A Note on the Boundary Shape of Matrix Polytope Products*

Florian Bünger
Institute for Reliable Computing, Hamburg University of Technology, Schwarzenbergstr. 95, D-21073 Hamburg, Germany

florian.buenger@tuhh.de

Abstract

Motivated by interval matrix multiplication we consider (matrix) polytopes $\boldsymbol{A} \subseteq \mathbb{R}^{m,n}$, $\boldsymbol{B} \subseteq \mathbb{R}^{n,k}$, $m, n, k \in \mathbb{N}$, and investigate the boundary shape of their pointwise product $\boldsymbol{AB} := \{AB \mid A \in \boldsymbol{A}, B \in \boldsymbol{B}\}$. We prove that $\boldsymbol{AB}$ cannot have outward curved boundary sections while inward curved sections may exist. This is achieved by a simple local extreme point analysis. Results are proved in a more general abstract setting for images of compact sets of (not necessarily finite dimensional) locally convex vector spaces under continuous multilinear mappings. They can be seen as extensions of the Zadeh-Desoer Mapping Theorem which is a fundamental tool in control theory.

Keywords: pointwise matrix polytope products, pointwise interval matrix products, extreme points
AMS subject classifications: 65G30 51M04 51M20 52B11

1 Introduction

Throughout, m, n, k always denote positive natural numbers, i.e., elements of $\mathbb{N}$. A set $\boldsymbol{A} \subseteq \mathbb{R}^{m,n}$ consisting of matrices of order (m, n) with real-valued entries is called an *interval matrix* if there exist matrices $\underline{A}, \overline{A} \in \mathbb{R}^{m,n}$ such that

$$\boldsymbol{A} = \{A \in \mathbb{R}^{m,n} \mid \underline{A}_{i,j} \le A_{i,j} \le \overline{A}_{i,j} \text{ for all } i = 1, ..., m,\ j = 1, ..., n\}\ .$$

The set of all such interval matrices is denoted by $\mathbb{IR}^{m,n}$. Interval matrices in $\mathbb{IR}^{m,1}$ are called *interval vectors* and the set of all such interval vectors is for simplicity

*Submitted: May 22, 2014; Revised: July 29, 2014; Accepted:October 11, 2014.

denoted by $\mathbb{IR}^{\mathrm{m}}$. Interval matrices and interval vectors are the basic objects of interval arithmetic for verified computation in linear algebra. Now, if $\boldsymbol{A} \in \mathbb{IR}^{\mathrm{m,n}}$ and $\boldsymbol{B} \in \mathbb{IR}^{\mathrm{n,k}}$ are interval matrices, their *pointwise product*

$$\boldsymbol{C} := \boldsymbol{A}\boldsymbol{B} := \{AB \mid A \in \boldsymbol{A}, B \in \boldsymbol{B}\}$$

is in general not an interval matrix. Moreover, even when $\boldsymbol{A}$ and $\boldsymbol{B}$ are polytopes, $\boldsymbol{C}$ can be not convex and not a union of finitely many polytopes. In computational practice of interval arithmetic $\boldsymbol{C}$ is therefore replaced by an appropriate interval matrix that includes $\boldsymbol{C}$. This replacement causes excess width in subsequent computations that might, using interval arithmetic in a naive way, accumulate quite rapidly and it is one of the major tasks in interval arithmetic to find subtle algorithms that control this overestimation. Thus, from the point of view of interval arithmetic it is near at hand to investigate the qualitative boundary shape of $\boldsymbol{C}$. Stimulated by a short discussion on the Reliable Computing mailing list initiated by Kelsey [8] who posed some questions and conjectures on this subject which were soon answered by Goldsztejn [7] and Kreinovich [10], we started to carry out numerical experiments.

Surprisingly, these experiments suggested that pointwise matrix polytope products do not contain outward curved boundary parts while, for example, inward curved boundary sections sometimes showed up. [1]

Trying to prove this fact by elementary differential geometry turned out to be tedious and not very promising. In order to avoid differentiability pitfalls this led us to an elementary nondifferential extreme point analysis: Let $\mathrm{ext}(\boldsymbol{A})$ and $\mathrm{ext}(\boldsymbol{B})$ denote the extreme points (vertices) of $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively, and let $\mathrm{locext}(\boldsymbol{C})$ denote the set of *local extreme points* of $\boldsymbol{C}$ which are those points of $\boldsymbol{C}$ that do not lie on an open line segment that is completely contained in $\boldsymbol{C}$. [2] Then, our main result for interval matrices reads, see Corollary 3.3:

$$\mathrm{locext}(\boldsymbol{C}) \subseteq \mathrm{ext}(\boldsymbol{A})\mathrm{ext}(\boldsymbol{B}).$$

In particular, $\mathrm{locext}(\boldsymbol{C})$ is finite. Since outward curved boundary sections contain infinitely many local extreme points, such boundary shapes cannot occur for pointwise interval matrix products or for matrix polytope products in general as was suggested by numerical evidence. This result will be proved in a more general setting for multilinear functions on (not necessarily finite dimensional) locally convex vector spaces. This includes interval matrix multiplication. The main theorem is:

Theorem 1.1 *Let $U_1, ..., U_n$ be real Hausdorff locally convex vector spaces, let V be a Hausdorff topological real vector space, and let $f : U_1 \times U_2 \times \cdots \times U_n \to V$ be a continuous multilinear function where $U_1 \times \cdots \times U_n$ is endowed with the associated product topology. Then, for arbitrary compact sets $K_i \subseteq U_i$, $i = 1, ..., n$, we have*

a) $\mathrm{ext}(f(K_1 \times ... \times K_n)) \subseteq f(\mathrm{ext}(K_1) \times ... \times \mathrm{ext}(K_n))$

b) $\mathrm{locext}(f(K_1 \times ... \times K_n)) \subseteq f(\mathrm{locext}(K_1) \times ... \times \mathrm{locext}(K_n))$

where $f(X_1 \times \cdots \times X_n) := \{f(x_1, ..., x_n) \mid x_i \in X_i, i = 1, ..., n\}$ for arbitrary sets $X_i \subseteq U_i$, $i = 1, ..., n$.

[1] The terms "outward curved" and "inward curved" are used in a naive, non-rigorous, descriptive way. For example, an outward curved boundary part of a three dimensional body may look like a boundary part of an ellipsoid. More formally, this means that in each point of that boundary part all main curvatures are positive.

[2] To our knowledge the notion of local extreme points is new while that of (global) extreme points is well-known, see Section 3 for details.

The proof of Theorem 1.1 is not very difficult and we emphasize that we consider the finding of the appropriate method of local extreme point analysis as the main effort of this note. Theorem 1.1 can be seen as an extension of the Zadeh-Desoer Mapping Theorem [12] (ZDMT) which is a fundamental tool in control theory.[3] The version of the ZDMT given in [2] reads:

Theorem 1.2 (Zadeh-Desoer Mapping Theorem)
Suppose $Q \subset \mathbb{R}^l$ is a box with extreme points $\{q^i \mid i = 1, ..., 2^l\}$ and $f : Q \to \mathbb{R}^k$ is multilinear. Let $f(Q) = \{f(q) \mid q \in Q\}$ denote the range of f .Then it follows that

$$\operatorname{conv}(f(Q)) = \operatorname{conv}(\{f(q^i) \mid i = 1, ..., 2^l\}). \tag{1}$$

The ZDMT is a special finite dimensional case of Theorem 1.1 a) since for the box $Q = \prod_{i=1}^{l} I_i$ with finite closed intervals $I_i \subset \mathbb{R}$ we have

$$\operatorname{ext}(Q) = \prod_{i=1}^{l} \operatorname{ext}(I_i) = \{q^i \mid i = 1, ..., 2^l\}$$

wherefore Theorem 1.1 a) with $n := l$, $K_i := I_i$, and $V := \mathbb{R}^k$ supplies

$$\operatorname{ext}(f(Q)) \subseteq f(\operatorname{ext}(Q)) = \{f(q^i) \mid i = 1, ..., 2^l\}. \tag{2}$$

Now, Minkowski's theorem, see Theorem 3.1 below, yields

$$\operatorname{conv}(f(Q)) = \operatorname{conv}(\operatorname{ext}(f(Q))).$$

Using (2) this implies (1). Conversely, (1) does not directly imply (2). Actually, part b) of Theorem 1.1 is the more important one for our purpose. First, for convex K_i it implies part a). Second and mainly, it shows that the multilinear image of the K_i cannot have locally outward curved boundary parts. This cannot be deduced from part a), see the example illustrated in Figure 10.

We mention that to the best of our knowledge the proofs for the ZDMT given in the literature which are known to us do not carry over to the proof of Theorem 1.1. For example, they use specific properties of boxes or induction on dimension. Our approach pursues a different way.

The article is organized as follows: Section 2 states basic facts on pointwise matrix polytope products. To our knowledge Lemma 2.1 and its corollaries, even though elementary, seem to be not explicitly mentioned in the literature and we therefore considered them as noteworthy. Furthermore, simple illustrative "counterexamples" of pointwise matrix polytope products that are not convex and have curved boundary parts are presented. (These results and examples already answer the questions and conjectures posed in [8].) Section 3 establishes the connection between the local and global extreme points of a pointwise product $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$ of compact sets $\boldsymbol{A}$, $\boldsymbol{B}$ and those of its factors. As mentioned before, this is proved in a more general context of multilinear mappings on (not necessarily finite dimensional) locally convex topological vector spaces. The main results are Theorems 1.1 and 3.3 where the latter extends the former to vector spaces over the complex numbers. The finite dimensional case of real or complex interval matrix multiplication will follow as a corollary. For readers who are less involved with locally convex spaces the appendix contains some basic explanations and references.

[3] Many thanks to one of the unknown referees for pointing to the Zadeh-Desoer Mapping Theorem which was not known to the author.

2 Basic Remarks and Some Examples

In the following text V denotes a real (not necessarily finite-dimensional) vector space. In the finite-dimensional case topological properties like compactness are always considered with respect to the natural Euclidean topology. For an arbitrary subset $A \subseteq V$ the *convex hull* of A in V is denoted by

$$\mathrm{conv}(A) := \left\{ \sum_{i=1}^{m} \alpha_i a_i \mid m \in \mathbb{N}, a_1, ..., a_m \in A, \alpha_1, ..., \alpha_m \in [0,1], \sum_{i=1}^{m} \alpha_i = 1 \right\}$$

and A is called *convex* if $A = \mathrm{conv}(A)$. A *polytope* C in V is simply the convex hull of finitely many points of V, i.e., there are $v_1, ..., v_n \in V$ such that

$$C = \mathrm{conv}(\{v_1, ..., v_n\}) = \left\{ \sum_{i=1}^{n} \alpha_i v_i \mid \alpha_1, ..., \alpha_n \in [0,1], \sum_{i=1}^{n} \alpha_i = 1 \right\}. \tag{3}$$

If n is minimal subject to (3) then $v_1, ..., v_n$ are the (uniquely determined) vertices of C. For the special case $n = 2$, $v_1 \neq v_2$, the closed line segment between v_1 and v_2 is abbreviated by

$$\overline{v_1 v_2} := \mathrm{conv}(\{v_1 v_2\}) = \{\alpha v_1 + (1-\alpha) v_2 \mid \alpha \in [0,1]\}.$$

Remark 2.1 *Let $\boldsymbol{A} \subseteq \mathbb{R}^{m,n}$ and $\boldsymbol{B} \subseteq \mathbb{R}^{n,k}$ be two matrix polytopes with vertices $A_1, \ldots, A_r \in \mathbb{R}^{m,n}$ and $B_1, \ldots, B_s \in \mathbb{R}^{n,k}$, respectively. Then, the convex hull of $\boldsymbol{AB} := \{AB \mid A \in \boldsymbol{A}, B \in \boldsymbol{B}\} \subseteq \mathbb{R}^{m,k}$ equals the convex hull of the vertex products $A_i B_j$, $i = 1, ..., r$, $j = 1, ..., s$, i.e.,*

$$\mathrm{conv}(\boldsymbol{AB}) = \mathrm{conv}(\{A_i B_j \mid i = 1, ..., r, j = 1, ..., s\}) =: \boldsymbol{C} \ .$$

In particular, $\mathrm{conv}(\boldsymbol{AB})$ *is again a matrix polytope.*

Proof: Let $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s \in [0,1]$ such that $\sum_{i=1}^{r} \alpha_i = 1 = \sum_{j=1}^{s} \beta_j$. Then,

$$\sum_{i=1}^{r} \sum_{j=1}^{s} \alpha_i \beta_j = \sum_{i=1}^{r} \alpha_i \sum_{j=1}^{s} \beta_j = 1$$

and therefore

$$\left(\sum_{i=1}^{r} \alpha_i A_i \right) \left(\sum_{j=1}^{s} \beta_j B_j \right) = \sum_{i=1}^{r} \sum_{j=1}^{s} (\alpha_i \beta_j) A_i B_j \in \boldsymbol{C}. \qquad \square$$

Corollary 2.1 *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ like in Remark 2.1. If $\boldsymbol{AB}$ is convex, then $\boldsymbol{AB}$ is again a matrix polytope.*

Remark 2.2 *Let $x, y \in \mathbb{R}$ with $xy \geq 0$, $u, v \in V$ and $\lambda \in [0,1]$. Then,*

$$\nu := \begin{cases} \frac{\lambda x}{\lambda x + (1-\lambda) y} & \text{if } \lambda x + (1-\lambda) y \neq 0, \\ 0 & \text{else,} \end{cases}$$

is contained in $[0,1]$ *and fulfills*

$$\lambda x u + (1-\lambda) y v = (\lambda x + (1-\lambda) y)(\nu u + (1-\nu) v) \ . \tag{4}$$

Proof: The real number $z := \lambda x + (1-\lambda)y$ lies between x and y. The assumption $xy \geq 0$ implies that x and y do not have opposite signs whence also λx and z do not have opposite signs and $|z| = \lambda|x| + (1-\lambda)|y| \geq \lambda|x|$. Thus $\nu = \frac{\lambda x}{z} \in [0,1]$ if $z \neq 0$. If $z = 0$, then, since its two addends λx and $(1-\lambda)y$ do not have opposite signs, $\lambda x = 0 = (1-\lambda)y$ holds and therefore both sides of (4) are zero. If $z \neq 0$, then

$$\lambda xu + (1-\lambda)yv = z\frac{\lambda x}{z}u + z(1 - \frac{\lambda x}{z})v = z(\nu u + (1-\nu)v).$$ □

Corollary 2.2 *Let $C \subseteq V$ be a convex set and $a, b \in \mathbb{R}$ with $a \leq b$ and $ab \geq 0$. Then, $[a,b]C := \{xu \mid x \in [a,b], u \in C\}$ is convex. In other words, the pointwise product of a convex set and an interval that does not contain zero as an inner point is again convex.*

Proof: Let $x, y \in [a,b]$, $u, v \in C$ and $\lambda \in [0,1]$. By Remark 2.2 there is a $\nu \in [0,1]$ such that $\lambda xu + (1-\lambda)yv = (\lambda x + (1-\lambda)y)(\nu u + (1-\nu)v) \in [a,b]C$. □

We remind of the following basic facts on sums and linear combinations of convex sets and polytopes:

Remark 2.3

a) *For convex $\boldsymbol{A}, \boldsymbol{B} \subseteq V$ the* Minkowski sum $\boldsymbol{A} + \boldsymbol{B} := \{a + b \mid a \in \boldsymbol{A}, b \in \boldsymbol{B}\}$ *is again a convex set.*

b) *Let $\boldsymbol{A}, \boldsymbol{B} \subseteq V$ be polytopes with vertices $a_1, ...a_r$ and $b_1, ...b_s$, respectively. Then, $\boldsymbol{A} + \boldsymbol{B}$ is a polytope and equals the convex hull $\boldsymbol{C}$ of the vertex sums $a_i + b_j$, $i = 1, ..., r$, $j = 1, ..., s$.*

c) *For convex sets (polytopes) $A_1, ..., A_n \subseteq V$ and $\alpha_1, ..., \alpha_n \in \mathbb{R}$, $\sum_{i=1}^n \alpha_i A_i$ is a convex set (a polytope).*

Lemma 2.1 *Let $\boldsymbol{a^{(1)}}, ..., \boldsymbol{a^{(n)}} \subseteq \mathbb{R}^m$ and $\boldsymbol{b} \subseteq \mathbb{R}^n$ be convex sets (polytopes) such that $\boldsymbol{b}$ is completely contained in one orthant of $\mathbb{R}^n$. For $i = 1, ..., n$ let $e_i \in \mathbb{R}^n$ denote the i-th standard unit vector and define $\boldsymbol{A} := \sum_{i=1}^n \boldsymbol{a^{(i)}} e_i^T \subseteq \mathbb{R}^{m,n}$. Then, $\boldsymbol{Ab} \subseteq \mathbb{R}^m$ is a convex set (polytope).* [4]

Proof: For $X, Y \in \boldsymbol{A}$, $x, y \in \boldsymbol{b}$, $\lambda \in [0,1]$ we will show that $\lambda Xx + (1-\lambda)Yy \in \boldsymbol{Ab}$. By definition, $X = \sum_{i=1}^n u^{(i)} e_i^T$ and $Y = \sum_{i=1}^n v^{(i)} e_i^T$ for suitable $u^{(i)}, v^{(i)} \in \boldsymbol{a^{(i)}}$. By Remark 2.2 for each $i \in \{1, ..., n\}$ there is a $\nu_i \in [0,1]$ such that

$$\begin{aligned} \lambda x_i u^{(i)} + (1-\lambda)y_i v^{(i)} &= (\lambda x_i + (1-\lambda)y_i)(\nu_i u^{(i)} + (1-\nu_i)v^{(i)}) \\ &= (\nu_i u^{(i)} + (1-\nu_i)v^{(i)}) e_i^T (\lambda x + (1-\lambda)y) \; . \end{aligned}$$

Since all $\boldsymbol{a^{(i)}}$ and $\boldsymbol{b}$ are convex, we have $w^{(i)} := (\nu_i u^{(i)} + (1-\nu_i)v^{(i)}) \in \boldsymbol{a^{(i)}}$ and $z := (\lambda x + (1-\lambda)y) \in \boldsymbol{b}$ whence

$$\lambda Xx + (1-\lambda)Yy = \left(\sum_{i=1}^n w^{(i)} e_i^T\right) z \in \boldsymbol{Ab}$$

which proves convexity of $\boldsymbol{Ab}$. In the case of polytopes, Remark 2.3 c) and Corollary 2.1 show that $\boldsymbol{Ab}$ is a polytope. □

[4]Recall that the *orthants* of $\mathbb{R}^n$ are the sets $\Omega_s := \{x \in \mathbb{R}^n \mid x_i s_i \geq 0, i = 1, ..., n\}$ where $s \in \{-1, 1\}^n$ is a signature vector.

Corollary 2.3 *Let $\boldsymbol{a}^{(1)},...,\boldsymbol{a}^{(n)} \subseteq \mathbb{R}^m$ and $\boldsymbol{b} \subseteq \mathbb{R}^n$ be connected finite unions of polytopes and define $\boldsymbol{A} := \sum_{i=1}^n \boldsymbol{a}^{(i)} e_i^T \subseteq \mathbb{R}^{m,n}$. Then, $\boldsymbol{Ab} \subseteq \mathbb{R}^m$ is a connected finite union of polytopes.*

Proof: By assumption $\boldsymbol{a}^{(i)} = \cup_{j=1}^{r_i} \boldsymbol{a}^{(i,j)}$, $\boldsymbol{b} = \cup_{j=1}^{s} \boldsymbol{b}^{(j)}$ for suitable $r_i, s \in \mathbb{N}$ and polytopes $\boldsymbol{a}^{(i,j)} \subseteq \mathbb{R}^m$, $\boldsymbol{b}^{(j)} \subseteq \mathbb{R}^n$. We can further subdivide each $\boldsymbol{b}^{(j)}$ into polytopes lying in one of the 2^n orthants of $\mathbb{R}^n$ so that without loss of generality we may already assume that each $\boldsymbol{b}^{(j)}$ is contained in one orthant of $\mathbb{R}^n$. By Lemma 2.1, $(\sum_{i=1}^n \boldsymbol{a}^{(i,j_i)} e_i^T)\boldsymbol{b}^{(j)}$ is a polytope for all $j_i \in \{1,...,r_i\}$, $i,j \in \{1,...,n\}$. Hence,

$$\boldsymbol{Ab} = \left(\sum_{i=1}^n \boldsymbol{a}^{(i)} e_i^T\right) \boldsymbol{b} = \cup_{j_1=1}^{r_1} \cdots \cup_{j_n=1}^{r_n} \cup_{j=1}^{s} \left(\sum_{i=1}^n \boldsymbol{a}^{(i,j_i)} e_i^T\right) \boldsymbol{b}^{(j)}$$

is a finite union of polytopes which by continuity of matrix multiplication is connected. □

Corollary 2.4 *Let $\boldsymbol{A} \in \mathbb{IR}^{\mathrm{m,n}}$ be an interval matrix and let $\boldsymbol{b} \subseteq \mathbb{R}^n$ be a convex set (polytope) that is completely contained in one orthant of $\mathbb{R}^n$. Then, $\boldsymbol{Ab}$ is a convex set (polytope). In particular, if $\boldsymbol{b} \in \mathbb{IR}^{\mathrm{n}}$ is an interval vector that is contained in one orthant of $\mathbb{R}^n$, then $\boldsymbol{Ab}$ is a polytope.*

Proof: The columns $\boldsymbol{a}^{(i)} := \boldsymbol{A}e^i \in \mathbb{IR}^{\mathrm{m}}$ of $\boldsymbol{A}$ are interval vectors. In particular, they are polytopes in $\mathbb{R}^m$ and fulfill

$$\boldsymbol{A} = \sum_{i=1}^n \boldsymbol{a}^{(i)} e_i^T \ .$$

Thus Lemma 2.1 yields the assertion. □

Corollary 2.5 *If $\boldsymbol{A} \in \mathbb{IR}^{\mathrm{m,n}}$ is an interval matrix and if $\boldsymbol{b} \subseteq \mathbb{R}^n$ is a connected finite union of polytopes (which, for example, is the case if $\boldsymbol{b} \in \mathbb{IR}^{\mathrm{n}}$ is an interval vector), then $\boldsymbol{Ab}$ is a connected finite union of polytopes.*

Proof: Like in the proof of the previous Corollary 2.4, the assertion follows directly from Lemma 2.1. □

The following example shows that in general the pointwise product $\boldsymbol{Ab}$ of a matrix polytope $\boldsymbol{A} \subseteq \mathbb{R}^{2,2}$ and a vector polytope $\boldsymbol{b} \subseteq \mathbb{R}^2$, both contained in the non-negative orthant, is not necessarily a finite union of polytopes. Consider

$$A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, a := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, b := \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The straight line $\boldsymbol{A} := \overline{AB} = \{\mathrm{diag}(\lambda, 1-\lambda) \mid \lambda \in [0,1]\} \subset \mathbb{R}^{2,2}$ joining A and B and the straight line $\boldsymbol{b} := \overline{ab} = \{(\mu, 1-\mu)^T \mid \mu \in [0,1]\} \subset \mathbb{R}^2$ joining a and b, are clearly polytopes (each contained in the non-negative orthant). Their pointwise product $\boldsymbol{Ab} = \{(\lambda\mu, (1-\lambda)(1-\mu))^T \mid \lambda, \mu \in [0,1]\}$ is the area enclosed by the x- and y-axis and the nonlinear curve $f(x) = (1-\sqrt{x})^2$, see Figure 1. In particular, $\boldsymbol{Ab}$ is not a finite union of polytopes.

This example might look a little bit simple and somehow degenerate since $\boldsymbol{A}$ and $\boldsymbol{b}$ are just lines and one might therefore be interested in higher dimensional polytopes

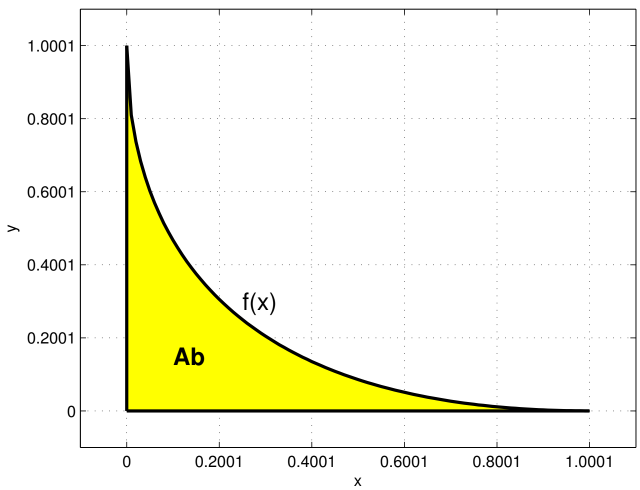

Figure 1: line product

$\boldsymbol{A}$ and thick interval vectors $\boldsymbol{b}$ serving as counterexamples. Such an example is given now: Consider the matrix polytope

$$\boldsymbol{A} := \left\{ \begin{bmatrix} \alpha - 1 & \beta \\ \beta - 1 & \alpha \end{bmatrix} \mid \alpha, \beta \in [0,1] \right\}$$

which has the four vertices $\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$, and the interval vector $\boldsymbol{b} = \begin{bmatrix} [0,1] \\ [0,1] \end{bmatrix}$. Then, the set

$$\boldsymbol{Ab} = \left\{ \begin{bmatrix} (\alpha - 1)u + \beta v \\ (\beta - 1)u + \alpha v \end{bmatrix} \mid \alpha, \beta, u, v \in [0,1] \right\}$$

has the shape shown in Figure 2 which is obviously non-convex. It is also not a finite union of polytopes since the boundary parts f_i, $i = 1, ..., 4$, are curved.

These curves are by symmetry all congruent and can be computed explicitly which is done best by rotating $\boldsymbol{Ab}$ by $\pi/4$ to the right and stretching it by a factor of $\sqrt{2}$. This is achieved by multiplying $\boldsymbol{Ab}$ with the matrix $Q := \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ from the left. The set

$$Q\boldsymbol{Ab} = \left\{ \begin{bmatrix} (\alpha + \beta)(v + u) - 2u \\ (\alpha - \beta)(v - u) \end{bmatrix} \mid \alpha, \beta, u, v \in [0,1] \right\}$$

is symmetric to the x-axis which simplifies the computation of the curved boundary sections $g_i := Qf_i$, $i = 1, ..., 4$, see Figure 3. Skipping basic calculations it turns out that

$$g_1(x) = \left(2 - \sqrt{2 - x}\,\right)^2 = -g_2(x)$$

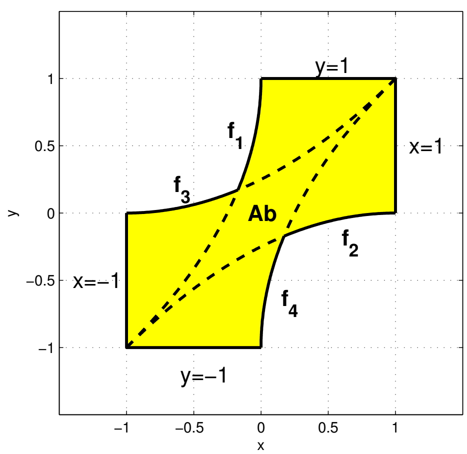

Figure 2: **Ab**

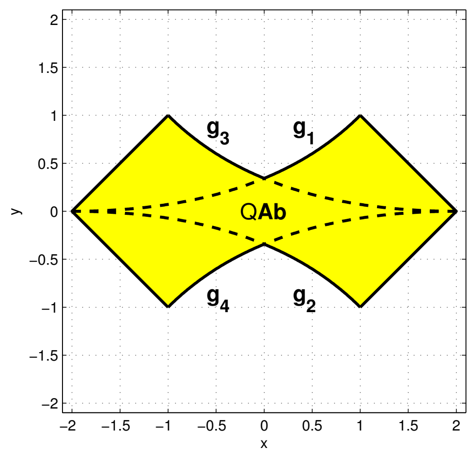

Figure 3: Q**Ab**

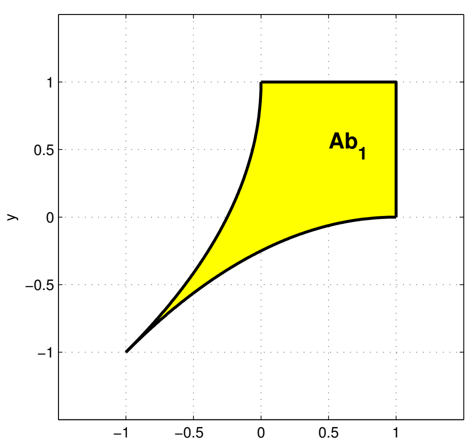

Figure 4: $\mathbf{Ab_1}$

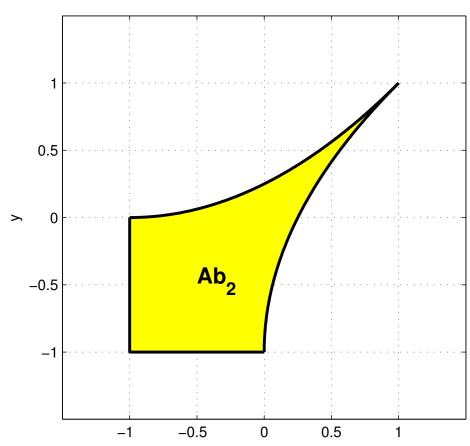

Figure 5: $\mathbf{Ab_2}$

where $x \in [0,1]$ and the dashed line continuing g_1 is of course obtained by taking $x \in [-2,0]$. Correspondingly, we have $g_3(x) = g_1(-x) = -g_4(x)$ for $x \in [-1,0]$.

It might additionally be interesting to note that for

$$\boldsymbol{b_1} := \begin{bmatrix} [0,1] \\ 1 \end{bmatrix}, \boldsymbol{b_2} := \begin{bmatrix} 1 \\ [0,1] \end{bmatrix} \subset \begin{bmatrix} [0,1] \\ [0,1] \end{bmatrix} = \boldsymbol{b}$$

– at first glance surprisingly – holds $\boldsymbol{Ab} = \boldsymbol{Ab_1} \cup \boldsymbol{Ab_2}$. We omit the elementary proof since it is not in the focus of this paper. The sets $\boldsymbol{Ab_1}$ and $\boldsymbol{Ab_2}$ are the two overlapping "sting ray" shapes shown in Figures 4, 5.

The next example shows that the pointwise product of two interval matrices, in general, is not a finite union of polytopes. Consider

$$\boldsymbol{A} := \begin{bmatrix} [0,1] \\ 1 \end{bmatrix} \in \mathbb{IR}^2, \quad \tilde{\boldsymbol{A}} := \begin{bmatrix} [0,1] & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{IR}^{2,2}.$$

Then,

$$\boldsymbol{C} := \boldsymbol{A}\boldsymbol{A}^T = \left\{ \begin{bmatrix} ab & a \\ b & 1 \end{bmatrix} \mid a,b \in [0,1] \right\}, \ \tilde{\boldsymbol{C}} := \tilde{\boldsymbol{A}}\tilde{\boldsymbol{A}} = \left\{ \begin{bmatrix} ab+1 & a \\ b & 1 \end{bmatrix} \mid a,b \in [0,1] \right\}.$$

Neglecting a constant shift, both product sets $\boldsymbol{A}\boldsymbol{A}^T$ and $\tilde{\boldsymbol{A}}\tilde{\boldsymbol{A}}$ can be visualized in three dimensions as the surface $\{(x,y,xy) \mid x,y \in [0,1]\}$ which is curved (and also non-convex) and therefore is not a finite union of polytopes, see Figure 6.

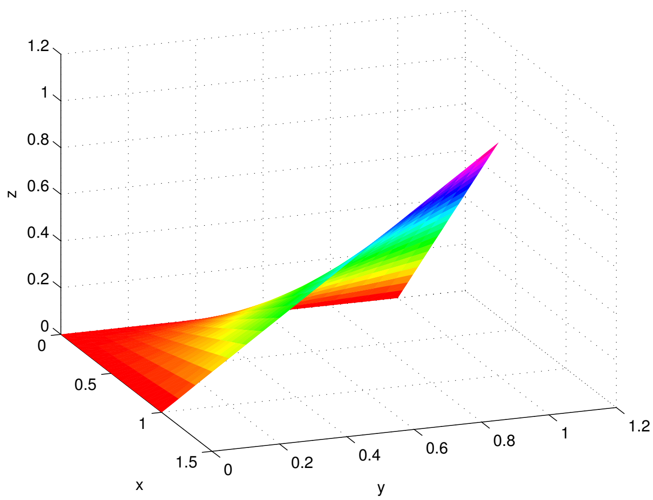

Figure 6: interval matrix product

3 Extreme Point Analysis

According to the previous examples the boundary shape of the pointwise matrix product $\boldsymbol{C} = \boldsymbol{AB}$ of arbitrary matrix polytopes $\boldsymbol{A}$ and $\boldsymbol{B}$ might be seen as rather random

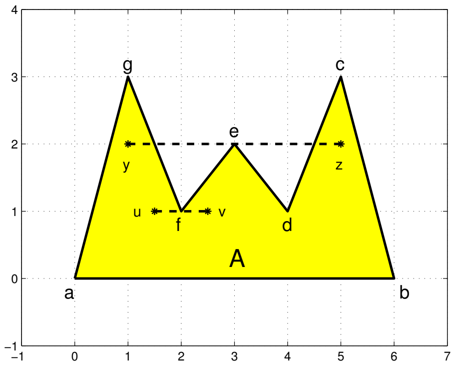

Figure 7: local and global extreme points

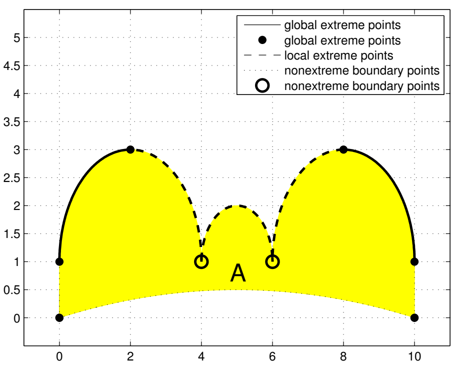

Figure 8: infinitely many extreme points

and it seems that quite arbitrary curved geometrical figures might be constructed in this way. But this is by no means true. For example, $\boldsymbol{C}$ can never have outward curved boundary sections like illustrated in Figure 8 for the two-dimensional case. This will be described precisely in the following text.

Definition 3.1 *Let A be an arbitrary subset of V. A point $x \in A$ is called an* extreme point *(or a* global extreme point*) of A if there do not exist two distinct points $y, z \in A \backslash \{x\}$ such that x lies on the line $\overline{yz}$ connecting y and z. A point $x \in A$ is called a* local extreme point *of A if there do not exist two distinct points $y, z \in A \backslash \{x\}$ such that $x \in \overline{yz} \subseteq A$. The set of all extreme points of A is denoted by* $\mathrm{ext}(A)$ *and the set of all local extreme points by* $\mathrm{locext}(A)$.

Note that each global extreme point is by definition also a local extreme point, i.e., $\mathrm{ext}(A) \subseteq \mathrm{locext}(A)$, but not vice versa. Figures 7, 8 visualize these definitions.

In Figure 7 the set A has the extreme points $\mathrm{ext}(A) = \{a, b, c, g\}$ and the local extreme points $\mathrm{locext}(A) = \mathrm{ext}(A) \cup \{e\}$. The local extreme point e is not a global one since it lies on the line $\overline{yz}$ with distinct $y, z \in A \backslash \{e\}$. Note also that the two internal vertices d, f are not local extreme points since for example f lies on the line $\overline{uv} \subseteq A$ with $u, v \neq f$. In Figure 8 we can intuitively see that outward curved boundary sections of A always contain infinitely many local extreme points.

For convex sets A global and local extreme points coincide and they are exactly those points $x \in A$ for which $A \backslash \{x\}$ is again convex. The later statement is actually a common way to define extreme points in the context of convex set theory but it does not make sense for non-convex sets. The following theorem is due to Minkowski [11].

Theorem 3.1 (Minkowski) *For a compact and convex set $K \subset \mathbb{R}^n$ and a subset $A \subseteq K$ the following statements are equivalent:*

a) *K is the convex hull of A.*

b) *The extreme points of K are contained in A.*

In formulae: $K = \mathrm{conv}(A) \quad \Leftrightarrow \quad \mathrm{ext}(K) \subseteq A$.
In particular, for $A := \mathrm{ext}(K)$ *this means* $K = \mathrm{conv}(\mathrm{ext}(K))$.

Corollary 3.1 *For compact $K \subset \mathbb{R}^n$, $\mathrm{conv}(K)$ is also compact and therefore*

$$\mathrm{conv}(K) = \mathrm{conv}(\mathrm{ext}(K)).$$

Minkowski's theorem was generalized by Krein and Milman [9] to their following celebrated result which is frequently used in functional analysis (see also [3], Chapter II, §7, Section 1, Theorem 1 and its corollary):

Theorem 3.2 (Krein-Milman) *In a Hausdorff locally convex vector space V every compact convex set $K \subseteq V$ is the topological closure of the convex hull of its extreme points, i.e., $K = \overline{\mathrm{conv}(\mathrm{ext}(K))}$ where $\overline{X}$ denotes the topological closure of a subset X in V.*[5]

In particular, the Krein-Milman Theorem ensures the existence of extreme points in nonempty compact convex sets in Hausdorff locally convex vector spaces. This also holds true for arbitrary nonempty compact but not necessarily convex sets, see [1], Chapter 7, Section 7.12, Corollary 7.66:

Lemma 3.1 *Every nonempty compact subset of a Hausdorff locally convex vector space has an extreme point.*

Only this fact will be used in the proof of the following lemma which will be used to prove our main Theorem 1.1.

Lemma 3.2 *Suppose that V is a Hausdorff locally convex vector space, that W is a closed linear subspace of V, and that $K \subseteq V$ is compact. Then,*

a) $\mathrm{ext}(K/W) \subseteq \mathrm{ext}(K)/W$

b) $\mathrm{locext}(K/W) \subseteq \mathrm{locext}(K)/W$

where K/W, $\mathrm{ext}(K)/W$, and $\mathrm{locext}(K)/W$ are the images of K, $\mathrm{ext}(K)$, and $\mathrm{locext}(K)$ in the factor space $V/W = \{v + W \mid v \in V\}$.

Proof: Let $x + W \in K/W$ with $x \in K$. Since K is compact and $x + W$ is closed, $K \cap (x + W)$ is compact and nonempty as $x \in K \cap (x + W)$. From Corollary 3.1 it follows that $\mathrm{ext}(K \cap (x + W)) \neq \emptyset$. (The use of Corollary 3.1 is the main point where we need that the topology of V is locally convex.) Thus we can choose an $x^* \in \mathrm{ext}(K \cap (x + W))$ which in particular fulfills $x^* + W = x + W$ and $x^* \in K$. This shows that we may have chosen apriori $x = x^*$ so that $x \in \mathrm{ext}(K \cap (x + W))$ can be assumed in the following.

a) Let us consider the case $x+W \in \mathrm{ext}(K/W)$ and assume that $x \notin \mathrm{ext}(K)$. Then, by definition, there exist distinct $y, z \in K\backslash\{x\}$ and $\lambda \in\ ]0, 1[$ such that $x = \lambda y + (1-\lambda)z$. Hence also $x+W = \lambda(y+W)+(1-\lambda)(z+W)$ and therefore $x+W \in \mathrm{ext}(K/W)$ implies $x+W = y+W = z+W$ which means that $y, z \in (K\cap(x+W))\backslash\{x\}$, a contradiction to $x \in \mathrm{ext}(K\cap(x+W))$. Hence $x \in \mathrm{ext}(K)$ holds true and therefore $x+W \in \mathrm{ext}(K)/W$.

b) Let us now consider the case $x+W \in \mathrm{locext}(K/W)$ and assume that $x \notin \mathrm{locext}(K)$. Then, by definition, there exist distinct $y, z \in K\backslash\{x\}$ such that $x \in \overline{yz} \subseteq K$. By linearity of the canonical mapping $V \to V/W,\ v \mapsto v + W$ it follows that

$$x + W \in \overline{yz}/W = \overline{(y + W)(z + W)} \subseteq K/W\ .$$

[5] Relevant definitions and properties of topological vector spaces are listed in the appendix.

Thus $x + W \in \text{locext}(K/W)$ implies $x + W = y + W = z + W$ and therefore $\overline{yz} \subseteq (K \cap (x+W))$. In particular, this means $y, z \in (K \cap (x+W)) \backslash \{x\}$. But this contradicts $x \in \text{ext}(K \cap (x+W))$. Hence $x \in \text{locext}(K)$ holds true which proves $x + W \in \text{locext}(K)/W$. □

Now, Theorem 1.1 will be proved:

a) **Case 1:** $n = 1$. Set $U := U_1$ and $K := K_1$. Then, $f : U \to V$ is linear.
Case 1.1: f is injective. Let $v \in \text{ext}(f(K))$ and choose $a \in K$ such that $v = f(a)$. If $a \in \text{ext}(K)$, then we are done, so that we may assume that $a \notin \text{ext}(K)$. Thus there are distinct $x, y \in K\backslash\{a\}$ such that $a \in \overline{xy}$. But then linearity of f implies $v = f(a) \in f(\overline{xy}) = \overline{f(x)f(y)}$ and $f(x), f(y) \in f(K)$. Therefore $v \in \text{ext}(f(K))$ yields $f(a) = v \in \{f(x), f(y)\}$, a contradiction as f is injective and $x \neq a \neq y$.
Case 1.2: f is not injective. Then, the kernel W of f is nontrivial. Since f is continuous and V is Hausdorff, W is closed in U. The quotient space $\tilde{U} := U/W = \{u + W \mid u \in U\}$ endowed with the quotient topology is a again a Hausdorff locally convex vector space and its subset $\tilde{K} := K/W = \{a + W \mid a \in K\}$ is also compact again. Moreover, the mapping $\tilde{f} : \tilde{U} \to V, \quad u + W \mapsto f(u)$ is well-defined, linear, injective and continuous. In particular, we have

$$\tilde{f}(\tilde{K}) = f(K) \tag{5}$$
$$\tilde{f}(\text{ext}(K)/W) = f(\text{ext}(K)) \ . \tag{6}$$

By Case 1.1 applied to $\tilde{f}$, Lemma 3.2 a) and Equations (5),(6) it follows that

$$\begin{aligned} \text{ext}(f(K)) &= \text{ext}(\tilde{f}(\tilde{K})) && \text{[Equation (5)]} \\ &\subseteq \tilde{f}(\text{ext}(\tilde{K})) && \text{[Case 1.1 applied to } \tilde{f}] \\ &\subseteq \tilde{f}(\text{ext}(K)/W) && \text{[Lemma 3.2 a)]} \\ &= f(\text{ext}(K)) && \text{[Equation (6)]} \ . \end{aligned}$$

Case 2: $n \geq 2$. Let $v \in \text{ext}(f(K_1 \times ... \times K_n))$ and choose $a_i \in K_i$, $i = 1, ..., n$ such that $v = f(a_1, ..., a_n)$. The function

$$g : K_1 \to V, \ x \mapsto f(x, a_2, ..., a_n)$$

is linear and continuous. Since $g(K_1) \subseteq f(K_1 \times ... \times K_n)$, clearly

$$v \in \text{ext}(f(K_1 \times ... \times K_n)) \cap g(K_1) \subseteq \text{ext}(g(K_1)) \ ,$$

and since we already proved the assertion for the linear case, we have

$$v \in \text{ext}(g(K_1)) \subseteq g(\text{ext}(K_1))$$

which means that there is an $a_1^* \in \text{ext}(K_1)$ such that

$$f(a_1^*, a_2, ..., a_n) = g(a_1^*) = v \ .$$

Proceeding in the same way for $a_2, a_3, ..., a_n$, we can replace all a_i by some $a_i^* \in \text{ext}(K_i)$, $i = 1, ..., n$, such that $f(a_1^*, a_2^*, ..., a_n^*) = v$. This finishes the proof of a).

The proof of b) is completely analogous to a). Simply replace all occurrences of "ext" by "locext". Only in Case 1.1 there is a slight variation: The points $x, y \in K\backslash\{a\}$

can be chosen such that $a^* \in \overline{xy} \subseteq K$. Then, linearity of f implies $v = f(a) \in f(\overline{xy}) = \overline{f(x)f(y)} \subseteq f(K)$ which as $v \in \text{locext}(f(K))$ again implies $f(a) = v \in \{f(x), f(y)\}$, a contradiction to f being injective. Furthermore, in Case 1.2, part b) of Lemma 3.2 must be used instead of a). □

We want to mention clearly that Theorem 1.1 a) appears for convex compact sets K_i in [6], § 8, p. 112, Lemma 8.11, with a totally different proof and in a totally different context concerning convergence in weak topologies of topological tensor products. It was used in [6] to generalize known results on this topic.

Theorem 1.1 easily extends to vector spaces over the complex numbers:

Theorem 3.3 *Let $U_1, ..., U_n$ be Hausdorff locally convex vector spaces over the complex numbers and let V be a Hausdorff topological vector space over the complex numbers. Furthermore, let*

$$f : U_1 \times U_2 \times \cdots \times U_n \to V$$

be a continuous, $\mathbb{R}$-multilinear function so that

$$f(\lambda_1 u_1, ..., \lambda_n u_n) = \prod_{i=1}^{n} \lambda_i \ f(u_1, ..., u_n)$$

for all $u_i \in U_i$, $\lambda_i \in \mathbb{R}$, $i = 1, ..., n$. Then, for arbitrary compact sets $K_i \subseteq U_i$, $i = 1, ..., n$, we have

a) $\text{ext}(f(K_1 \times ... \times K_n)) \subseteq f(\text{ext}(K_1) \times ... \times \text{ext}(K_n))$

b) $\text{locext}(f(K_1 \times ... \times K_n)) \subseteq f(\text{locext}(K_1) \times ... \times \text{locext}(K_n))$.

Proof: All vector spaces $U_1, ..., U_n, V$ considered as real vector spaces remain locally convex. Moreover, the function f is $\mathbb{R}$-multilinear, so that the assertion follows from Theorem 1.1. □

Theorems 1.1 and 3.3 can be easily extended to unions of convex compact sets:

Corollary 3.2 *Take $n = 2$, U_1, U_2, V, and f like in Theorem 1.1 or Theorem 3.3, and let $(A_i)_{i \in I}$ and $(B_j)_{j \in J}$ be two families of compact convex sets in U_1 and U_2 respectively where I and J are arbitrary (not necessarily finite or countable) index sets. Then,* $\text{locext}(f(\cup_{i \in I} A_i, \cup_{j \in J} B_j)) \subseteq \cup_{i \in I} \cup_{j \in J} f(\text{ext}(A_i), \text{ext}(B_j))$.

Proof: Let $w \in \text{locext}(f(\cup_{i \in I} A_i, \cup_{j \in J} B_j)) \subseteq f(\cup_{i \in I} A_i, \cup_{j \in J} B_j)$. Then, there are $i \in I$, $j \in J$, $a \in A_i$, $b \in B_j$ such that $w = f(a, b)$. By definition of local extreme points and by Theorems 1.1 and 3.3 applied to the convex compact sets A_i and B_j it follows that

$$\begin{aligned} w \in f(A_i, B_j) \cap \text{locext}(f(\cup_{i \in I} A_i, \cup_{j \in J} B_j)) &\subseteq \text{locext}(f(A_i, B_j)) \\ &\subseteq f(\text{ext}(A_i), \text{ext}(B_j)) \subseteq \cup_{i \in I} \cup_{j \in J} f(\text{ext}(A_i), \text{ext}(B_j)) \ . \end{aligned}$$ □

Considering real or complex matrix polytopes and taking matrix multiplication as $\mathbb{R}$-bilinear mapping f, Corollary 3.2 implies

Corollary 3.3

a) *For matrix polytopes* $\boldsymbol{A} \subseteq \mathbb{C}^{m,n}$ *and* $\boldsymbol{B} \subseteq \mathbb{C}^{n,k}$ *we have* $\operatorname{locext}(\boldsymbol{AB}) \subseteq \operatorname{ext}(\boldsymbol{A})\operatorname{ext}(\boldsymbol{B})$. *In particular,* $\operatorname{locext}(\boldsymbol{AB})$ *is finite.*

b) *For connected finite unions* $\boldsymbol{A} = \cup_{i=1}^{a} \boldsymbol{A}_i \subseteq \mathbb{C}^{m,n}$ *and* $\boldsymbol{B} = \cup_{i=1}^{b} \boldsymbol{B}_i \subseteq \mathbb{C}^{n,k}$ *of matrix polytopes* $\boldsymbol{A}_i \subseteq \mathbb{C}^{m,n}$ *and* $\boldsymbol{B}_j \subseteq \mathbb{C}^{n,k}$ *we have*

$$\operatorname{locext}(\boldsymbol{AB}) \subseteq \cup_{i=1}^{a} \cup_{j=1}^{b} \operatorname{ext}(\boldsymbol{A}_i)\operatorname{ext}(\boldsymbol{B}_j).$$

In particular, $\operatorname{locext}(\boldsymbol{AB})$ *is finite.*

Corollary 3.3 rigorously formalizes the illustrative but imprecise initially stated phrase that outward curved boundary sections cannot occur in the pointwise matrix product of two (real or complex) matrix polytopes because such boundary shapes contain infinitely many local extreme points.

We end with an examples that illustrates this fact once more. Consider the following interval matrix and interval vector:

$$\boldsymbol{A} \quad = \quad \begin{bmatrix} [-1,1] & [-1,1] \\ [1,2] & [0,1] \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} [-1,2] \\ [0,1] \end{bmatrix}.$$

Their pointwise product $\boldsymbol{Ab}$ is by Corollary 2.5 a connected finite union of polytopes. Since for a connected finite union of polytopes the local extreme points build a subset of its finitely many vertices, we already know implicitly without using Corollary 3.3 that $\boldsymbol{Ab}$ only has finitely many local extreme points. The vertices of $\boldsymbol{Ab}$ in counterclockwise order are stated in the columns of the following table, see Figure 9:

x	2	2	1.5	3	3	−3	−3	−1.5	−2	−2
y	−2	0	0.5	2	5	5	2	0.5	0	−2

Clearly, the two vertices $\begin{bmatrix} \pm 2 \\ 0 \end{bmatrix}$ are local but not global extreme points.

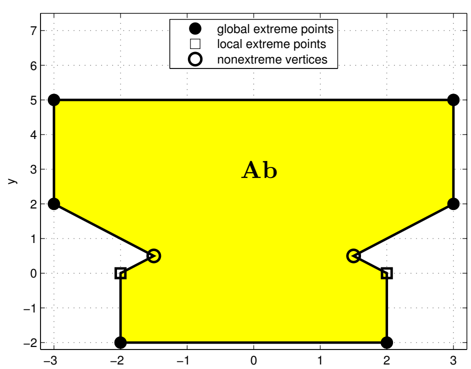

Figure 9: **Ab**

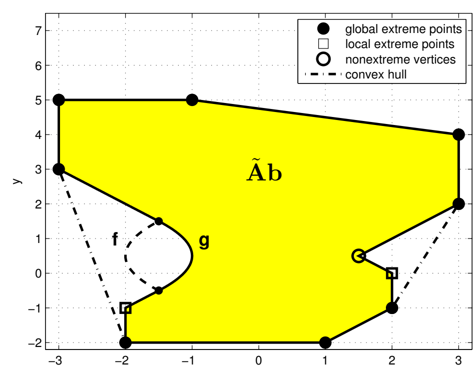

Figure 10: $\tilde{\mathbf{A}}\mathbf{b}$

Next we consider the subpolytope

$$\tilde{\boldsymbol{A}} := \bigcup \left\{ \begin{bmatrix} 2\alpha - 1 & [-1,1] \\ [1,2] & 1-\alpha \end{bmatrix} \mid \alpha \in [0,1] \right\}$$

of $\boldsymbol{A}$ which has the vertices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $(a,d) \in \{(-1,1),(1,0)\}$, $b \in \{-1,1\}$, $c \in \{1,2\}$. Clearly $\tilde{\boldsymbol{A}}\boldsymbol{b} \subseteq \boldsymbol{A}\boldsymbol{b}$ but it turns out that $\tilde{\boldsymbol{A}}\boldsymbol{b}$ is not a finite union of polytopes anymore. Figure 10 shows that $\tilde{\boldsymbol{A}}\boldsymbol{b}$ has a curved boundary part g. Omitting basic calculations it can be shown that g has the analytic description

$$x = g(y) = -\frac{1}{2}\left(y - \frac{1}{2}\right)^2 - 1 \quad \text{for } y \in [-\frac{1}{2}, \frac{3}{2}] \; .$$

All other boundary parts of $\tilde{\boldsymbol{A}}\boldsymbol{b}$ are piecewise straight line segments. The set $\tilde{\boldsymbol{A}}\boldsymbol{b}$ still has two local extreme points which are not global ones, namely $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, but in this case we cannot deduce apriori, without Corollary 3.3, like we did for $\boldsymbol{A}\boldsymbol{b}$, that $\tilde{\boldsymbol{A}}\boldsymbol{b}$ only has finitely many local extreme points. For example, without using Corollary 3.3 it seems apriori not clear at all that $\tilde{\boldsymbol{A}}\boldsymbol{b}$ cannot have an outward curved boundary part f mirror symmetric to g instead of g like sketched by the dashed curve in Figure 10. Note carefully that f does not cross the boundary of $\operatorname{conv}(\tilde{\boldsymbol{A}}\boldsymbol{b})$ which is drawn as a broken-dotted line in Figure 10. This means that we also could not use the basic Remark 2.1 to exclude apriori an outward curved boundary shape like f.

We hope that this example supports the relevance of Theorem 1.1 and and its corollaries.

4 Appendix

In this appendix we briefly recall definitions and relevant facts of topological vector spaces and especially locally convex vector spaces. Details can be found, for example, in [3], [4], [5].

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ be endowed with the Euclidean topology. An $\mathbb{F}$-vector space V endowed with a topology τ is a *topological vector space* if the vector summation $V \times V \to V$, $(u,v) \mapsto u+v$ and also the scalar multiplication $\mathbb{F} \times V \to V$, $(\lambda, v) \mapsto \lambda v$ are both continuous. Here $V \times V$ and $\mathbb{F} \times V$ are considered as topological spaces endowed with the corresponding product topologies. Then, in particular, $(V,+)$ is an abelian *topological group*.

A topological vector space (V,τ) is called *locally convex* if there exists a fundamental system of neighborhoods of 0 consisting of convex sets or, equivalently, if the topology τ is defined by a set $\{p_i \mid i \in I\}$ of *seminorms* on V where I is some index set. The latter means that for each $i \in I$, $p_i : V \to [0,+\infty[$ fulfills $p_i(\lambda v) = |\lambda| p_i(v)$ and $p_i(u+v) \leq p_i(u) + p_i(v)$ for all $\lambda \in \mathbb{F}$ and $u, v \in V$ and the sets $B_{J,\varepsilon} := \cap_{j \in J}\{v \in V \mid p_j(v) < \varepsilon\}$ where J is a finite subset of I and $\varepsilon > 0$ form a fundamental system of convex neighborhoods of 0. In other words: τ is the *initial topology* of the seminorms p_i, $i \in I$, which is the coarsest topology on V such that all p_i are continuous. Note that a topological vector space (V,τ) over the complex numbers is locally convex if and only if (V,τ) considered as a vector space over the real numbers is locally convex.

Recall also that a topological space (V,τ) is *Hausdorff* if for all distinct $u, v \in V$ there are neighborhoods $A, B \in \tau$ such that $u \in A$, $v \in B$ and $A \cap B = \emptyset$.

Now if U is a (linear) subspace of a topological vector space (V,τ), then the *quotient topology* on the *quotient space* $V/U = \{v + U \mid v \in V\}$ is the finest topology such that the canonical linear mapping $\varphi_U : V \to V/U,\ v \mapsto v + U$ is continuous which means that the quotient topology is the so-called *final topology* of φ_U induced on V/U. If W is any other topological space, then a function $g : V/U \to W$ is continuous if and only if $g \circ \varphi_U : V \to W$ is continuous. The quotient topology is Hausdorff if and only if U is closed in V. Moreover, if V is locally convex, then also V/U is locally convex.

Next, like any abelian topological group, a topological vector space (V, τ) carries a *uniform structure* which is compatible with its topology τ and it is therefore possible to speak of completeness or incompleteness of V. This uniform structure is obtained in the following way: Let $\mathcal{R}$ be the set of all neighborhoods of 0 and for $R \in \mathcal{R}$ define $N_R := \{(u, v) \in V \times V \mid u - v \in R\}$. The sets N_R build a fundamental system for a *uniformity* $\mathcal{N}$ on V such that the topology $\tau_{\mathcal{N}}$ induced by $\mathcal{N}$ on V equals τ. Now the *uniform space* $(V, \mathcal{N})$ is called *complete* if each Cauchy filter in V with respect to $\mathcal{N}$ converges in V. A *Fréchet space* is a complete metrisable locally convex space. In particular, each Banach space is a Fréchet space and therefore of course also a locally convex space.

Acknowledgments

I thank Prof. S.M. Rump for helpful and inspiring discussions of the subject and the two unknown referees for their valuable hints and comments that led to improvements of this paper.

References

[1] Aliprantis, C.D., Border, K.C.: *Infinite Dimensional Analysis: A Hitchhiker's Guide*, Springer 2006.

[2] Barmish, B.R.: *New Tools for Robustness of Linear Systems*, Chapter 14, Macmillan, 1979.

[3] Bourbaki, N.: *Topological Vector Spaces*, Chapters 1-5, Springer, 1987.

[4] Bourbaki, N.: *General Topology*, Chapters 1-4, Springer, 1989.

[5] Bourbaki, N.: *General Topology*, Chapters 5-10, Springer, 1989.

[6] Floret, K.: Weakly Compact Sets, *Lecture Notes in Mathematics*, 801, Springer, 1980.

[7] Goldsztejn, A.: e-mail to the *Reliable Computing* mailing list, July 18, 2012. `http://dir.gmane.org/gmane.comp.mathematics.reliable-computing`

[8] Kelsey, R.: Matrix and Vector Product Conjectures, e-mail to the *Reliable Computing* mailing list, July 18, 2012. `http://dir.gmane.org/gmane.comp.mathematics.reliable-computing`

[9] Krein, M.G., and Milman, D.P.: On extreme points of regular convex sets, *Studia Mathematica*, 9:133–138, 1940.

[10] Kreinovich, V.: e-mail to the *Reliable Computing* mailing list, July 19, 2012.

[11] Minkowski, H.: *Gesammelte Abhandlungen*, Bd. II., Kap. XXV. Theorie der konvexen Körper, insbesondere Begründung ihres Oberflächenbegriffes, § 12. Die extremen Punkte eines konvexen Bezirkes, Teubner, Leipzig 1911.

[12] Zadeh, L.A., Desoer, C.A.: *Linear System Theory: The State Space Approach*, Chapter 9, Robert E. Krieger, 1979.