Preface

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Figure 1: Sergei Natanovich Bernstein (1880–1968). Photograph reproduced from the Russian Academy of Sciences website — see http://www.ras.ru.

In 1912[1] Sergei Natanovich Bernstein's (1880–1968, see Fig. 1) paper *Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités* appeared in the *Communications de la Société Mathématique de Kharkov* 2. Series XIII No. 1, pp. 1–2. In this short note[2] Bernstein introduced, for a given degree l, the polynomials

$$B_i^l(x) \;=\; \binom{l}{i} x^i (1-x)^{l-i}, \;\; i = 0, \ldots, l, \;\; x \in \mathbb{R} \tag{1}$$

which are now called *Bernstein polynomials*, in order to present a short proof of the Weierstrass Approximation Theorem. The subsequent history is well documented, see, e.g., [29] for the period up to 1955, the monograph [18] published in 1953, and the survey article [9] which appeared on the occasion of the hundredth anniversary of the above paper by Bernstein. Since the latter publication provides a historical perspective on the evolution of the polynomials (1) and a synopsis of the current state of associated algorithms and applications, we will

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[1] In this year the first issue, no.1, appeared, although the bound volume 13 of this journal carries the year 1913, see [29].

[2] This paper can be found at www.math.technion.ac.il/hat/papers/P03.PDF, at the homepage of *History of Approximation Theory*, which also provides useful related material.

focus here on the use of Bernstein polynomials in reliable computing, which covers the years since 1966, beginning just over a half-century after 1912.

The starting point was the paper [4] in which the *range enclosing property* of the Bernstein polynomials was first given: Since these polynomials form a basis for the space of the degree l polynomials, we can represent a given polynomial

$$p(x) \; = \; \sum_{i=0}^{l} a_i x^i \tag{2}$$

over $I = [0, 1]$ as

$$p(x) \; = \; \sum_{i=0}^{l} b_i B_i^l(x). \tag{3}$$

Then we have for the range of p over I the enclosure

$$\min_i \{b_i\} \; \leq \; p(x) \; \leq \; \max_i \{b_i\}, \;\; x \in I. \tag{4}$$

Property (4) is a consequence of the fact that the polynomials (1) are non-negative over I and form a partition of unity, i.e., they sum up to 1. For further properties of the polynomials (1) see, e.g., [9]. Rivlin [23] proved (linear) convergence of the bounds when the degree of the expansion is elevated and considered the case of complex polynomial coefficients. In a series of papers including [24, 25, 26], Rokne extended the results to (real and complex) interval polynomials. Lane and Riesenfeld [16] introduced subdivision, which exhibits quadratic convergence of the bounds, see, e.g., [10, 12].

The Bernstein expansion (3) was extended from the univariate to the n-variate case in two ways: Over the unit box I^n by tensorial Bernstein polynomials [12] and over the unit simplex in $\mathbb{R}^n$ by simplicial Bernstein polynomials, see, e.g., [12, 17]. The fact that the enclosure (4) remains in force in the multivariate case opened the way for a broad application of the Bernstein expansion in many fields where verification of the results is required. Furthermore, the use of interval arithmetic provides a guarantee of the enclosure also in the presence of rounding errors, e.g., [10, 28].

It is known that the representation (3) is numerically stable with respect to perturbations of the coefficients of the polynomial (2) and to rounding errors occurring during floating-point computations, e.g., [9]. "The importance of this attribute stems from the high premium placed on the 'robustness' (i.e., accuracy and consistency) of the geometrical computations performed in CAD systems. Unlike most other forms of scientific or engineering computation, the output of CAD systems — geometric models — are not ends in themselves. Such models are rather the point of departure for downstream applications (meshing for finite-element analysis, path planning for manufacturing, etc.) that cannot succeed without accurate and consistent geometrical representations." [9, p. 394].

Besides its optimal stability for evaluation, the Bernstein basis (1) has optimal shape preserving properties, minimal conditioning of its collocation matrices

and fastest convergence rates of the corresponding iteration approximation, e.g., [8, 9].

We list here those applications that are, in our opinion, the most important; in each case we give a few references, where the focus is on papers from this issue:

1. Root isolation for polynomials, e.g., [16, 19], or more generally, the enclosure of the solutions of systems of polynomial equations and inequalities, e.g., [1, 11, 14, 28].

2. Computer aided geometric design: This includes, e.g., the computation of intersection points of planar algebraic curves and algebraic surfaces as an application of item 1. and the approximation by interval Bézier curves [27].

3. Robust control, e.g., invariance of stability properties of polynomials under polynomial parameter dependency [30].

4. Dynamic systems, e.g., computation of the reachable set of a polynomial dynamic system [6].

5. Global optimization: This includes (unconstrained) global minimization of polynomials over the standard simplex [17] or a box; for a list of applications in the quadratic case see [7]. In the constrained case, bound functions for the objective and constraint functions which may be used as relaxations in a branch and bound framework can be constructed by using the Bernstein expansion, see, e.g., [22] for constant and [13, 28] for affine bound functions.

6. Analysis and optimization of programs, e.g., memory requirement estimation [5].

7. Automatic theorem proving: This recent application includes proof of nonlinear inequalities using the functional programming language Haskell in the *flyspeck project*, which aims at a formal proof of the Kepler conjecture [15], and implementations using the mechanical theorem prover *Prototype Verification System* (PVS) [21] and the CoQ system with SSREFLECT extension [2].

By a very recent and somewhat surprising result [20], tight bounds on the range of a multivariate rational function over a box can be computed from the Bernstein enclosure of the ranges of the numerator and denominator polynomials. This will allow one to expand the range of problems which can be treated by the Bernstein approach.

We have collected eleven papers on Bernstein polynomials and have divided them into two groups: The first concerns mainly new properties of these polynomials, whereas the second group focuses on new applications.

We would like to thank all the authors for their rapid submissions and valuable contributions and the referees (see the list below) for their timely reviews

which have made it possible that this issue appears in the year of the hundredth anniversary. By a serendipitous coincidence, 2012 marks the occasion of the centennial anniversary not only of Bernstein's proof of the Weierstrass approximation theorem but also of Brouwer's fixed point theorem [3] on which many proofs of results in interval mathematics rely.

Constance, Germany, December 2012

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