m}^k + \overline{M}^k}{\overline{m}^k + \underline{M}^k}\right) w(\boldsymbol{x}^k).$$

From Case 2 and Case 3, we have

$$\begin{aligned}
w(\boldsymbol{x}^{k+1}) &\leq \left(1 - \frac{\underline{m}^k + \overline{M}^k}{\overline{m}^k + \underline{M}^k}\right) w(\boldsymbol{x}^k) \\
&\leq \left(1 - \frac{\underline{m}^k + \overline{M}^k}{\overline{m}^k + \underline{M}^k}\right)\left(1 - \frac{\underline{m}^{k-1} + \underline{M}^{k-1}}{\overline{m}^{k-1} + \overline{M}^{k-1}}\right) w(\mathbf{x}^{k-1}) \\
&\quad \dots \\
&\leq \left(1 - \frac{\underline{m}^k + \overline{M}^k}{\overline{m}^k + \underline{M}^k}\right)\left(1 - \frac{\underline{m}^{k-1} + \underline{M}^{k-1}}{\overline{m}^{k-1} + \overline{M}^{k-1}}\right) \cdots \left(1 - \frac{\underline{m}^0 + \underline{M}^0}{\overline{m}^0 + \overline{M}^0}\right) w(\mathbf{x}^0).
\end{aligned}$$

Since $\frac{\underline{m}^k+\overline{M}^k}{\overline{m}^k+\underline{M}^k} < 1$ for every k, so $w(\boldsymbol{x}^{k+1}) \to 0$ as $k \to \infty$. Since $\hat{\xi} \subseteq \boldsymbol{x}^k$ for every k and $w(\boldsymbol{x}^{k+1}) \to 0$ as $k \to \infty$, the sequence $\{\boldsymbol{x}^k\}$ generated by Algorithm 3.1 converges to $\widehat{\xi}$. $\square$

Theorem 3. *Suppose the assumptions of Theorem 2 hold. If f' is Lipschitz continuous in $\boldsymbol{x}^0$, the sequence $\{\boldsymbol{x}^k\}$ generated by Algorithm 3.1 converges to $\widehat{\xi}$ cubically.*

Proof. Since $f'(x)$ is a Lipschitz continuous function, $f'(\mathbf{x}^0)$ is a Lipschitz interval extension in $\mathbf{x}^0$ (see Note 2). Hence from Definition 4, there exists $L_1 > 0$ such that

$$w(f'(\mathbf{x})) \leq L_1 w(\mathbf{x}) \text{ for all } \mathbf{x} \subseteq \mathbf{x}^0.$$

$\widetilde{\boldsymbol{x}}^k$ is computed using Step 3 of Algorithm 3.1, which is the classical interval Newton iterate (8). From the proof of Theorem 1 (for a detailed proof, see Theorem 4 of Chapter 7 in [3] for the classical interval Newton method), we borrow the result

$$w(\widetilde{\boldsymbol{x}}^k) \leq \frac{L_1}{\overline{m}^k} w(\boldsymbol{x}^k)^2. \tag{9}$$

Expanding f by Taylor's formula about ξ,

$$f(x) = f(\xi) + (x-\xi)f'(\xi) + O((x-\xi)^2) = (x-\xi)f'(\xi) + O((x-\xi)^2).$$

For any $\widetilde{x}^k \in \widetilde{\boldsymbol{x}}^k$,

$$\begin{aligned} |f(\widetilde{x}^k)| &\leq |\widetilde{x}^k - \xi||f'(\xi)| + O((\widetilde{x}^k-\xi)^2) \\ &\leq w(\widetilde{\boldsymbol{x}}^k)|f'(\xi)| + O(w(\widetilde{\boldsymbol{x}}^k)^2) \\ &\leq \frac{L_1|f'(\xi)|}{\overline{m}^k} w(\boldsymbol{x}^k)^2 + O(w(\boldsymbol{x}^k)^4) \quad \text{(from (9))}. \end{aligned} \tag{10}$$

Next,

$$\begin{aligned} w(N(\boldsymbol{x}^k)) &= w\left(\widehat{\widetilde{x}^k} - \frac{2f(\widetilde{x}^k)}{f'(\boldsymbol{x}^k) + f'(\widetilde{\boldsymbol{x}}^k)}\right) \\ &= w\left(\frac{2f(\widetilde{x}^k)}{f'(\boldsymbol{x}^k) + f'(\widetilde{\boldsymbol{x}}^k)}\right) \text{(using (1) and (5))} \\ &= \frac{2|f(\widetilde{x}^k)|}{(\underline{m}^k + \overline{M}^k)(\overline{m}^k + \underline{M}^k)} \Big(w(f'(\boldsymbol{x}^k)) + w(f'(\widetilde{\boldsymbol{x}}^k))\Big) \text{(using (1), (2), and (3))} \\ &\leq \frac{2L_1|f'(\xi)| \Big(w(\boldsymbol{x}^k)^2 + O(w(\boldsymbol{x}^k)^4)\Big)}{\overline{m}^k(\underline{m}^k + \overline{M}^k)(\overline{m}^k + \underline{M}^k)} \Big(L_1 w(\boldsymbol{x}^k) + L_1 w(\widetilde{\boldsymbol{x}}^k)\Big) \text{(from (10))} \\ &\leq \frac{2L_1^2|f'(\xi)|}{\overline{m}^k(\underline{m}^k + \overline{M}^k)(\overline{m}^k + \underline{M}^k)} \Big(w(\boldsymbol{x}^k)^3 + O(w(\boldsymbol{x}^k)^4)\Big). \end{aligned} \tag{11}$$

Hence,

$$\begin{aligned} w(\boldsymbol{x}^{k+1}) &= w(\boldsymbol{x}^k \bigcap N(\boldsymbol{x}^k)) \\ &\leq \min\{w(\boldsymbol{x}^k), w(N(\boldsymbol{x}^k))\} \text{(by (6))} \\ &\leq w(N(\boldsymbol{x}^k)) \\ &\leq \frac{2L_1^2|f'(\xi)|}{\overline{m}^k(\underline{m}^k + \overline{M}^k)(\overline{m}^k + \underline{M}^k)} \Big(w(\boldsymbol{x}^k)^3 + O(w(\boldsymbol{x}^k)^4)\Big) \text{(from (11))}. \end{aligned}$$

Hence by Definition 7, $\{\boldsymbol{x}^k\}$ converges to $\widehat{\xi}$ at least with cubic order. (Here, $\mathbf{x}^*$ is the degenerate interval $\widehat{\xi}$, so $w(\mathbf{x}^*) = 0$.) $\square$

Theorem 4. *Suppose the assumptions of Theorem 2 hold. If f' and f'' are Lipschitz continuous in $\boldsymbol{x}^0$, and $0 \in f''(\boldsymbol{x}^k)$ for all k, then the sequence $\{\boldsymbol{x}^k\}$ generated by Algorithm 3.1 converges to $\widehat{\xi}$ with order of convergence four.*

Proof. Since $f''(x)$ is a Lipschitz continuous function, $f''(\boldsymbol{x}^0)$ is a Lipschitz interval extension in $\boldsymbol{x}^0$ (see Note 2). Hence from Definition 4, there exists $L_2 > 0$ such that

$$w(f''(\mathbf{x})) \leq L_2 w(\mathbf{x}) \text{ for all } \mathbf{x} \subseteq \mathbf{x}^0. \tag{12}$$

Using Taylor's expansion of f' about ξ, for every $x^k \in \boldsymbol{x}^k$, $f'(x^k) = f'(\xi) + (x^k - \xi) f''(\theta)$, $\theta \in (x^k, \xi)$. Using image extensions of f' and f'' in $\boldsymbol{x}^k$, we have

$$\begin{aligned} f'(\boldsymbol{x}^k) &\subseteq \widehat{f'(\xi)} + (\boldsymbol{x}^k - \widehat{\xi}) f''(\boldsymbol{x}^k), \\ w(f'(\boldsymbol{x}^k)) &\leq w\left(\widehat{f'(\xi)} + (\boldsymbol{x}^k - \widehat{\xi}) f''(\boldsymbol{x}^k)\right) \\ &= w((\boldsymbol{x}^k - \widehat{\xi}) f''(\boldsymbol{x}^k)) \\ &\leq |(\boldsymbol{x}^k - \widehat{\xi})| w(f''(\boldsymbol{x}^k)) + |f''(\boldsymbol{x}^k)| w(\boldsymbol{x}^k - \widehat{\xi}) \text{ (using (4))}. \end{aligned} \tag{13}$$

From the definition of absolute value of an interval (see Section 2), we get

$$|(\boldsymbol{x}^k - \widehat{\xi})| = max\{|\underline{x}^k - \xi|, |\overline{x}^k - \xi|\} \leq w(\boldsymbol{x}^k). \tag{14}$$

Since $0 \in f''(\boldsymbol{x}^k)$,

$$|f''(\boldsymbol{x}^k)| \leq w(f''(\boldsymbol{x}^k)) \leq L_2 w(\boldsymbol{x}^k). \tag{15}$$

Using (12), (14), and (15), Inequality (13) can be simplified to

$$w(f'(\boldsymbol{x}^k)) \leq 2L_2 w(\boldsymbol{x}^k)^2. \tag{16}$$

Similarly from (9),

$$w(f'(\boldsymbol{x}^k)) \leq 2L_2 w(\widetilde{\boldsymbol{x}}^k) \leq \frac{2L_1^2 L_2}{\overline{m}^{k\,2}} w(\boldsymbol{x}^k)^2. \tag{17}$$

Next,

$$\begin{aligned} w(\boldsymbol{x}^{k+1}) &= w(\boldsymbol{x}^k \bigcap N(\boldsymbol{x}^k)) \\ &\leq \min\{w(\boldsymbol{x}^k), w(N(\boldsymbol{x}^k))\} \text{ (by (6))} \\ &\leq w(N(\boldsymbol{x}^k)) \\ &= w\left(\widehat{\widetilde{x}^k} - \frac{2f(\widetilde{x}^k)}{f'(\boldsymbol{x}^k) + f'(\widetilde{\boldsymbol{x}}^k)}\right). \end{aligned}$$

Using results (1), (2), (3), and (5),

$$w(\boldsymbol{x}^{k+1}) \leq \frac{2|f(\widetilde{x}^k)|}{(\underline{m}^k + \overline{M}^k)(\overline{m}^k + \underline{M}^k)} \Big(w(f'(\widetilde{x}^k)) + w(f'(\widetilde{\boldsymbol{x}}^k)) \Big). \tag{18}$$

Using (10), (16), and (17), Inequality (18) can be simplified to

$$\begin{aligned} w(\boldsymbol{x}^{k+1}) &\leq \frac{2L_1|f'(\xi)|w(\boldsymbol{x}^k)^2 + O(w(\boldsymbol{x}^k)^4)}{\overline{m}^k(\underline{m}^k + \overline{M}^k)(\overline{m}^k + \underline{M}^k)} \left(2L_2 w(\boldsymbol{x}^k)^2 + \frac{2L_1^2 L_2}{\overline{m}^{k^2}} w(\boldsymbol{x}^k)^2 \right) \\ &= \frac{4L_1 L_2 |f'(\xi)|}{\overline{m}^k(\underline{m}^k + \overline{M}^k)(\overline{m}^k + \underline{M}^k)} \Big(w(\boldsymbol{x}^k)^4 \Big) + O(w(\boldsymbol{x}^k)^5). \end{aligned}$$

Hence by Definition 7, $\{\boldsymbol{x}^k\}$ converges to $\widehat{\xi}$ at least bi-quadratically. (Here, $\mathbf{x}^*$ is the degenerate interval $\widehat{\xi}$, so $w(\mathbf{x}^*) = 0$.) $\square$

Remark 1. *We have considered the set image of a function as an interval extension, but computing a set image is not always easy for some complicated functions. In this situation, one may consider some suitable Lipschitz inclusion isotonic interval extension of the given function and execute Algorithm 3.1 without affecting the convergence of the algorithm.*

4 Example

There are several higher order root fining methods in the literature (see Section 1). In this section, we illustrate some advantages of our algorithm using examples. The existing algorithms are based on certain restrictions on the selection of initial interval.

- The algorithm of [18] converges cubically if the conditions of Theorem 3.1 of that paper are satisfied for the initial interval, but it is difficult to search for such an initial interval. Our algorithm has weaker assumptions on the initial interval.
- Some algorithms, e.g., [18, 26] need the second derivative f''. Our algorithm needs only the first derivative f', so it requires less computation.
- Some algorithms are for polynomials only, e.g., [18, 26], but our algorithm works for first order differentiable functions.
- The algorithm in [27] and our algorithm are distinct approaches. The algorithm in [27] converges cubically, whereas our algorithm converges cubically (Theorem 3) under certain conditions and bi-quadratically under the stronger conditions of Theorem 4. In addition, our algorithm is simpler than the algorithm provided in [27].

We illustrate these algorithms in Example 4.1. We consider a function $f(x) = x^3 + x$ and solve using both the algorithms [18] and [27] with initial interval [0.5, 2] (with imaginary part zero in algorithm of [18]). The condition of Theorem 3.1 of [18] is satisfied for this interval. Our algorithm converges bi-quadratically, more rapidly than the algorithm of [18], and it requires fewer iterations.

To compare the results using the proposed algorithm and algorithms in [18] and [27], we may express an interval $\boldsymbol{x} = [\underline{x}, \overline{x}]$ in midpoint-radius form as $\boldsymbol{x} = \{c : r\}$, where $c = m(\boldsymbol{x})$ and $r = \text{rad}(\boldsymbol{x}) = w(\boldsymbol{x})/2$. Hence, $\boldsymbol{x} = [c - r, c + r]$.

Example 4.1. *Consider* $f(x) = x^3 + x$, $\boldsymbol{x}^0 = [-0.49, 0.51]$, $\epsilon = 10^{-200}$.

Using our algorithm, we obtain the inclusion intervals $\{m(\boldsymbol{x}^k) : rad(\boldsymbol{x}^k)\}$

$$\begin{aligned}
m(\boldsymbol{x}^1) &= 0.0003074212598 \qquad \mathrm{rad}(\boldsymbol{x}^1) = 0.00030743177\\
m(\boldsymbol{x}^2) &= 4.11900800473 \times 10^{-17} \qquad \mathrm{rad}(\boldsymbol{x}^2) = 4.119008005 \times 10^{-17}\\
m(\boldsymbol{x}^3) &= 1.7785023313 \times 10^{-81} \qquad \mathrm{rad}(\boldsymbol{x}^3) = 1.7785023313 \times 10^{-81}\\
m(\boldsymbol{x}^4) &= 2.669091444 \times 10^{-403} \qquad \mathrm{rad}(\boldsymbol{x}^4) = 2.669091444 \times 10^{-403}
\end{aligned}$$

Using two step interval method in [27], we obtain the inclusion intervals $\{m(\boldsymbol{x}^k) : rad(\boldsymbol{x}^k)\}$

$$\begin{aligned}
m(\boldsymbol{x}^1) &= 0.00048008113 \qquad \mathrm{rad}(\boldsymbol{x}^1) = 0.00048009165\\
m(\boldsymbol{x}^2) &= 7.65090586827989 \times 10^{-16} \qquad \mathrm{rad}(\boldsymbol{x}^2) = 7.65090586828 \times 10^{-16}\\
m(\boldsymbol{x}^3) &= 7.86476134093 \times 10^{-75} \qquad \mathrm{rad}(\boldsymbol{x}^3) = 7.864761340932 \times 10^{-75}\\
m(\boldsymbol{x}^4) &= 9.0271154416 \times 10^{-370} \qquad \mathrm{rad}(\boldsymbol{x}^4) = 9.0271154416 \times 10^{-370}
\end{aligned}$$

Using the method in [18], we obtain the inclusion intervals $\{m(\boldsymbol{x}^k) : rad(\boldsymbol{x}^k)\}$

$$\begin{aligned}
m(\boldsymbol{x}^1) &= -1.333692914 \times 10^{-6} \qquad \mathrm{rad}(\boldsymbol{x}^1) = 4.2573862392 \times 10^{-6}\\
m(\boldsymbol{x}^2) &= 2.368373121 \times 10^{-18} \qquad \mathrm{rad}(\boldsymbol{x}^2) = 1.0708343125 \times 10^{-17}\\
m(\boldsymbol{x}^3) &= -1.32633853 \times 10^{-53} \qquad \mathrm{rad}(\boldsymbol{x}^3) = 5.996546905 \times 10^{-53}\\
m(\boldsymbol{x}^4) &= 2.3295239102 \times 10^{-159} \qquad \mathrm{rad}(\boldsymbol{x}^4) = 1.05320769066 \times 10^{-185}\\
m(\boldsymbol{x}^5) &= -3.790451123 \times 10^{-476} \qquad \mathrm{rad}(\boldsymbol{x}^5) = 5.70627084977 \times 10^{-476}
\end{aligned}$$

Example 4.2. *Consider the function*

$$\begin{aligned}
f(x) \quad = \quad & x^{17} - x^{16} + 28x^{15} - 390x^{14} + 6002x^{13} - 10762x^{12} - 29484x^{11} + 846040x^{10}\\
& -76809707x^9 + 130583427x^8 - 2113327216x^7 + 24795890990x^6\\
& -339342802696x^5 + 178957763336x^4 + 7226702364672x^3\\
& -88957569392640x^2 + 1984671888998400x - 1902803374080000,
\end{aligned}$$

with $\boldsymbol{x}^0 = [0.5, 2]$ *and* $\epsilon = 10^{-150}$.

Our algorithm yields the inclusion intervals $\{m(\boldsymbol{x}^k) : rad(\boldsymbol{x}^k)\}$

$$\begin{aligned}
m(\boldsymbol{x}^1) &= 1 + 4.83243235 \times 10^{-5}, \qquad \mathrm{rad}(\boldsymbol{x}^1) = 0.0002490466531\\
m(\boldsymbol{x}^2) &= 1 + 1.621527 \times 10^{-16}, \qquad \mathrm{rad}(\boldsymbol{x}^2) = 1.393803591252234 \times 10^{-15}\\
m(\boldsymbol{x}^3) &= 1 + 6.124747 \times 10^{-51}, \qquad \mathrm{rad}(\boldsymbol{x}^3) = 8.780408858477 \times 10^{-50}\\
m(\boldsymbol{x}^4) &= 1 + 3.300505 \times 10^{-154}, \qquad \mathrm{rad}(\boldsymbol{x}^4) = 7.8914357407525282 \times 10^{-153},
\end{aligned}$$

while the two step interval method in [27] yields the interval inclusion

$$\begin{aligned}
m(\boldsymbol{x}^1) &= 1.00012370843177, \qquad \mathrm{rad}(\boldsymbol{x}^1) = 0.0004943042\\
m(\boldsymbol{x}^2) &= 1 + 5.44258215 \times 10^{-15}, \qquad \mathrm{rad}(\boldsymbol{x}^2) = 3.6264859955 \times 10^{-14}\\
m(\boldsymbol{x}^3) &= 1 + 4.6319176 \times 10^{-46}, \qquad \mathrm{rad}(\boldsymbol{x}^3) = 5.147435737112 \times 10^{-45}\\
m(\boldsymbol{x}^4) &= 1 + 2.855146 \times 10^{-139}, \qquad \mathrm{rad}(\boldsymbol{x}^4) = 5.29184999001 \times 10^{-138}\\
m(\boldsymbol{x}^5) &= 1 + 6.686992 \times 10^{-419}, \qquad \mathrm{rad}(\boldsymbol{x}^5) = 2.06708865402 \times 10^{-417}.
\end{aligned}$$

From this explicit calculation that at every iteration, one may observe that the radius of the interval from our algorithm is less than half the radius of the algorithm of [27]. The conditions of Theorem 3.1 in [18] are not satisfied in the initial interval $[0.5, 2]$, so the algorithm of [18] is not applicable for this interval, while our algorithm performs well in this case.

Example for which classical Newton method fails and interval Newton method works: Consider the following numerical example from [22]. The classical Newton method fails for some initial points, but the interval Newton method works for any initial interval containing the root being sought (provided extended interval arithmetic is implemented appropriately in the case $f'(\boldsymbol{x})$ contains 0.). For $0 < a < b$, $f_{nic}(x)$ is

$$f_{nic}(x) = \begin{cases} \frac{3}{2}\frac{x}{\sqrt{a}} - \frac{1}{2}\frac{x^2}{a^{\frac{3}{2}}}, & 0 \le x \le a; \\ \sqrt{x}, & a \le x \le b; \\ (\frac{3}{2})^{\frac{1}{3}} b^{\frac{1}{6}} (x-b)^{\frac{1}{3}}, & b \le x; \\ -f_{nic}(-x), & x \le 0. \end{cases}$$

Here $f_{nic} \in \mathcal{C}^1(-\infty, \infty)$, and 0 is the only root of $f_{nic}(x) = 0$. The classical Newton method fails if $|x^0| \ge a$, while the interval Newton method provides a convergent sequence if 0 belongs to the initial interval regardless of the initial point (see [9, 16], provided we use extended interval arithmetic correctly). Our modified interval Newton method yields a convergent sequence whose order of convergence is higher than that of the classical interval Newton method. We have solved this problem for $a = 4$ and $b = 64$ using both the classical interval Newton method, our modified interval Newton method, and the method in [27].

Other differences in the of number of iterations may be seen in numerical results summarized in Table 1. For $x^k = m(\boldsymbol{x}^k)$, $\widetilde{x}^k = m(\widetilde{\boldsymbol{x}}^k)$, and stopping criteria $w(\mathbf{x}^k) < \epsilon = 10^{-100}$ or 10^{-30}, we ran programs in INTLAB Version 6 (under MATLAB 2012b and the Multiprecision Computing Toolbox) for some Lipschitz continuous functions including f_{nic}. Our algorithm is compared with the classical interval Newton algorithm, the method by Milošević et al. [18], and two step method by Petković [27]. Algorithm 3.1 required fewer iterations than the classical interval Newton algorithm. Moreover, the method proposed in [18] fails for transcendental equations.

5 Conclusion and Future Work

In this paper, we present a new algorithm using interval analysis for finding a simple root of a non-linear equation. The classical interval Newton method is used implicitly for greater efficiency. Our algorithm has the following advantages:

- Has higher order of convergence than existing interval Newton methods.
- Avoids the burden of selection of a suitable initial point for classical point Newton methods.
- Is less restrictive in selection of initial interval than the methods in [18, 26].

However, there remain opportunities for improvement. Our algorithm is valid for finding simple roots only, and we have not addressed the solution of systems of equations. While univariate functions often can be evaluated in a fraction of second, that is not necessarily true for functions of several variables. Our objective is to develop a higher order algorithm in several variables, so we have begun with a higher order algorithm for univariate functions with an intent to extend it.

Function	$\mathbf{x}^0$	ϵ	No. of iterations in MINM	No. of iterations in INM	No. of iterations in [18]	No.of iterations in [27]	Root
f_{nic}	[-5,70]	10^{-100}	6	9	N. A.	6	0
$x^3 + 4x^2 - 10$	[1,2]	”	4	7	5	5	1.36523001
$x^5 + x - 10000$	[6,6.5]	”	4	7	N. C.	4	6.3087771
$x^3 - 10$	[2,3]	”	5	7	N. C.	5	2.15443469
$(x-1)^3 - 1$	[1.5,3]	”	5	8	6	5	2
$x^3 - 3x + 2.001$	[-3,-1.5]	”	5	9	5	5	-2.0001
$x^2 - e^x - 3x + 2$	[0,1]	”	5	9	N. A.	5	0.2575302
$e^{x^2+7x-30} - 1$	[1,3.5]	”	8	14	N. A.	8	3
$(x^3 - 27)e^{x/10} + \cos(3-x) - 1$	[2.3,3.3]	10^{-30}	3	5	N.A.	3	1.4044916482
$xe^{x^2} - \sin^2(x) + 3\cos(x) + 5$	[-2,-1]	”	3	6	N. A.	3	-1.20764782713
$\sin^2(x) - x^2 + 1$	[1,3.5]	”	3	5	N. A.	3	1.4044916482

Table 1: Comparison for different functions
MINM: Modified Interval Newton Method
INM: Interval Newton Method
N. A.: Not Applicable
N. C.: Not Convergent

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