A Unifying Framework to Uncertainty Quantification of Polynomial Systems Subject to Aleatory and Epistemic Uncertainty*

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Abstract

This article presents a unifying framework to uncertainty quantification for systems subject to several design requirements that depend polynomially on both aleatory and epistemic uncertainties. This methodology, which is based on the Bernstein expansions of polynomials, enables calculating bounding intervals for the range of means, variances and failure probabilities of response metrics corresponding to all possible epistemic realizations. Moreover, it enables finding sets that contain the critical combination of epistemic uncertainties leading to the best-case and worst-case results, e.g., cases where the failure probability attains its smallest and largest value. These bounding intervals and sets, whose analytical structure renders them free of approximation error while eliminating the possibility of convergence to non-global optima, can be made arbitrarily tight with additional computational effort. This framework enables the consideration of arbitrary and

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possibly dependent aleatory variables as well as the efficient accommodation for changes in the models used to describe the uncertainty.

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1 Introduction

Uncertainty Quantification (UQ) is the process of determining the effect of parameter uncertainties on response metrics of interest. Denote by $\boldsymbol{p}$ the parameter vector whose value is uncertain. Uncertain parameters can be classified as either *aleatory*, which are parameters subject to inherent and irreducible variability, or *epistemic*, which are reducible uncertainties resulting from a lack of knowledge [8]. While being aleatory or not is an intrinsic property of a parameter, being epistemic depends upon the knowledge the analyst has on the value(s) of the parameter. Consequently, these two classes of uncertainty are not mutually exclusive.

This article studies the performance and reliability of a system whose response metrics are polynomial functions of the uncertain parameters. Regarding performance, the acceptability of the system depends upon the most likely outcomes of a response metric. The performance analysis of a system consists of evaluating low-order moments of the performance function for a given uncertainty model of $\boldsymbol{p}$. These models are commonly prescribed by random variables via Probability Density Functions (PDF) or Cumulative Distribution Functions (CDF). Regarding reliability, the acceptability of the system depends upon its ability to satisfy several design requirements simultaneously. These requirements, which are represented by a set of inequality constraints, depend on $\boldsymbol{p}$. The system is deemed acceptable if all constraints are satisfied. Hence, the reliability analysis of a system consists of evaluating the probability of violating at least one of the requirements, a.k.a. the probability of failure.

The most common practice in UQ is to model all uncertainties as random variables and calculate some statistics of the response metrics. In performance analysis these statistics are the mean and the variance, while in reliability analysis this statistic is the probability of failure. Statistical moments can be readily approximated using sampling-based methods. The failure probability is also commonly approximated using these methods [7] as well as approaches based on asymptotic approximations of the failure domain [9].

Methods for propagating epistemic uncertainties include interval-value probability [3], second order probability [4], imprecise probability theory and Dempster-Shafer theory of evidence [11]. A common approach to quantifying the effects of both aleatory and epistemic uncertainties is to perform nested sampling. This involves drawing samples of the epistemic variables in an outer loop and performing an UQ for the aleatory variables in an inner loop. In this fashion, ensembles of statistics are generated by performing an UQ analysis for each realization of the epistemic variables. Nested iteration tends to be computationally expensive. Consequently, the nested sampling must often be under-resolved, particularly at the epistemic outer loop, resulting in an under-prediction of the range of statistics. Methods that replace the outer sampling loop with an optimization loop have been developed [1]. Even though these methods are more efficient than their sampling-based counterparts, they lead to very same under-predictions when convergence to non-global optima occurs.

Most of the approaches handling epistemic uncertainty treat it in a manner that is inconsistent with its definition. Specifically, epistemic uncertainties are often described as uniform random variables or probability boxes, a.k.a. p-boxes. A reducible uncertainty, should, in principle, be reducible to a *constant*. The value of that constant is, indeed, unknown in advance. However, the ignorance on such a value does not make the variable or the response metric random. This distinction has significant implications in the resulting UQ analysis. For example, let a be an aleatory variable modelled as a uniform random variable in $[0, 1]$, while e is an unknown constant in $[0, 1]$. In this setting we want to evaluate the response metric $g = a + e$ for various uncertainty models of e. When e is incorrectly modelled as a uniform random variable, g is a triangular distribution in $[0, 2]$. When e is incorrectly modelled as a p-box with vertical envelopes at 0 and 1, g is a p-box with lower and upper CDF envelopes g and $g+1$ supported in $[0, 1]$ and $[1, 2]$ respectively. When e is correctly modelled as an unknown constant, g is uniform in $[e, e+1]$. Let us look into the mean, variance and probability of g exceeding $3/2$. Denote by $E[\cdot]$, $V[\cdot]$ and $P[\cdot]$ the mean, variance and probability operators on the probability space of a. While the probabilistic model leads to $E[g] = 1$, $V[g] = 3/18$ and $P[g > 3/2] = 1/16$, the p-box model leads to $\text{Range}(E[g]) = [1/2, 3/2]$, $\text{Range}(V[g]) = [0, 13/6]$ and $\text{Range}(P[g > 3/2]) = [0, 1/2]$. The correct model on the other hand, yields $\text{Range}(E[g]) = [1/2, 3/2]$, $\text{Range}(V[g]) = 1/12$ and $\text{Range}(P[g > 3/2]) = [0, 1/2]$. Since the probabilistic model yields results that are not representative of the extreme values taken by the statistics, its usage may result in unsound and misleading UQ assessments. As for the p-box model, the UQ analysis yields conservative bounds for the range of the variance. Note that this conservatism is irreducible. For general g's, the range of statistics obtained by using this approach will suffer from irreducible conservatism even if the propagation is exact. This is a consequence of the uncertainty model not being sufficiently restrictive; e.g., in the example above the probability box describing the interval allows for unintended probabilistic models of e. This trivial example, where the resulting distributions and corresponding statistics are markedly different, illustrates the significant effects that the assumed uncertainty model has on the UQ analysis. Studies where epistemic uncertainties are modelled as uniform random variables or p-boxes are abundant. As the above example demonstrates, these models fail to properly describe the ubiquitous family of uncertainties that are *unknown constants*. In contrast to the developments in [3, 4, 11], epistemic variables will be modelled as such hereafter. Note that the values taken by aleatory variables are a function of the operating conditions affecting the system, while those of the epistemic variables are not. Therefore, only the aleatory variables can cause variations in the system's response.

A key feature of the present article is that the distinction between aleatory and epistemic variables is made consistently both qualitatively and quantitatively. While aleatory uncertainties are manipulated according to long-standing concepts of probability theory, epistemic uncertainties are manipulated using properties of the Bernstein expansion of polynomials. In the proposed context, each response metric becomes a random process. The family of random variables associated with this process is parametrized by the value of the epistemic variables. In regard to the management of aleatory uncertainty, the framework proposed enables bounding tightly and rigorously the value of statistics supporting conventional UQ analyses; e.g., tight, formally verifiable ranges containing the failure probability range are calculated. In regard to the management of both aleatory and epistemic uncertainties, the framework proposed enables bounding tightly and rigorously the range of statistics corresponding to all possible realizations of the epistemic variables (e.g., the range of failure probabilities), as well as the epistemic realizations where the statistics take on extreme values (e.g., the combination of epistemic variables leading to the largest failure probability). These realizations will be referred to as *critical epistemic points.* The bounds of the ranges and of the critical epistemic points resulting from the approach proposed can be made arbitrarily small with additional computational effort regardless of the manner in which the response metrics depend on $\boldsymbol{p}$.

The strategies proposed here yield exact results. The computed bound of an extreme statistic does not suffer from approximation error, while the convergence to the critical epistemic point(s) is guaranteed. Standard probabilistic methods, such as polynomial chaos, Monte Carlo sampling, imprecise probabilities, FORM, etc., cannot bound the approximation error present in their estimate of statistics, nor can they identify the extreme values attained by such statistics when epistemic variables are present. As compared to methods based on interval analysis [5], whose results are also formally verifiable, the bounds proposed are better since they can always be made to converge to the exact range. Bounds based on interval arithmetic suffer from irreducible conservatism when the requirement functions have repeated uncertain parameters. As compared to methods where the search for critical epistemic values is carried out using nonlinear programming or sampling, the methods developed here eliminate the possibility of under-predicting the range of the statistic of interest; i.e., the search for the extrema of non-convex functions via nonlinear optimization may converge to non-global optima, thereby producing a UQ assessment that under-predicts the range of the statistic of interest. Furthermore, this framework enables the consideration of uncertainty models comprised of arbitrary and possibly dependent aleatory variables, as well as the accommodation for changes in such a model with a small amount of computational effort, e.g., evaluating the change in the failure probability that stems from changing the random variables that prescribe the aleatory variables. On the down side, as with all the UQ methods listed above, “the curse of dimensionality” restricts the applicability of the proposed strategies to systems with a moderate number of uncertainties.

This article is organized as follows. Basic concepts and notions are introduced in Sections 2 and 3. Section 4 presents strategies for UQ in the presence of aleatory uncertainties. Section 5 presents developments for UQ in the presence of epistemic and aleatory uncertainties. Low-dimensional, easily reproducible examples are used to facilitate the presentation of the results and benchmark the scope of framework. Finally, a few concluding remarks are made.

2 Basic Concepts and Notions

In a system that depends on the uncertain parameter $\boldsymbol{p}$, response metrics are real-valued functions defined on a *master domain* $\mathcal{D} \subseteq \mathbb{R}^s$. These metrics include performance functions and requirements functions. A performance function, such as the cost of a system, is a metric used to evaluate a system's performance. This function will be denoted by $g(\boldsymbol{p})$, where $g : \mathcal{D} \to \mathbb{R}$. On the other hand, a reliability requirement is an admissible range of variation of a performance metric, e.g., cost not exceeding the available budget. The functions prescribing reliability requirements will be called requirements functions. These requirements are prescribed by the vector inequality[1] $\boldsymbol{g}(\boldsymbol{p}) < \boldsymbol{0}$, where $\boldsymbol{g} : \mathcal{D} \to \mathbb{R}^v$.

The *failure domain*, denoted as $\mathcal{F} \subset \mathbb{R}^s$, is comprised of the uncertain parameter realizations that fail to satisfy at least one of the reliability requirements. Specifically, the failure domain is given by

$$\mathcal{F} = \bigcup_{i=1}^{v} \{\boldsymbol{p} : \boldsymbol{g}_i(\boldsymbol{p}) \geq 0\} = \{\boldsymbol{p} : w(\boldsymbol{p}) \geq 0\}, \tag{1}$$

where $w(\boldsymbol{p}) = \max_{i \leq v}\{\boldsymbol{g}_i(\boldsymbol{p})\}$ is the *worst-case requirement function*. When the components of $\boldsymbol{g}$ are polynomials, w is in general piecewise polynomial. The *safe domain*, given by $\mathcal{S} = C(\mathcal{F})$, where $C(\cdot)$ denotes the *complement* set operator over the universal set $\mathcal{D}$, e.g., $\mathcal{S} = \mathcal{D} \setminus \mathcal{F}$, consists of the parameter realizations satisfying all the design requirements.

Techniques for bounding $\mathcal{F}$ and $\mathcal{S}$ will be presented below. The resulting bounding sets are comprised of hyper-rectangles. The *hyper-rectangle* $\mathcal{R} \in \mathbb{R}^s$, whose "lower left" and "upper right" corners are at $\boldsymbol{x}$ and $\boldsymbol{y}$ with $\boldsymbol{x} < \boldsymbol{y}$, is given by

$$\mathcal{R}(\boldsymbol{x}, \boldsymbol{y}) = \{\boldsymbol{p} : \boldsymbol{x} < \boldsymbol{p} \leq \boldsymbol{y}\} = (\boldsymbol{x}_1, \boldsymbol{y}_1] \times (\boldsymbol{x}_2, \boldsymbol{y}_2] \times \cdots \times (\boldsymbol{x}_s, \boldsymbol{y}_s], \tag{2}$$

where the latter expression is the Cartesian product of intervals which exclude the left endpoint and include the right. Hyper-rectangles can be subdivided into smaller hyper-rectangles without overlap, so that each point of the original hyper-rectangle falls into exactly one of the subdividing hyper-rectangles. Under these circumstances, the larger hyper-rectangle is said to have been *partitioned* or *subdivided* into smaller hyper-rectangles. If $\rho(\mathcal{R}) = \{\mathcal{R}_1, \ldots, \mathcal{R}_t\}$ is a pairwise disjoint collection of hyper-rectangles where $\mathcal{R} = \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_j$, then ρ is a partition of $\mathcal{R}$. Multiple subdivision schemes are possible. For instance, a bisection-based subdivision of $\mathcal{R}$ in the ith direction, $i \leq s$, is given by

$$\rho(\mathcal{R}(\boldsymbol{x}, \boldsymbol{y})) = \{\mathcal{R}(\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{w}), \mathcal{R}(\boldsymbol{y} - \boldsymbol{w}, \boldsymbol{y})\}, \tag{3}$$

where $\boldsymbol{w} = [\boldsymbol{y}_1 - \boldsymbol{x}_1, \ldots, (\boldsymbol{y}_i - \boldsymbol{x}_i)/2, \ldots, \boldsymbol{y}_s - \boldsymbol{x}_s]$. Throughout this manuscript the input to ρ is either a single hyper-rectangle or a list of several of them. In the latter case, each sub-rectangle comprising the input set will be bisected using Equation (3) where the bisecting direction is a maximal side of $\mathcal{R}$.

The uncertainties considered in this article are classified as either epistemic or aleatory. Denote by $\boldsymbol{e} \in \mathbb{R}^e$ a sub-vector of $\boldsymbol{p}$ containing the epistemic variables, and by $\boldsymbol{a} \in \mathbb{R}^a$ a sub-vector of $\boldsymbol{p}$ containing the aleatory variables such that $a + e = s$. Without loss of generality, we will perform this analysis under the assumption that $\boldsymbol{p} = \langle \boldsymbol{a}, \boldsymbol{e} \rangle$ and $a > 0$.

[1]Throughout this article, it is assumed that vector inequalities hold component-wise, super-indices denote a particular vector or set, and sub-indices refer to vector components; e.g., $\boldsymbol{p}_i^{(j)}$ is the ith component of the vector $\boldsymbol{p}^{(j)}$.

In this article, we will use probabilistic models for $\boldsymbol{a}$ and non-probabilistic ones for $\boldsymbol{e}$. Epistemic variables will be modeled by a set. This set, called the *support set* of the epistemic variables, will be denoted as $\Delta_{\boldsymbol{e}} \subseteq \mathbb{R}^e$. Each epistemic variable is modeled by providing an interval within which its value lies. Therefore, $\Delta_{\boldsymbol{e}}$ is the Cartesian product of these intervals. Aleatory parameters will be modeled as random variables. A probabilistic uncertainty model assigns a measure of probability to each member of the support set. This model is fully prescribed by the joint PDF $f(\boldsymbol{a}) : \Delta_{\boldsymbol{a}} \subseteq \mathbb{R}^a \to \mathbb{R}$, or equivalently, by the CDF $F(\boldsymbol{a}) : \Delta_{\boldsymbol{a}} \subseteq \mathbb{R}^a \to [0, 1]$. Note that $\Delta_{\boldsymbol{a}} \times \Delta_{\boldsymbol{e}} \subseteq \mathcal{D}_{\boldsymbol{a}} \times \mathcal{D}_{\boldsymbol{e}} = \mathcal{D}$. If $\text{Proj}_{\boldsymbol{a}}(\mathcal{X})$ denotes the projection of $\mathcal{X} \subseteq \mathbb{R}^s$ onto the aleatory subspace and $\text{Proj}_{\boldsymbol{e}}(\mathcal{X})$ denotes the projection of the same set onto the epistemic subspace, $\mathcal{D}_{\boldsymbol{e}} = \text{Proj}_{\boldsymbol{e}}(\mathcal{D})$ and $\mathcal{D}_{\boldsymbol{a}} = \text{Proj}_{\boldsymbol{a}}(\mathcal{D})$.

When epistemic and aleatory uncertainties are present, the failure domain can be described as

$$\mathcal{F} = \bigcup_{\boldsymbol{e} \in \Delta_{\boldsymbol{e}}} \mathcal{F}_a(\boldsymbol{e}), \tag{4}$$

where $\mathcal{F}_{\boldsymbol{a}}(\boldsymbol{e}) \triangleq \{\boldsymbol{p} = \langle \boldsymbol{a}, \boldsymbol{e} \rangle : \boldsymbol{g}(\boldsymbol{p}) > \boldsymbol{0}\}$, is a set value function of the epistemic variable. The mean, variance and failure probability associated with the uncertainty model of the aleatory variables are

$$E[g(\boldsymbol{p})] = \int_{\Delta_{\boldsymbol{a}}} g(\boldsymbol{p}) f(\boldsymbol{a}) d\boldsymbol{a}, \tag{5}$$

$$V[g(\boldsymbol{p})] = E[g^2(\boldsymbol{p})] - E[g(\boldsymbol{p})]^2, \tag{6}$$

$$P[\mathcal{F}_{\boldsymbol{a}}] = \int_{\mathcal{F}_{\boldsymbol{a}}} f(\boldsymbol{a}) d\boldsymbol{a}. \tag{7}$$

When all uncertain parameters are aleatory, these statistics are constants. Conversely, when there are both epistemic and aleatory variables, these statistics are random processes parametrized by the value of the epistemic variable. The subscript of $\mathcal{F}_{\boldsymbol{a}}$ will be omitted hereafter since its meaning can be easily inferred by context.

3 Bernstein Expansion

The image of a hyper-rectangle when mapped by a multivariable polynomial is a bounded interval. By expanding that polynomial using the Bernstein basis over the rectangle, rigorous bounds to such an interval can be calculated by mere algebraic manipulations. Bernstein polynomials [2, 12, 6] will be used for determining if a hyper-rectangle $\mathcal{R}$ is fully contained in the failure/safe domain or not. The outcome of the set containment test presented below depends exclusively on how much refinement of $\mathcal{R}$ the analyst is willing to perform. The refinement of $\mathcal{R}$ is determined by the number and size of sub-rectangles in a partition of $\mathcal{R}$. Better refinements can always be used to render the set containment tests conclusive. The mathematical foundation of this approach is presented next.

The Bernstein expansion first requires mapping $\mathcal{R}$ into the unit hyper cube. Let $\mathcal{R} \subseteq \mathcal{D}$ be an arbitrary hyper-rectangle in the master domain. Denote by $\boldsymbol{u} = U(\boldsymbol{p})$ the affine transformation that maps the hyper-rectangle $\mathcal{R}$ onto the unit hyper-cube $\mathcal{U} = \mathcal{R}(\boldsymbol{0}, \boldsymbol{1})$.

Let g be an arbitrary polynomial in $\boldsymbol{p}$. Then $h(\boldsymbol{u}) \triangleq g(U^{-1}(\boldsymbol{u}))$ is a polynomial in $\mathcal{U}$. Note that the extrema of g on $\mathcal{R}$ are identical to the extrema of h in $\mathcal{U}$.

For simplicity in the presentation we first consider a univariate polynomial. Since $\boldsymbol{p}$, $\boldsymbol{u}$ and $\boldsymbol{g}$ are scalars here, we will represent them as p, u and g without the bold font. The transformation of g into the unit cube leads to

$$h(u) = \sum_{i=0}^{n} a_i u^i, \tag{8}$$

whose Bernstein expansion is given by

$$h(u) = \sum_{i=0}^{n} b_i(\mathcal{R}, g) B_i^n(u), \tag{9}$$

where

$$B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i}, \tag{10}$$

is the ith Bernstein polynomial of degree n (i.e., an element of the basis) and

$$b_i(\mathcal{R}, g) = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{n}{j}} a_j, \tag{11}$$

is the ith Bernstein coefficient. Some fundamental properties of this basis are $\sum B_i^n(u) = 1$ (partition of unity), $0 \leq B_i^n(u) \leq 1$ for $0 \leq u \leq 1$ (boundedness), and $B_i^n(u) = B_{n-i}^n(1-u) > 0$ (symmetry). Some of the Bernstein coefficients assume the same value taken by the polynomial at the vertices of the hyper-rectangular domain. This leads to the "free function evaluation property"[2] at the interval's endpoints, which is given by

$$h(0) = b_0(\mathcal{R}, g), \tag{12}$$

$$h(1) = b_n(\mathcal{R}, g). \tag{13}$$

The range enclosing property is described next. Suppose $\mathcal{R}$ is a hyper-rectangle and $\{b_i(\mathcal{R}, g) : 0 \leq i \leq n\}$ are the Bernstein coefficients of g on $\mathcal{R}$. The range enclosing property dictates that, for $p \in \mathcal{R}$, $\min_{0 \leq i \leq n} b_i(\mathcal{R}, g) \leq g(p) \leq \max_{0 \leq i \leq n} b_i(\mathcal{R}, g)$. Tighter bounds are obtained if $\mathcal{R}$ is subdivided. In particular, if $\rho(\mathcal{R}) = \{\mathcal{R}_1, \ldots, \mathcal{R}_t\}$, we have that, for all $p \in \mathcal{R}$,

$$\underline{g}(p, \rho) \leq g(p) \leq \overline{g}(p, \rho), \tag{14}$$

where

$$\underline{g}(p, \rho) = \sum_{j=1}^{t} \min_{0 \leq i \leq n} \{b_i(\mathcal{R}_j, g)\} \, I(p\,; \mathcal{R}_j), \tag{15}$$

$$\overline{g}(p, \rho) = \sum_{j=1}^{t} \max_{0 \leq i \leq n} \{b_i(\mathcal{R}_j, g)\} \, I(p\,; \mathcal{R}_j), \tag{16}$$

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[2] This property is also referred to as "end points interpolation" in the literature [2]. We chose the above name to emphasize that the selected Bernstein coefficients *exactly* match the values of the polynomial at the endpoints of the interval.

where $I(\cdot\,;\mathcal{R}_j)$ is the *indicator function* of $\mathcal{R}_j$. This function is defined as follows. $I(p\,;\mathcal{R}_j)=1$ if $p\in\mathcal{R}_j$, $I(p\,;\mathcal{R}_j)=0$, otherwise. Each of the bounding functions $\underline{g}$ and $\overline{g}$ is constant on each set $\mathcal{R}_j$. We call $\underline{g}$ and $\overline{g}$ the *Bernstein lower* and *upper function bounds*, respectively, of g.

The multivariate polynomial case is considered next. Define the multi-index $\boldsymbol{i}$ to be a vector of non-negative integers of length s. The monomial $\boldsymbol{u}_1^{\boldsymbol{i}_1}\boldsymbol{u}_2^{\boldsymbol{i}_2}\cdots\boldsymbol{u}_s^{\boldsymbol{i}_s}$ is abbreviated as $\boldsymbol{u}^{\boldsymbol{i}}$. A s-variate polynomial can be represented as

$$h(\boldsymbol{u})=\sum_{\boldsymbol{0}\le\boldsymbol{i}\le\boldsymbol{n}} a_{\boldsymbol{i}}\boldsymbol{u}^{\boldsymbol{i}}, \tag{17}$$

where $\boldsymbol{u}\in\mathcal{U}$. The Bernstein expansion of (17) is given by

$$h(\boldsymbol{u})=\sum_{\boldsymbol{0}\le\boldsymbol{i}\le\boldsymbol{n}} b_{\boldsymbol{i}}(\mathcal{R},g)B_{\boldsymbol{i}}^{\boldsymbol{n}}(\boldsymbol{u}), \tag{18}$$

where

$$B_{\boldsymbol{i}}^{\boldsymbol{n}}(\boldsymbol{u})=B_{\boldsymbol{i}_1}^{\boldsymbol{n}_1}(\boldsymbol{u}_1)\cdots B_{\boldsymbol{i}_s}^{\boldsymbol{n}_s}(\boldsymbol{u}_s) \tag{19}$$

is the $\boldsymbol{i}$th Bernstein polynomial of degree $\boldsymbol{n}$ and

$$b_{\boldsymbol{i}}(\mathcal{R},g)=\sum_{\boldsymbol{j}\le\boldsymbol{i}}\left(\prod_{k=1}^{p}\frac{\binom{\boldsymbol{i}_k}{\boldsymbol{j}_k}}{\binom{\boldsymbol{n}_k}{\boldsymbol{j}_k}}\right)a_{\boldsymbol{j}}, \tag{20}$$

for $\boldsymbol{0}\le\boldsymbol{i}\le\boldsymbol{n}$, is the $\boldsymbol{i}$th Bernstein coefficient. In this setting the free function evaluation property is

$$h\left(\langle\boldsymbol{i}_1/\boldsymbol{n}_1,\ldots,\boldsymbol{i}_s/\boldsymbol{n}_s\rangle\right)=b_{\boldsymbol{i}}(\mathcal{R},g), \tag{21}$$

where $\boldsymbol{i}$ is an element of $\{0,\boldsymbol{n}_1\}\times\cdots\times\{0,\boldsymbol{n}_s\}$. The range enclosing property is as follows: suppose $\mathcal{R}$ is a hyper-rectangle and $\{b_{\boldsymbol{i}}(\mathcal{R},g):\boldsymbol{0}\le\boldsymbol{i}\le\boldsymbol{n}\}$ are the Bernstein coefficients of g on $\mathcal{R}$. The range enclosing property dictates that, for $\boldsymbol{p}\in\mathcal{R}$, $\min_{\boldsymbol{0}\le\boldsymbol{i}\le\boldsymbol{n}} b_{\boldsymbol{i}}(\mathcal{R},g)\le g(\boldsymbol{p})\le\max_{\boldsymbol{0}\le\boldsymbol{i}\le\boldsymbol{n}} b_{\boldsymbol{i}}(\mathcal{R},g)$. As before, tighter bounds are obtained if Bernstein expansions over partitions of $\mathcal{R}$ are calculated. Specifically,

$$\underline{g}(\boldsymbol{p},\rho)\le g(\boldsymbol{p})\le\overline{g}(\boldsymbol{p},\rho), \tag{22}$$

for all $\boldsymbol{p}\in\mathcal{R}$ where

$$\underline{g}(\boldsymbol{p},\rho)=\sum_{j=1}^{t}\min_{\boldsymbol{0}\le\boldsymbol{i}\le\boldsymbol{n}}\{b_{\boldsymbol{i}}(\mathcal{R}_j,g)\}\,I(\boldsymbol{p}\,;\mathcal{R}_j), \tag{23}$$

$$\overline{g}(\boldsymbol{p},\rho)=\sum_{j=1}^{t}\max_{\boldsymbol{0}\le\boldsymbol{i}\le\boldsymbol{n}}\{b_{\boldsymbol{i}}(\mathcal{R}_j,g)\}\,I(\boldsymbol{p}\,;\mathcal{R}_j). \tag{24}$$

As before, $\underline{g}$ and $\overline{g}$ are constant on each set $\mathcal{R}_j$. If $\boldsymbol{p}^{(j)}$ is the center of $\mathcal{R}_j$, these constant values are equal to $\underline{g}(\boldsymbol{p}^{(j)})$ and $\overline{g}(\boldsymbol{p}^{(j)})$. Since g is a continuous function on a connected set, we have

$$\text{Range}(g)=\left[\min_{\boldsymbol{p}\in\mathcal{R}} g(\boldsymbol{p}),\max_{\boldsymbol{p}\in\mathcal{R}} g(\boldsymbol{p})\right]\subseteq\left[\min_{\boldsymbol{p}\in\mathcal{R}}\underline{g}(\boldsymbol{p}),\max_{\boldsymbol{p}\in\mathcal{R}}\overline{g}(\boldsymbol{p})\right]. \tag{25}$$

If the partition of $\mathcal{R}$ is successively refined so that the maximum volume of the subsets approaches zero, the endpoints of the bounding interval converge to the global extrema of g.

A strategy for bounding the values of $\boldsymbol{p}$ where the global extrema of a polynomial occur is presented next. These values will be referred to as *critical* since they attain the extreme values of the function within the domain. Let ρ be a partition of $\mathcal{R}$ and let $\boldsymbol{p}^{\min}$ and $\boldsymbol{p}^{\max}$ denote the point sets where $\boldsymbol{g}$ achieves its global minimum and global maximum, respectively, over the hyper-rectangle. The lower and upper bounding functions in (23-24) can be used to calculate supersets of $\boldsymbol{p}^{\min}$ and $\boldsymbol{p}^{\max}$. These supersets, denoted hereafter as $\mathcal{P}^{\min}$ and $\mathcal{P}^{\max}$, are given by

$$\mathcal{P}^{\min} = \bigcup_j \left\{ \mathcal{R}_j : \underline{g}(\boldsymbol{p}^{(j)}) \leq \min_{\boldsymbol{p}\in\mathcal{R}} \overline{g}(\boldsymbol{p}) \right\} \supset \boldsymbol{p}^{\min}, \tag{26}$$

$$\mathcal{P}^{\max} = \bigcup_j \left\{ \mathcal{R}_j : \overline{g}(\boldsymbol{p}^{(j)}) \geq \max_{\boldsymbol{p}\in\mathcal{R}} \underline{g}(\boldsymbol{p}) \right\} \supset \boldsymbol{p}^{\max}. \tag{27}$$

Therefore, the superset containing the point where $g(\boldsymbol{p})$ attains its global minima is comprised of all rectangles where the minima of the upper bounding function $\overline{g}$ is larger than the lower bounding function $\underline{g}$. Because the optimizations at the right-hand side of the inequalities in (26) and (27) entail finding the extrema of piecewise constant functions over a finite partition of $\mathcal{R}$, they can be calculated exactly in a finite number of steps[3]. The supersets $\mathcal{P}^{\min}$ and $\mathcal{P}^{\max}$ approach $\boldsymbol{p}^{\min}$ and $\boldsymbol{p}^{\max}$ respectively as the partition of $\mathcal{R}$ becomes finer. The mathematical foundation supporting the polynomial bounds guarantees that $\mathcal{P}^{\min}$ and $\mathcal{P}^{\max}$ will contain all the critical epistemic points regardless of their number and their location. Equations (25) yield lower and upper bounds of the extrema of $g(\boldsymbol{p})$ while Equations (26) and (27) are supersets of the epistemic realizations where such extrema occur.

The preceding analysis of a single polynomial function can be applied on a component by component basis to the vector $\boldsymbol{g}$ prescribing the failure domain. For a given partition of $\mathcal{R}$, the preceding analysis is applied to each component of $\boldsymbol{g}$ to determine the bounding functions in (22). These functions can be used to calculate bounds of the worst-case requirement function in (1). The lower and upper bounding functions of w, denoted as $\underline{w}$ and $\overline{w}$, are given by

$$\underline{w}(\boldsymbol{p}, \rho) = \sum_{j=1}^{t} \left(\max_{i \leq v} \underline{\boldsymbol{g}_i}(\boldsymbol{p}^{(j)}) \right) I(\boldsymbol{p}, \mathcal{R}_j), \tag{28}$$

$$\overline{w}(\boldsymbol{p}, \rho) = \sum_{j=1}^{t} \left(\max_{i \leq v} \overline{\boldsymbol{g}_i}(\boldsymbol{p}^{(j)}) \right) I(\boldsymbol{p}, \mathcal{R}_j), \tag{29}$$

where $\rho(\mathcal{R}) = \{\mathcal{R}_1, \ldots, \mathcal{R}_t\}$. As before, $\underline{w}$ and $\overline{w}$ are piecewise constant on each member of the partition. The following theorems, which make use of these bounding functions,

[3]Smaller supersets $\mathcal{P}^{\min}$ and $\mathcal{P}^{\max}$ result from replacing the arguments of the min and max operators in Equations (26) and (27) with a vector comprised of all free function evaluations, i.e., Equation (21), for all the elements of the partition. This practice, however, cannot be carried out when the metric of interest is a statistic of $g(\boldsymbol{a})$.

enable determining whether a set $\mathcal{Z} \in \mathbb{R}^s$ is fully contained (or not) in the safe domain $\mathcal{S}$ or failure domain $\mathcal{F}$.

Theorem 1 (Set Containment in the Safe Domain)**.** *Let $w(\boldsymbol{p})$ be the worst-case requirement function defined in (1) and $\rho(\mathcal{H}) = \{\mathcal{R}_1, \ldots, \mathcal{R}_t\}$ be a partition of the bounding set $\mathcal{H}$ satisfying $\mathcal{Z} \subseteq \mathcal{H}$. The set containment condition $\mathcal{Z} \subseteq \mathcal{S}$ holds if*

$$\max_{\boldsymbol{p}} \overline{w}(\boldsymbol{p}, \rho) < 0. \tag{30}$$

Furthermore, $\mathcal{Z} \not\subseteq \mathcal{S}$ if there exists a $k \leq v$, a $j \leq t$, and a multi-index $\boldsymbol{i} \in \{0, \boldsymbol{n}_1\} \times \cdots \times \{0, \boldsymbol{n}_s\}$ such that

$$b_{\boldsymbol{i}}(\mathcal{R}_j, \boldsymbol{g}_k) \geq 0. \tag{31}$$

While Formula (30) results from using $\underline{w}(\boldsymbol{p}) \leq w(\boldsymbol{p})$ for all $\boldsymbol{p} \in \mathcal{H}$ in Equation (1), Formula (31) results from applying the free function evaluation property (21).

Theorem 2 (Set Containment in the Failure Domain)**.** *Let $w(\boldsymbol{p})$ be the worst-case requirement function defined in (1) and $\rho(\mathcal{H}) = \{\mathcal{R}_1, \ldots, \mathcal{R}_t\}$ be a partition of the bounding set $\mathcal{H}$ satisfying $\mathcal{Z} \subseteq \mathcal{H}$. The set containment condition $\mathcal{Z} \subseteq \mathcal{F}$ holds if*

$$\min_{\boldsymbol{p}} \underline{w}(\boldsymbol{p}, \rho) \geq 0. \tag{32}$$

Furthermore, $\mathcal{Z} \not\subseteq \mathcal{F}$ if there exists a $k \leq v$, a $j \leq t$, and a multi-index $\boldsymbol{i} \in \{0, \boldsymbol{n}_1\} \times \cdots \times \{0, \boldsymbol{n}_s\}$ such that

$$b_{\boldsymbol{i}}(\mathcal{R}_j, \boldsymbol{g}_k) < 0. \tag{33}$$

While Formula (32) results from using $w(\boldsymbol{p}) \leq \overline{w}(\boldsymbol{p})$ for all $\boldsymbol{p} \in \mathcal{H}$ in Equation (1), Formula (33) results from applying the free function evaluation property (21).

The implicit formulation for calculating Bernstein coefficients proposed in [10] was adopted. This formulation is much more efficient than using Equation (11). Further efficiency can be realized by using the subdividing logic in Equation (3) along with the algorithms of [6] that relate the Bernstein coefficients of a hyper-rectangle with those of its subsets.

4 UQ for Aleatory Uncertainties

In this section we develop strategies for bounding failure probabilities, means, and variances of piecewise polynomial response metrics that depend exclusively on aleatory variables.

4.1 Reliability Analysis

In the context of aleatory uncertainties only, by reliability analysis we refer to the quantification or bounding of the probability of failure. The key developments in this section are the calculation of inner and outer bounding sets to the failure domain and the calculation of their probabilities. These sets are comprised by hyper-rectangles whose membership into such domains is established using Theorems 1 and 2 above.

This section presents an algorithm to generate and sequentially expand subsets of the failure and safe domains. These sets are unions of disjoint hyper-rectangles chosen from a partition Q of $\mathcal{D}$. Let $\mathcal{F}^{\text{sub}}$ and $\mathcal{S}^{\text{sub}}$ denote subsets (i.e., inner approximations) of the failure and safe domains formed from selected elements of Q. Note that $\emptyset \subseteq \mathcal{F}^{\text{sub}} \subseteq \mathcal{F} \subseteq C(\mathcal{S}^{\text{sub}}) \subseteq \mathcal{D}$ and that the failure domain boundary, denoted as $\partial\mathcal{F}$, lies in the region between the interiors of $\mathcal{F}^{\text{sub}}$ and $\mathcal{S}^{\text{sub}}$.

The sequences of inner bounding sets $\{\mathcal{S}_1^{\text{sub}}, \mathcal{S}_2^{\text{sub}}, \ldots\}$ and $\{\mathcal{F}_1^{\text{sub}}, \mathcal{F}_2^{\text{sub}}, \ldots\}$ are generated by the algorithm below. These sequences are made to converge to the domain being bounded. In particular, the algorithm iteratively generates indexed partitions Q_i of $\mathcal{D}$ and indexed sets $\mathcal{S}_i^{\text{sub}}$, $\mathcal{F}_i^{\text{sub}}$ and Λ_i which are unions of hyper-rectangles from Q_i, where $\mathcal{S}_i^{\text{sub}}$ is an inner approximation to the safe domain, $\mathcal{F}_i^{\text{sub}}$ is an inner approximation to the failure domain, and Λ_i is a region comprised by the rectangles of Q_i that are not in $\mathcal{S}_i^{\text{sub}}$ or $\mathcal{F}_i^{\text{sub}}$. Note that while Q_i is a list of hyper-rectangles, $\mathcal{S}_i^{\text{sub}}$, $\mathcal{F}_i^{\text{sub}}$ and Λ_i are sets comprised by the union of some of these rectangles. The algorithm proceeds by successively selecting each of the component hyper-rectangles $\mathcal{R}$ of Λ_i. Then, Theorems 1 and 2 are used to determine if $\mathcal{R}$ is contained by the failure domain or the safe domain. If the tests determine that $\mathcal{R} \subseteq \mathcal{S}$ or $\mathcal{R} \subseteq \mathcal{F}$, then $\mathcal{R}$ is removed from Λ_i and added to $\mathcal{S}_i^{\text{sub}}$ or $\mathcal{F}_i^{\text{sub}}$, respectively. If neither of these determinations can be made, $\mathcal{R}$ is subdivided, and the resulting sub-rectangles replace $\mathcal{R}$ in the partition. The algorithm terminates when the volume of Λ_i is sufficiently small. The algorithmic representation of this procedure is as follows.

Algorithm 1: Let the inequality constraint $\boldsymbol{g}(\boldsymbol{p}) < \boldsymbol{0}$ defined over $\boldsymbol{p} \in \mathcal{D}$ prescribe the system requirements and $F(\boldsymbol{a})$ be the uncertainty model of $\boldsymbol{p}$ in $\Delta_{\boldsymbol{a}} \subseteq \mathcal{D}$. Set $i = 1$, $Q_1 = \{\mathcal{D}\}$, $\Lambda_1 = \mathcal{D}$, $\mathcal{F}_1^{\text{sub}} = \emptyset$, and $\mathcal{S}_1^{\text{sub}} = \emptyset$. Pick a convergence criterion $\epsilon > 0$.

1. Let L contain the elements of Q_i comprising Λ_i.

2. Apply Theorems 1 and 2 to each hyper-rectangle in L without partitioning it, to determine which elements of L are contained in the safe domain and which ones are contained in the failure domain. Denote by U the list of elements contained by the safe domain, by V the list of elements contained by the failure domain, and by W the list of elements that are not in U nor V. Furthermore, let $\mathcal{U}$ and $\mathcal{V}$ be the union of the elements in U and V respectively.

3. Make $\mathcal{S}_{i+1}^{\text{sub}} = \mathcal{S}_i^{\text{sub}} \cup \mathcal{U}$; $\mathcal{F}_{i+1}^{\text{sub}} = \mathcal{F}_i^{\text{sub}} \cup \mathcal{V}$; and $\Lambda_{i+1} = \Lambda_i \setminus (\mathcal{U} \cup \mathcal{V})$.

4. If $\text{Vol}(\Lambda_{i+1}) < \epsilon$ stop. Otherwise, make $Q_{i+1} = (Q_i \setminus W) \cup \rho(W)$, increase i by one, and go to Step (1).

As the number of iterations increases, $\mathcal{S}_i^{\text{sub}}$ and $\mathcal{F}_i^{\text{sub}}$ approach the safe and failure domain. Notice that the subdividing algorithm only partitions boxes whose containment in $\mathcal{S}$ or in $\mathcal{F}$ has not been established. The partitioning logic, prescribed by ρ, can be set arbitrarily. Further notice that the closure of Λ_i not only covers the boundary of $\mathcal{F}$ but also approaches that boundary more and more closely as i increases.

If a particular CDF defined over $\boldsymbol{p} \in \Delta \subseteq \mathcal{D}$ is prescribed, bounds to the failure probability, given by

$$P[\mathcal{F}_i^{\rm sub}] \leq P[\mathcal{F}] \leq 1 - P[\mathcal{S}_i^{\rm sub}], \tag{34}$$

can be readily calculated. These expressions are evaluated by adding up the probability of the rectangles comprising the bounding sets. Note that $P[\mathcal{F}_i^{\rm sub}]$ and $P[\mathcal{S}_i^{\rm sub}]$ are monotonically increasing functions of i; while $P[\Lambda_i]$ is a monotonically decreasing function of i. The choice of ρ made at Step (4) implies that these bounds converge to $P[\mathcal{F}]$ as i increases. Therefore, all the conservatism in the bounds is reducible by additional computational effort. Further notice that once the bounding sets are available, failure probability bounds corresponding to any distribution supported in a subset of $\mathcal{D}$ can be calculated with minimal computational effort.

The values taken by $\underline{w}$ and $\overline{w}$ over the members of the partition ρ, which were required to compute the bounding sets $\mathcal{S}^{\rm sub}$ and $\mathcal{F}^{\rm sub}$, can be used to bound the CDF of the worst-case requirement function w. In particular, if $\rho(\mathcal{D}) = Q = \{\mathcal{R}_1, \ldots, \mathcal{R}_t\}$, F_w is bounded by

$$F_{\overline{w}}(x) = \sum_{j \in \overline{j}(x)} P[\mathcal{R}_j] \leq F_w(x) \leq \sum_{j \in \underline{j}(x)} P[\mathcal{R}_j] = F_{\underline{w}}(x), \tag{35}$$

where $\overline{j}(x) = \{j : 1 \leq j \leq t, \overline{w}(\boldsymbol{p}^{(j)}) \leq x\}$ and $\underline{j}(x) = \{j : 1 \leq j \leq t, \underline{w}(\boldsymbol{p}^{(j)}) \leq x\}$ for all x in the range of w. A few manipulations lead to

$$1 - F_{\underline{w}}(0) \leq P[\mathcal{F}] \leq 1 - F_{\overline{w}}(0). \tag{36}$$

This expression is equivalent to (34). The evaluation of (35) for each member of the partition sequence Q_i resulting from the algorithm above yields a sequence of CDF bounds. These CDFs become tighter near $w = 0$ as i increases.

Example 1: Consider the requirement functions

$$\boldsymbol{g}_1 = \boldsymbol{p}_1^2\boldsymbol{p}_2^4 + \boldsymbol{p}_1^4\boldsymbol{p}_2^2 - 3\boldsymbol{p}_1^2\boldsymbol{p}_2^2 - \boldsymbol{p}_1\boldsymbol{p}_2 + \frac{\boldsymbol{p}_1^6 + \boldsymbol{p}_2^6}{200} - \frac{7}{100}, \tag{37}$$

$$\boldsymbol{g}_2 = -\frac{\boldsymbol{p}_1^2\boldsymbol{p}_2^4}{2} - \boldsymbol{p}_1^4\boldsymbol{p}_2^2 + 3\boldsymbol{p}_1^2\boldsymbol{p}_2^2 + \frac{\boldsymbol{p}_1^5\boldsymbol{p}_2^3}{10} - \frac{9}{10}, \tag{38}$$

defined over the master domain $\mathcal{D} \in [-2, 2] \times [-2, 2]$. The parameters $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are aleatory uncertain parameters, i.e., $\boldsymbol{a} = \{\boldsymbol{p}_1, \boldsymbol{p}_2\}$ and $\boldsymbol{e} = \emptyset$, distributed according to the joint PDF

$$f_{\boldsymbol{p}_1\boldsymbol{p}_2}(\boldsymbol{p}_1, \boldsymbol{p}_2) = \frac{\cos^2(\boldsymbol{p}_1\boldsymbol{p}_2)}{8 + \mathrm{Si}(8)}, \tag{39}$$

where $\mathrm{Si}(\cdot)$ is the sine integral. Note that the uncertainty models of $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, shown in Figure 1, are strongly dependent. For instance, the conditional probability density function of $\boldsymbol{p}_1$ given $\boldsymbol{p}_2$ is uniform at $\boldsymbol{p}_2 = 0$, and it is tri-modal at $\boldsymbol{p}_2 = -2$. In this case, the probability of a hyper-rectangle can be calculated analytically.

Figure 2 shows the set approximations $\mathcal{F}_i^{\rm sub}$ and $\mathcal{S}_i^{\rm sub}$ for a fixed value of i as well as the failure domain boundary $\partial\mathcal{F}$. Boxes comprising $\mathcal{F}_i^{\rm sub}$ are colored in red, those comprising $\mathcal{S}_i^{\rm sub}$ are colored in green, and boxes comprising Λ_i are colored in white. Note

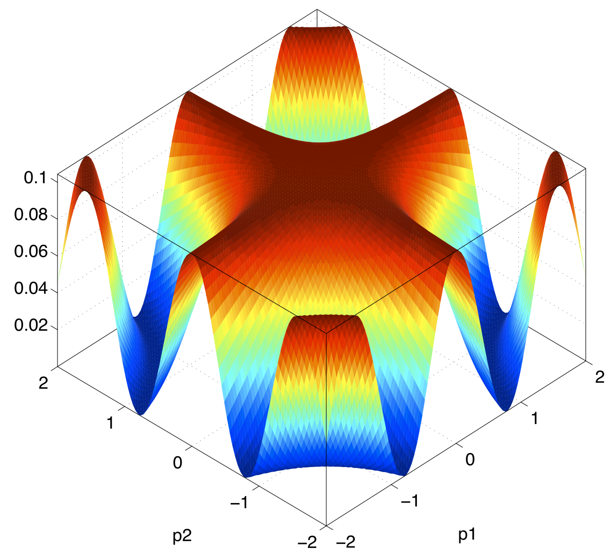

Figure 1: Joint PDF of aleatory uncertainties.

that Λ_i is a tight approximation of $\partial\mathcal{F}$. Further notice that the density of boxes per unit of area increases with the closeness to $\partial\mathcal{F}$.

The failure probability bounds in (34) corresponding to this partition are $0.492 \leq P[\mathcal{F}] \leq 0.571$. Note that this probability cannot be calculated using standard UQ methods, including sampling-based techniques. The difficulty stems from the inability to sample the joint PDF in (39) and transforming it to another probability space. Note that with the approximations $\mathcal{F}_i^{\text{sub}}$ and $\mathcal{S}_i^{\text{sub}}$ at hand, we can calculate bounds corresponding to any other joint PDF having a support set within $\mathcal{D}$ with minimal computational effort.

4.2 Performance Analysis

If M denotes the expected value or variance operators, the problem of interest is to calculate or bound $M[g(\boldsymbol{p})]$, where $\boldsymbol{p}$ is an aleatory variable distributed according to the CDF $F(\boldsymbol{a})$.

If the performance function g is a single polynomial in $\boldsymbol{p}$ and the aleatory parameters are independent random variables, moments of any order can be calculated exactly. When g is piecewise polynomial and $\boldsymbol{p}$ is comprised of arbitrarily distributed parameters, these moments cannot in general be calculated exactly. Lower and upper bounds of moments of any performance function that depends on aleatory variables within a bounded support set can be calculated from Equation (22). A technique for bounding $E[g(\boldsymbol{p})]$ and $V[g(\boldsymbol{p})]$,

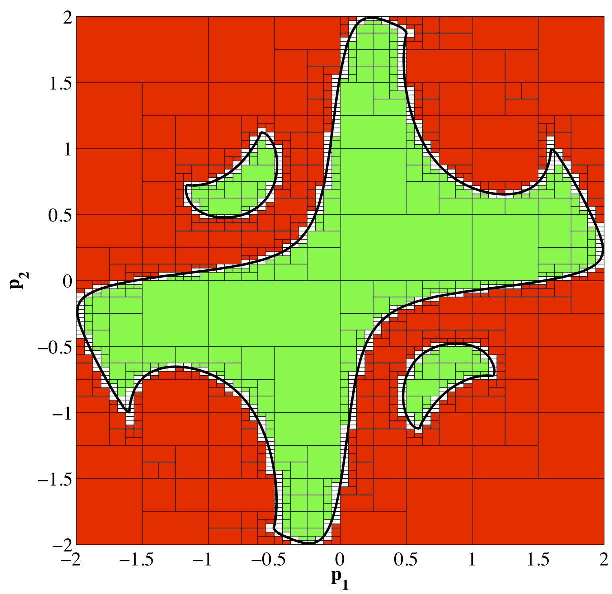

Figure 2: $\mathcal{F}^{\rm sub}$ (red) and $\mathcal{S}^{\rm sub}$ (green) and failure domain boundary (thick line).

applicable to the case mentioned above, is developed next.

In regard to the mean, the application of the expected value operator to Equation (22) yields

$$\sum_{j=1}^{t} \underline{g}(\boldsymbol{p}^{(j)})P[\mathcal{R}_j] = E[\underline{g}] \leq E[g] \leq E[\overline{g}] = \sum_{j=1}^{t} \overline{g}(\boldsymbol{p}^{(j)})P[\mathcal{R}_j], \tag{40}$$

where $\rho(\mathcal{D}) = \{\mathcal{R}_1, \ldots, \mathcal{R}_t\}$.

Bounds on the variance are considered next. Interval arithmetic will be used to simplify the notation. The starting point is the equation $V[g] = E[g^2] - E[g]^2$ and the interval inclusion $g \in [\underline{g}, \overline{g}]$. We recall the formulae for interval subtraction and interval squaring: $[a, b] - [c, d] \triangleq [a - d, b - c]$ and

$$[a, b]^2 \triangleq \begin{cases} [a^2, b^2] & \text{if } 0 < a, \\ [b^2, a^2] & \text{if } b < 0, \\ [0, \max\{a^2, b^2\}] & \text{otherwise.} \end{cases} \tag{41}$$

The bounds of $E[g^2]$, given by $E\left[[\underline{g},\overline{g}]^2\right] = [\mu,\nu]$, are

$$\mu = \sum_{j\in u} \underline{g}(\boldsymbol{p}^{(j)})^2 P[\mathcal{R}_j] + \sum_{j\in v} \overline{g}(\boldsymbol{p}^{(j)})^2 P[\mathcal{R}_j], \tag{42}$$

$$\nu = \sum_{j\in u\cup x} \overline{g}(\boldsymbol{p}^{(j)})^2 P[\mathcal{R}_j] + \sum_{j\in v\cup y} \underline{g}\boldsymbol{p}^{(j)})^2 P[\mathcal{R}_j], \tag{43}$$

where

$$u = \{1 \le j \le t : \underline{g}(\boldsymbol{p}^{(j)}) \ge 0\}, \tag{44}$$

$$v = \{1 \le j \le t : \overline{g}(\boldsymbol{p}^{(j)}) < 0\}, \tag{45}$$

$$x = \{1 \le j \le t : \underline{g}(\boldsymbol{p}^{(j)})\overline{g}(\boldsymbol{p}^{(j)}) < 0, |\underline{g}(\boldsymbol{p}^{(j)})| \le |\overline{g}(\boldsymbol{p}^{(j)})|)\}, \text{ and} \tag{46}$$

$$y = \{1 \le j \le t : j \notin (u \cup v \cup x)\}. \tag{47}$$

The bounds corresponding to $V[g] \in E\left[[\underline{g},\overline{g}]^2\right] - \left[E[\underline{g}], E[\overline{g}]\right]^2$ are given by

$$\max\{0,\alpha\} \le V[g] \le \beta, \tag{48}$$

where

$$\alpha = \begin{cases} \mu - E[\overline{g}]^2 & \text{if } E[\underline{g}] > 0, \\ \mu - E[\underline{g}]^2 & \text{if } E[\overline{g}] < 0, \\ \mu - \max\left\{E[\underline{g}]^2, E[\overline{g}]^2\right\} & \text{otherwise,} \end{cases} \tag{49}$$

$$\beta = \begin{cases} \nu - E[\underline{g}]^2 & \text{if } E[\underline{g}] > 0, \\ \nu - E[\overline{g}]^2 & \text{if } E[\overline{g}] < 0, \\ \nu & \text{otherwise .} \end{cases} \tag{50}$$

An algorithm to generate and sequentially refine outer bounds to the mean and variance of a performance function is presented next. In contrast to Algorithm 1, this algorithm does not generate bounding sets of the failure and safe domains.

Algorithm 2: Let $\boldsymbol{g}(\boldsymbol{p})$ be a performance function defined over $\boldsymbol{p} \in \mathcal{D}$ and $F(\boldsymbol{a})$ be the uncertainty model of $\boldsymbol{p}$ in $\Delta_{\boldsymbol{a}} \subseteq \mathcal{D}$. Set $i = 1$ and $Q_1 = \{\mathcal{D}\}$. Pick a selection criterion $0 < \eta < 1$ and a convergence criterion $\epsilon > 0$.

1. Calculate the bounding functions $\underline{g}(\boldsymbol{p})$ and $\overline{g}(\boldsymbol{p})$ over $\mathcal{D}$.
2. Calculate the limits of the bounding interval of M via Equations (40) or (48).
3. Find the elements of the partition Q_i satisfying[4] $\overline{g} - \underline{g} > \eta \max\left\{\overline{g} - \underline{g}\right\}$. Denote by L be the list of rectangles satisfying this condition.

[4]Alternatively, $E[\overline{g} - \underline{g} \,|\, \mathcal{R}] > \eta \max\left\{E[\overline{g} - \underline{g} \,|\, \mathcal{R}]\right\}$, where $\mathcal{R}$ is an element of Q_i, can also be used.

4. If $M[\overline{q} - \underline{g}] < \epsilon$ stop. Otherwise, make $Q_{i+1} = (Q_i \setminus L) \cup \rho(L)$, and go to Step (1).

As with the probability of failure, the bounds on the moments converge monotonically to their actual value as the partition becomes finer.

Example 2: For the same problem statement of Example 1, here we want to bound the expected value and variance of the worst-case requirement function w. This is a piecewise polynomial function whose discontinuities cannot be calculated in closed form. Figures 3 and 4 show the set approximations $\mathcal{F}_i^{\rm sub}$ and $\mathcal{S}_i^{\rm sub}$ for a fixed value of i corresponding to two qualitatively different partitions. Note that the set containment conditions, thus the coloring of the rectangles, are not required to calculate the bounding intervals. The partition used to generate Figure 3 only subdivides boxes where $\overline{w}(\boldsymbol{p}) - \underline{w}(\boldsymbol{p})$, the spread between the bounding functions, is in the top 50%. This partition leads to $3.21052 \le E[w] \le 4.71449$ and $80.97456 \le V[w] \le 145.11712$. In this case, the density of boxes per unit of area

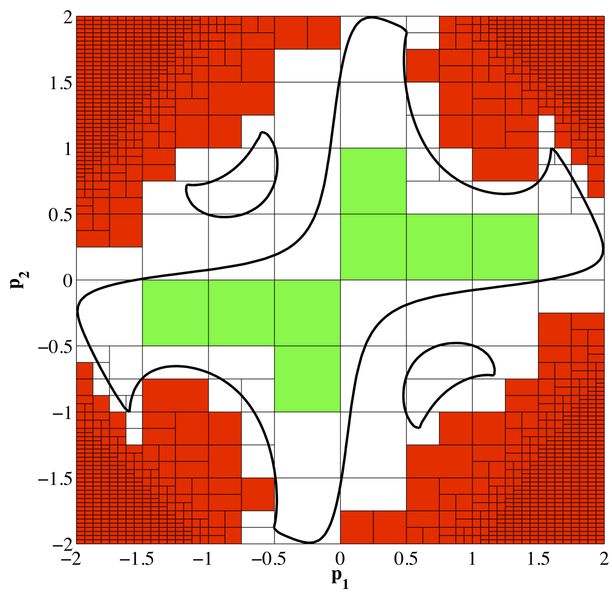

Figure 3: $\mathcal{F}^{\rm sub}$ (red) and $\mathcal{S}^{\rm sub}$ (green) and $\partial\mathcal{F}$ (thick line).

increases with the separation between $\underline{w}$ and $\overline{w}$. This separation is an indicator of the size of the range of w over a box. Consequently, the size of a box tends to be inversely proportional to the magnitude of the gradient of w within the box. The partition used to generate Figure 4 only subdivides boxes where $E[\overline{w}(\boldsymbol{p}) - \underline{w}(\boldsymbol{p}) \,|\, \mathcal{R}]$, where $\mathcal{R}$ is a member of the partition of $\mathcal{D}$, is in the top 50%. This yields $3.40700 \le E[w] \le 4.50007$ and

$82.9050 \le V[w] \le 143.5641$. These bounds are tighter than those from Figure 3 because the partition they are based on takes the uncertainty model into account. In this case, the size of a box tends to be inversely proportional to the magnitude of the probability-weighted gradient of w within the box. As before, standard UQ methods are inapplicable

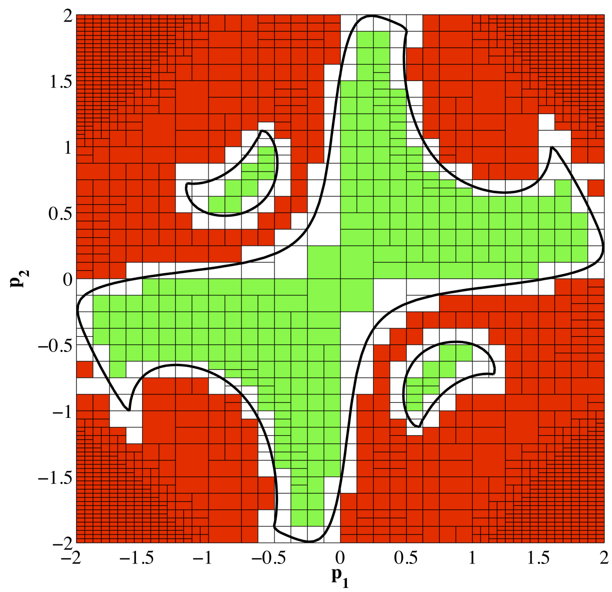

Figure 4: $\mathcal{F}^{\rm sub}$ (red) and $\mathcal{S}^{\rm sub}$ (green) and $\partial\mathcal{F}$ (thick line).

to this problem.

5 UQ for Mixed Uncertainties

In this section we develop strategies for bounding the range of failure probabilities, means and variances of piecewise polynomial response metrics that depend on both aleatory and epistemic parameters. As before, the aleatory variables can be arbitrarily distributed.

Recall that the propagation of probabilistic uncertainty in the aleatory variables through the response metrics yields random processes that are are parametrized by the epistemic variables. Note that for a fixed value of the epistemic variable, the UQ methods of Section 4 apply. While for the aleatory-only case the statistics take on a constant value, taking into account the epistemic intervals of uncertainty spreads each aleatory statistic over its own interval of uncertainty. Each value in that interval is a possible realization of the statistic. The values of the epistemic realizations prescribing the lower and upper interval's limits will be referred to as *best-case* and *worst-case*. These values will be denoted hereafter as $\boldsymbol{e}^{\min}$ and $\boldsymbol{e}^{\max}$. In this section we use the Bernstein expansion approach to bound the range of statistics of the response metric and calculate supersets of the critical epistemic points $\boldsymbol{e}^{\min}$ and $\boldsymbol{e}^{\max}$. The mathematical background for this is presented next.

5.1 Reliability Analysis

The reliability analysis of a system subject to the design requirements $\boldsymbol{g}(\boldsymbol{a},\boldsymbol{e}) < \boldsymbol{0}$ consists of calculating the failure probability range

$$\text{Range}\,(P[\mathcal{F}(\boldsymbol{e})]) = \left[\min_{\boldsymbol{e}\in\Delta_{\boldsymbol{e}}} P[w(\boldsymbol{a},\boldsymbol{e}) > 0], \max_{\boldsymbol{e}\in\Delta_{\boldsymbol{e}}} P[w(\boldsymbol{a},\boldsymbol{e}) > 0]\right], \tag{51}$$

and locating the sets of epistemic points that realize this interval's endpoints. While it may not be feasible to calculate the critical epistemic points exactly, the techniques presented here will enclose them in a sequence of supersets. This sequence is made to converge to all critical parameter points regardless of their number and location.

As in Section 4.1, a key development in this section is the calculation of bounding sets of the failure domain. An algorithm for generating and sequentially expanding subsets of the failure and safe domains is developed. Even though this algorithm has the same rationale of Algorithm 1, the logic governing the mechanism by which the master domain is partitioned is different. This logic segregates the epistemic subspace, where the critical epistemic realizations and their supersets are located, from the aleatory subspace, where the probability calculations are carried out. The expressions required to bound the failure probability range in Equation (51) and calculate supersets of the epistemic realizations attaining extreme failure probabilities are derived next.

Recall that the worst-case requirement function is bounded by

$$\underline{w}(\boldsymbol{a},\boldsymbol{e},\rho) \leq w(\boldsymbol{a},\boldsymbol{e}) \leq \overline{w}(\boldsymbol{a},\boldsymbol{e},\rho), \tag{52}$$

where ρ is a partition of the master domain $\mathcal{D} = \mathcal{D}_{\boldsymbol{a}} \times \mathcal{D}_{\boldsymbol{e}}$. This expression implies that

$$F_{\overline{w}}(x;\boldsymbol{e}) \leq F_w(x;\boldsymbol{e}) \leq F_{\underline{w}}(x;\boldsymbol{e}). \tag{53}$$

where $F_w(\cdot\,;\boldsymbol{e})$ (resp. $F_{\overline{w}}(\cdot\,;\boldsymbol{e})$ and $F_{\underline{w}}(\cdot\,;\boldsymbol{e})$) is the family of CDFs of $w(\boldsymbol{a},\boldsymbol{e})$ (resp. $\overline{w}(\boldsymbol{a},\boldsymbol{e})$ and $\underline{w}(\boldsymbol{a},\boldsymbol{e})$) corresponding to the CDF F of $\boldsymbol{a}$ and all possible realizations of $\boldsymbol{e}$ in $\Delta_{\boldsymbol{e}}$. As before, simple manipulations of (53) lead to

$$1 - F_{\underline{w}}(0;\boldsymbol{e}) \leq P[\mathcal{F}(\boldsymbol{e})] \leq 1 - F_{\overline{w}}(0;\boldsymbol{e}). \tag{54}$$

Observe that the evaluation of (54) and (58) at a fixed value of $\boldsymbol{e}$ yields the expressions (34) and (35) of Section 4.1.

Developments for bounding the epistemic realizations $\boldsymbol{e}^{\min}$ and $\boldsymbol{e}^{\max}$ leading to extreme failure probabilities are presented next. Expression (54) yields the outer bounding interval X given by

$$\text{Range}\,(P[\mathcal{F}(\boldsymbol{e})]) \subseteq X \subseteq \left[1 - \max_{\boldsymbol{e}\in\Delta_{\boldsymbol{e}}} F_{\underline{w}}(0;\boldsymbol{e}), 1 - \min_{\boldsymbol{e}\in\Delta_{\boldsymbol{e}}} F_{\overline{w}}(0;\boldsymbol{e}))\right]. \tag{55}$$

Since $\underline{w}$ and $\overline{w}$ are piecewise constant on a finite partition of $\Delta_{\boldsymbol{e}}$, the minimum and maximum in (55) can be calculated exactly in a finite number of steps. Supersets of $\boldsymbol{e}^{\min}$ and $\boldsymbol{e}^{\max}$, where the extreme failure probabilities occur, are given by

$$\mathcal{E}^{\min} = \bigcup_{j \in \underline{j}(0,\boldsymbol{e})} \left\{ \text{Proj}_{\boldsymbol{e}}(\mathcal{R}_j) : F_{\underline{w}}(0;\boldsymbol{e}) \geq \max_{\boldsymbol{e} \in \Delta_{\boldsymbol{e}}} F_{\overline{w}}(0;\boldsymbol{e}) \right\} \supset \boldsymbol{e}^{\min}, \tag{56}$$

$$\mathcal{E}^{\max} = \bigcup_{j \in \overline{j}(0,\boldsymbol{e})} \left\{ \text{Proj}_{\boldsymbol{e}}(\mathcal{R}_j) : F_{\overline{w}}(0;\boldsymbol{e}) \leq \min_{\boldsymbol{e} \in \Delta_{\boldsymbol{e}}} F_{\underline{w}}(0;\boldsymbol{e}) \right\} \supset \boldsymbol{e}^{\max}. \tag{57}$$

These expressions result from applying Equations (26-27) to the bounds in (54).

Equations (55), (56) and (57) are evaluated as follows. If $\rho(\mathcal{D}) = \{\mathcal{R}_1, \ldots \mathcal{R}_t\}$ is a partition of the master domain, the bounds in (53) are given by

$$F_{\underline{w}}(x;\boldsymbol{e}) = \sum_{j \in \underline{j}(x,\boldsymbol{e})} P\left[\text{Proj}_{\boldsymbol{a}}(\mathcal{R}_j)\right], \tag{58}$$

$$F_{\overline{w}}(x;\boldsymbol{e}) = \sum_{j \in \overline{j}(x,\boldsymbol{e})} P\left[\text{Proj}_{\boldsymbol{a}}(\mathcal{R}_j)\right], \tag{59}$$

where first $k(\boldsymbol{e}) = \{j : 1 \leq j \leq t \text{ and } \boldsymbol{e} \in \text{Proj}_{\boldsymbol{e}}(\mathcal{R}_j)\}$ contains the indices of the members of the partition whose projection onto the epistemic subspace contains the epistemic realization $\boldsymbol{e}$, and then $\overline{j}(x,\boldsymbol{e}) = \{j \in k(\boldsymbol{e}) : \overline{w}(\boldsymbol{p}^{(j)}) \leq x\}$ and $\underline{j}(x,\boldsymbol{e}) = \{j \in k(\boldsymbol{e}) : \underline{w}(\boldsymbol{p}^{(j)}) \leq x\}$ pick out those members of the partition which contribute to the respective CDFs. Note the similarity between the bounds in (35) and those in (58) and (59). A few manipulations lead to

$$F_{\underline{w}}(0;\boldsymbol{e}) = 1 - \sum_{j \in \underline{q}(\boldsymbol{e})} P\left[\text{Proj}_{\boldsymbol{a}}(\mathcal{R}_j)\right], \tag{60}$$

$$F_{\overline{w}}(0;\boldsymbol{e}) = \sum_{j \in \overline{q}(\boldsymbol{e})} P\left[\text{Proj}_{\boldsymbol{a}}(\mathcal{R}_j)\right], \tag{61}$$

where $\underline{q}(\boldsymbol{e}) = \{j \in k(\boldsymbol{e}) : \mathcal{R}_j \subseteq \mathcal{F}^{\text{sub}}\}$ and $\overline{q}(\boldsymbol{e}) = \{j \in k(\boldsymbol{e}) : \mathcal{R}_j \subseteq \mathcal{S}^{\text{sub}}\}$. The bounds $F_{\underline{w}}(0;\boldsymbol{e})$ and $F_{\overline{w}}(0;\boldsymbol{e})$ assume a constant value over the subsets $\text{Proj}_{\boldsymbol{e}}(\mathcal{R}_j)$ of $\mathcal{D}_{\boldsymbol{e}}$. Consequently, the minimum and maximum in (55-57) can be calculated exactly in a finite number of steps.

When the volume of the undetermined region Λ of the partition Q approaches zero, X converges to the failure probability range, while $\mathcal{E}^{\min}$, and $\mathcal{E}^{\max}$ converge to $\boldsymbol{e}^{\min}$ and $\boldsymbol{e}^{\max}$ respectively. The supersets $\mathcal{E}^{\min}$ and $\mathcal{E}^{\max}$ contain all the epistemic points where the global extrema of $P[\mathcal{F}(\boldsymbol{e})]$ occur. While the expressions above apply to arbitrary partitions of the master domain, there are partitions leading to tighter bounding intervals and smaller supersets. An algorithm for generating a sequence of improved partitions, having the same rationale of Algorithm 1, is presented next.

Algorithm 3: Let the inequality constraint $\boldsymbol{g}(\boldsymbol{p}) < \boldsymbol{0}$ defined over $\boldsymbol{p} \in \mathcal{D}$ prescribe the system requirements. Prescribe the epistemic range $\Delta_{\boldsymbol{e}} \subseteq \mathbb{R}^e$ and the CDF of the aleatory variable $F(\boldsymbol{a}) : \Delta_{\boldsymbol{a}} \subseteq \mathbb{R}^a$, where $\Delta_{\boldsymbol{a}} \times \Delta_{\boldsymbol{e}} \subseteq \mathcal{D}$. Set $i = 1$, $Q_1 = \{\mathcal{D}\}$, $\Lambda_1 = \mathcal{D}$, $\mathcal{F}_1^{\rm sub} = \emptyset$, $\mathcal{S}_1^{\rm sub} = \emptyset$, $X_1 = [0, 1]$, $\mathcal{E}_1^{\rm min} = \Delta_{\boldsymbol{e}}$ and $\mathcal{E}_1^{\rm max} = \Delta_{\boldsymbol{e}}$. Pick a selection criterion $0 < \eta < 1$ and a convergence criterion $\epsilon > 0$.

1. Let L contain the elements of the partition Q_i comprising Λ_i whose projection onto the epistemic subspace is contained by $\mathcal{E}_i^{\rm min} \cup \mathcal{E}_i^{\rm max}$. Let $\mathcal{L}$ be the union of the rectangles in L.
2. Apply Theorems 1 and 2 to each hyper-rectangle in L without partitioning it, to determine which elements of L are contained in the safe domain and which ones are contained in the failure domain. Denote by U the list of elements contained by the safe domain, by V the list of elements contained by the failure domain, and by W the list of elements that are not in U nor V. Furthermore, let $\mathcal{U}$ and $\mathcal{V}$ be the union of the elements in U and V respectively.
3. Make $\mathcal{S}_{i+1}^{\rm sub} = \mathcal{S}_i^{\rm sub} \cup \mathcal{U}$, $\mathcal{F}_{i+1}^{\rm sub} = \mathcal{F}_i^{\rm sub} \cup \mathcal{V}$, and $\Lambda_{i+1} = \Lambda_i \setminus (\mathcal{U} \cup \mathcal{V})$.
4. Calculate the bounding interval X_{i+1} and the supersets $\mathcal{E}_{i+1}^{\rm min}$ and $\mathcal{E}_{i+1}^{\rm max}$ by using Equations (55), (56) and (57).
5. If Volume $\left[\mathcal{E}_{i+1}^{\rm min} \cup \mathcal{E}_{i+1}^{\rm max}\right] < \epsilon$ stop. Otherwise, make $Q_{i+1} = (Q_i \setminus W) \cup \rho(W)$, increase i by one, and go to Step (1).

Note the similarities between this algorithm and Algorithm 1. Step (1) ensures that only those regions of the epistemic space where the critical epistemic realizations may be located are partitioned further. Step (4) enforces a stopping criterion based on the supersets of $\boldsymbol{e}^{\rm min}$ and $\boldsymbol{e}^{\rm max}$ being sufficiently small. The stopping criterion Volume $[X_i \setminus X_{i-1}] < \epsilon$, which requires a converged bounding interval, can also be used. As before, the supersets converge monotonically to their target value(s), and X converges monotonically to the failure probability range; as the partition becomes finer. In regard to the operator ρ in Step (6), note that while finer partitions in the aleatory dimensions will tighten the bounding interval X, finer partitions in the epistemic dimensions will tighten the supersets $\mathcal{E}^{\rm min}$ and $\mathcal{E}^{\rm max}$.

Example 3: Consider the requirement functions

$$\boldsymbol{g}_1 = -3\boldsymbol{p}_1^2\boldsymbol{p}_2^5 + 3\boldsymbol{p}_1^2 - \boldsymbol{p}_2^2 - 4\boldsymbol{p}_2 - 15, \tag{62}$$

$$\boldsymbol{g}_2 = -\boldsymbol{p}_1^{11}\boldsymbol{p}_2 + \boldsymbol{p}_1^2\boldsymbol{p}_2^7 - \frac{1}{10}, \tag{63}$$

$$\boldsymbol{g}_3 = -2\boldsymbol{p}_1^4 - \boldsymbol{p}_2^6 - \boldsymbol{p}_1^2\boldsymbol{p}_2 - \frac{\boldsymbol{p}_1\boldsymbol{p}_2^5}{10} + \frac{1}{500}, \tag{64}$$

defined over the master domain $\mathcal{D} \in [-2, 2] \times [-2, 2]$. To start, $\boldsymbol{p}_1$ will be considered a Beta-distributed aleatory variable with parameters $\langle 4, 4\rangle$ supported in $\Delta_{\boldsymbol{a}} = [-2, 2]$ while $\boldsymbol{p}_2$ will be epistemic supported in $\Delta_{\boldsymbol{e}} = [-2, 2]$. Therefore, $\boldsymbol{a} = \{\boldsymbol{p}_1\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_2\}$. In this setting we want to calculate X, $\mathcal{E}^{\rm max}$, and $\mathcal{E}^{\rm max}$.

Figure 5 shows terms of the sequences $\mathcal{E}_i^{\rm min}$ and $\mathcal{E}_i^{\rm max}$ as a function of the iteration number. This figure illustrates how the supersets are progressively refined. In both cases, there seems to be a single extrema. The existence of an isolated region in the vicinity of

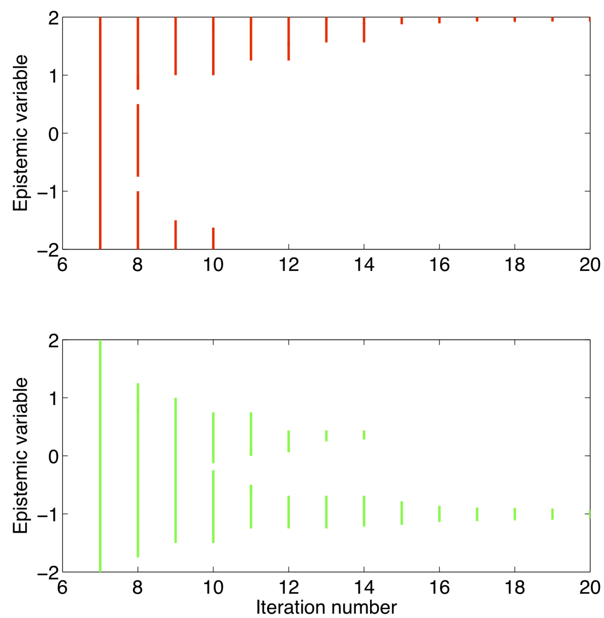

Figure 5: $\mathcal{E}_i^{\max}$ (top) and $\mathcal{E}_i^{\min}$ (bottom) for $\boldsymbol{a} = \{\boldsymbol{p}_1\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_2\}$.

$\boldsymbol{p}_2 = 0$ for $\mathcal{E}^{\max}$ at iteration number 8 and of $\boldsymbol{p}_1 = 0.39$ for $\mathcal{E}^{\max}$ at iteration number 14, and their subsequent disappearance, suggests the existence of non-global extrema. Figure 6 shows a fine partition of the parameter space. Note that Λ, the region containing the failure domain boundary colored in white, has not been uniformly refined. Further notice that the density of boxes in the epistemic direction $\boldsymbol{p}_2$ increases with the proximity to the global extrema. These extrema occur within $\mathcal{E}^{\max} = [1.9336, 2]$ and $\mathcal{E}^{\min} = [-1.07, -0.941]$ at iteration $i = 20$.

The top of Figure 7 shows the probability of Λ as a function of the epistemic variable $\boldsymbol{p}_2$ for the partition in Figure 6. The bottom shows the failure probability bounds in (54) over the epistemic domain. The supersets $\mathcal{E}^{\min}$ and $\mathcal{E}^{\max}$ corresponding to the last iteration in Figure 5 are displayed as vertical strips. Note that $P[\Lambda]$ approaches zero at the supersets of the best- and worst- case epistemic values. By design, these are the very same regions where the spread between the upper and lower failure probability bounds is minimal. Values of $\boldsymbol{p}_2$ where $P[\Lambda]$ is comparatively large are regions of no interest.

Finally, Figure 8 shows the ensemble of all CDF bounds in (53), as well as those corresponding to epistemic values in $\mathcal{E}^{\min}$ and $\mathcal{E}^{\max}$. While the lower CDF bound of the worst-case representative crosses $w = 0$ at $F_{\underline{w}}(0; \boldsymbol{e}) = 1 - 0.97148$, the upper CDF bound of the best-case representative crosses $w = 0$ at $F_{\overline{w}}(0; \boldsymbol{e}) = 1 - 0.076931$. These two values determine the endpoints of the bounding interval $X = [0.076931, 0.97148]$. Note

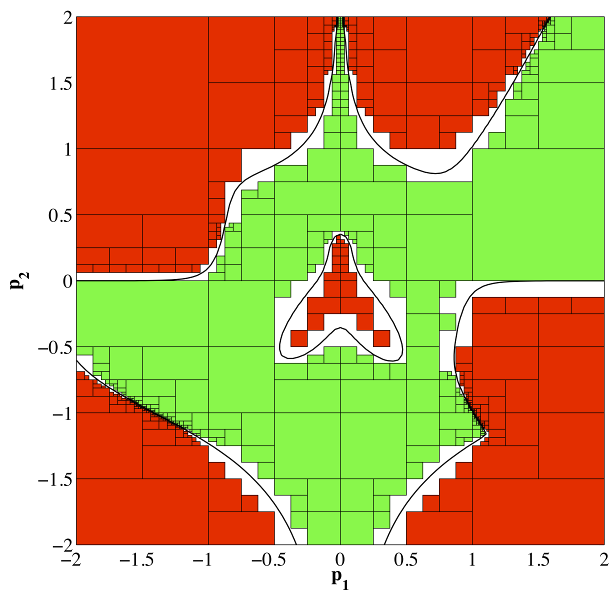

Figure 6: Partition of the master domain for $\boldsymbol{a} = \{\boldsymbol{p}_1\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_2\}$.

that the separation between the lower and upper CDF bounds of the best- and worst-case representatives decreases with the proximity to $w = 0$ by design.

We will now repeat this example by considering $\boldsymbol{p}_1$ as an epistemic variable supported in $\Delta_{\boldsymbol{e}} = [-2, 2]$ while $\boldsymbol{p}_2$ is an aleatory variable modelled as a Beta random variable with parameters $\langle 4, 6\rangle$ supported in $\Delta_{\boldsymbol{a}} = [-2, 2]$. Therefore $\boldsymbol{a} = \{\boldsymbol{p}_2\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_1\}$. Figure 9 shows terms of the sequences $\mathcal{E}_i^{\max}$ and $\mathcal{E}_i^{\min}$ as a function of the iteration number. In contrast to the previous example, the $\mathcal{E}^{\max}$ sequence suggests the existence of two global minima. Recall that the exact nature of the approach guarantees that all global extrema will be found and that no prior knowledge of the location or number of such extrema is required. Figure 10 shows the corresponding partition of the parameter space. Note that the density of boxes in the epistemic direction $\boldsymbol{p}_1$ increases with the proximity to $\mathcal{E}^{\max} = [1.2578, 2]$ and $\mathcal{E}^{\min} = [-0.48434, -0.46094] \cup [0.46094, 0.48434]$. These supersets correspond to iteration $i = 20$. Figure 11 and 12 are analogous to Figures 7 and 8. While the lower CDF bound of the worst-case representative crosses $w = 0$ near $F_{\underline{w}}(0; \boldsymbol{e}) = 1 - 0.74623$, the upper CDF bound of the best-case representative crosses $w = 0$ at $F_{\overline{w}}(0; \boldsymbol{e}) = 1 - 0.017921$. Hence, the bounding interval of the failure probability range is $X = [0.017921, 0.74623]$.

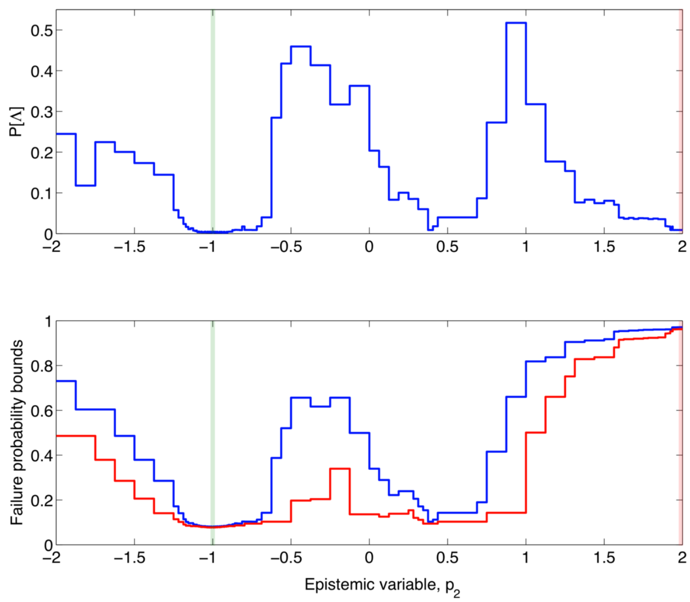

Figure 7: Probability of Λ and bounds of $P[\mathcal{F}]$ for $\boldsymbol{a} = \{\boldsymbol{p}_1\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_2\}$.

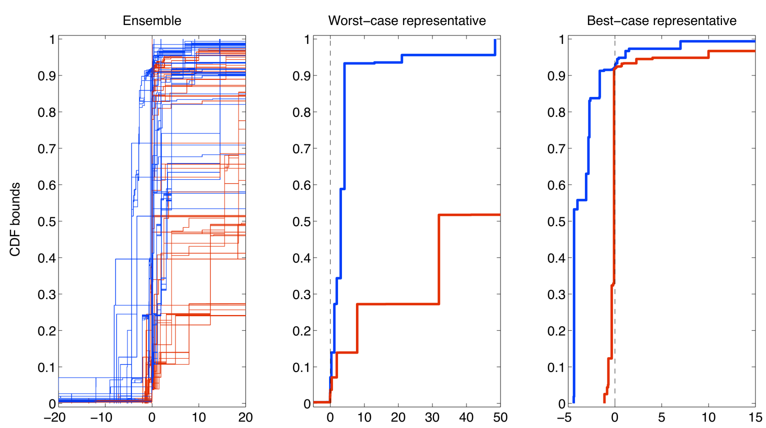

Figure 8: Ensemble of CDF bounds, worst- and best-case CDF bounds representatives for $\boldsymbol{a} = \{\boldsymbol{p}_1\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_2\}$.

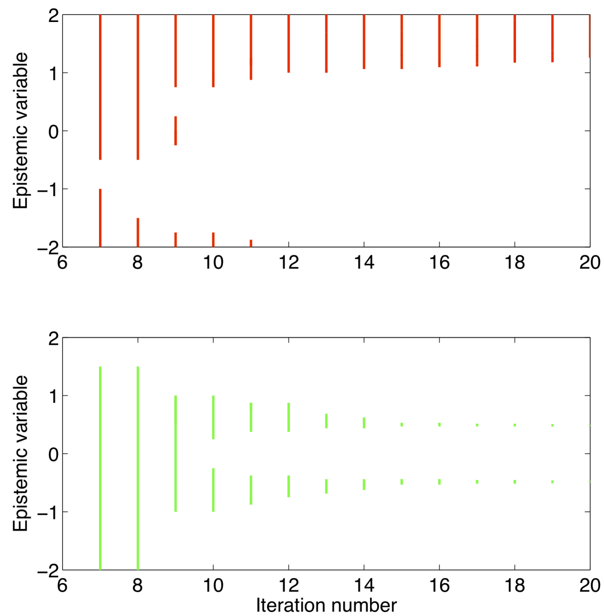

Figure 9: $\mathcal{E}_i^{\max}$ (top) and $\mathcal{E}_i^{\min}$ (bottom) for $\boldsymbol{a} = \{\boldsymbol{p}_2\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_1\}$.

5.2 Performance Analysis

The problem of interest is to calculate the range of M

$$\text{Range}\,(M[g(\cdot, \boldsymbol{e})]) = \left[\min_{\boldsymbol{e}\in\Delta_{\boldsymbol{e}}} M[g(\boldsymbol{a}, \boldsymbol{e})], \max_{\boldsymbol{e}\in\Delta_{\boldsymbol{e}}} M[g(\boldsymbol{a}, \boldsymbol{e})]\right], \tag{65}$$

and locate the sets of epistemic points that realize these interval's endpoints. While it may not be feasible to calculate the critical epistemic points exactly, the techniques presented here will enclose them in a sequence of supersets. As before, this sequence converges to all critical parameter points regardless of their number and location.

$E[g(\boldsymbol{a}, \boldsymbol{e})]$ and $V[g(\boldsymbol{a}, \boldsymbol{e})]$ can be calculated analytically when the aleatory parameters are independent random variables and g is a polynomial. In such a case one can solve for the extreme value of the moments and the corresponding best- and worst- case epistemic realizations by applying the analysis of Section 3 to the analytical expressions for the moments. These expressions will be polynomials in the variable $\boldsymbol{e}$. If g is a piecewise polynomial function of $\boldsymbol{p}$, and the components of $\boldsymbol{a}$ are arbitrarily and possibly dependently distributed, an approach analogous to the one in the preceding section can be applied. This approach yields bounds of the range in Equation (65) as well as supersets of the critical epistemic realizations. The expressions required to carry out this analysis are derived next.

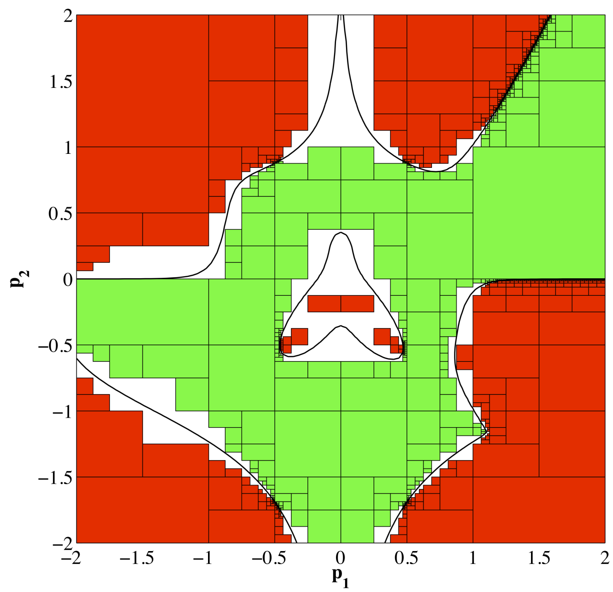

Figure 10: Partition of the master domain for $\boldsymbol{a} = \{\boldsymbol{p}_2\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_1\}$.

The bounding functions of w yield the outer bounding interval of the mean given by

$$\text{Range}\,(E[g(\cdot,\boldsymbol{e})]) \subseteq X = \left[\min_{\boldsymbol{e}\in\Delta_{\boldsymbol{e}}} E[\underline{g}(\cdot,\boldsymbol{e})], \max_{\boldsymbol{e}\in\Delta_{\boldsymbol{e}}} E[\overline{g}(\cdot,\boldsymbol{e})]\right], \tag{66}$$

where

$$E[\underline{g}(\cdot,\boldsymbol{e})] = \sum_{j\in k(\boldsymbol{e})} \underline{g}(\boldsymbol{p}^{(j)}) P\left[\text{Proj}_{\boldsymbol{a}}(\mathcal{R}_j)\right], \tag{67}$$

$$E[\overline{g}(\cdot,\boldsymbol{e})] = \sum_{j\in k(\boldsymbol{e})} \overline{g}(\boldsymbol{p}^{(j)}) P\left[\text{Proj}_{\boldsymbol{a}}(\mathcal{R}_j)\right], \tag{68}$$

and $k(\boldsymbol{e}) = \{j : 1 \le j \le t, \boldsymbol{e} \in \text{Proj}_{\boldsymbol{e}}(\mathcal{R}_j)\}$ contains the indices of the members of the partition whose projection onto the epistemic subspace contains the epistemic realization $\boldsymbol{e}$. Supersets of the epistemic realizations where the extreme values of $E[g(\cdot,\boldsymbol{e})]$ occur are

$$\mathcal{E}^{\min} = \bigcup_{j\in k(\boldsymbol{e})} \left\{\text{Proj}_{\boldsymbol{e}}(\mathcal{R}_j) : E[\underline{g}(\cdot,\boldsymbol{e})] \le \min_{\boldsymbol{e}\in\Delta_{\boldsymbol{e}}} E[\overline{g}]\right\} \supset \boldsymbol{e}^{\min}, \tag{69}$$

$$\mathcal{E}^{\max} = \bigcup_{j\in k(\boldsymbol{e})} \left\{\text{Proj}_{\boldsymbol{e}}(\mathcal{R}_j) : E[\overline{g}(\cdot,\boldsymbol{e})] \ge \max_{\boldsymbol{e}\in\Delta_{\boldsymbol{e}}} E[\underline{g}]\right\} \supset \boldsymbol{e}^{\max}. \tag{70}$$

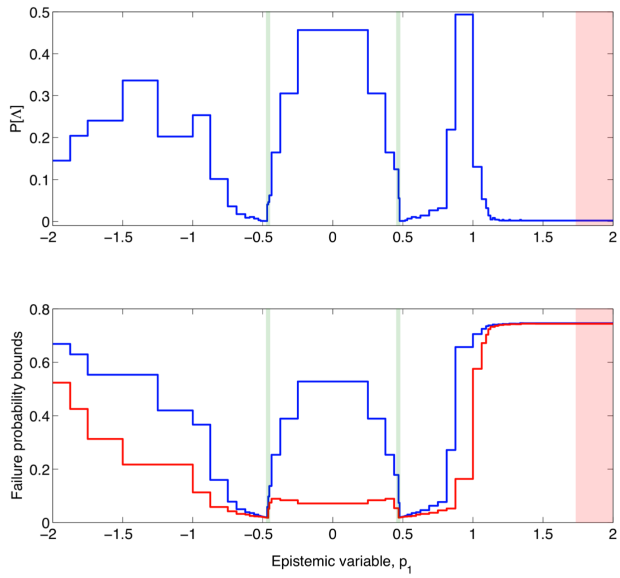

Figure 11: Probability of Λ and bounds of $P[\mathcal{F}]$ for $\boldsymbol{a} = \{\boldsymbol{p}_2\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_1\}$.

These expressions result from applying Equations (26-27) to the bounding functions in (66).

An outer bounding interval of the variance is

$$\text{Range}\left(V[g(\cdot,\boldsymbol{e})]\right) \subseteq X = [\max\{0, \alpha(\boldsymbol{e})\}, \beta(\boldsymbol{e})], \tag{71}$$

where $\alpha(\boldsymbol{e})$ and $\beta(\boldsymbol{e})$ are given by Equations (49) and (50) with the expected values given by Equations (67) and (68), and with $\mu(\boldsymbol{e})$ and $\nu(\boldsymbol{e})$ given by

$$\mu(\boldsymbol{e}) = \sum_{j\in u(\boldsymbol{e})} \underline{g}(\boldsymbol{p}^{(j)})^2 P\left[\text{Proj}_{\boldsymbol{a}}(\mathcal{R}_j)\right] + \sum_{j\in v(\boldsymbol{e})} \overline{g}(\boldsymbol{p}^{(j)})^2 P\left[\text{Proj}_{\boldsymbol{a}}(\mathcal{R}_j)\right], \tag{72}$$

$$\nu(\boldsymbol{e}) = \sum_{j\in u(\boldsymbol{e})\cup x(\boldsymbol{e})} \overline{g}(\boldsymbol{p}^{(j)})^2 P\left[\text{Proj}_{\boldsymbol{a}}(\mathcal{R}_j)\right] + \sum_{j\in v(\boldsymbol{e})\cup y(\boldsymbol{e})} \underline{g}(\boldsymbol{p}^{(j)})^2 P\left[\text{Proj}_{\boldsymbol{a}}(\mathcal{R}_j)\right], \tag{73}$$

where u, v, x and y are defined in Equations (44-47) and $\boldsymbol{e} \in \text{Proj}_{\boldsymbol{e}}(\mathcal{R}_j)$. Supersets of the epistemic realizations where the extreme variances occur are

$$\mathcal{E}^{\min} = \bigcup_{j\in k(\boldsymbol{e})} \left\{\text{Proj}_{\boldsymbol{e}}(\mathcal{R}_j) : \max\{0, \alpha(\boldsymbol{e})\} \leq \min_{\boldsymbol{e}\in\Delta_{\boldsymbol{e}}} \beta(\boldsymbol{e})\right\} \supset \boldsymbol{e}^{\min}, \tag{74}$$

$$\mathcal{E}^{\max} = \bigcup_{j\in k(\boldsymbol{e})} \left\{\text{Proj}_{\boldsymbol{e}}(\mathcal{R}_j) : \beta(\boldsymbol{e}) \geq \max_{\boldsymbol{e}\in\Delta_{\boldsymbol{e}}} \{\max\{0, \alpha(\boldsymbol{e})\}\}\right\} \supset \boldsymbol{e}^{\max}. \tag{75}$$

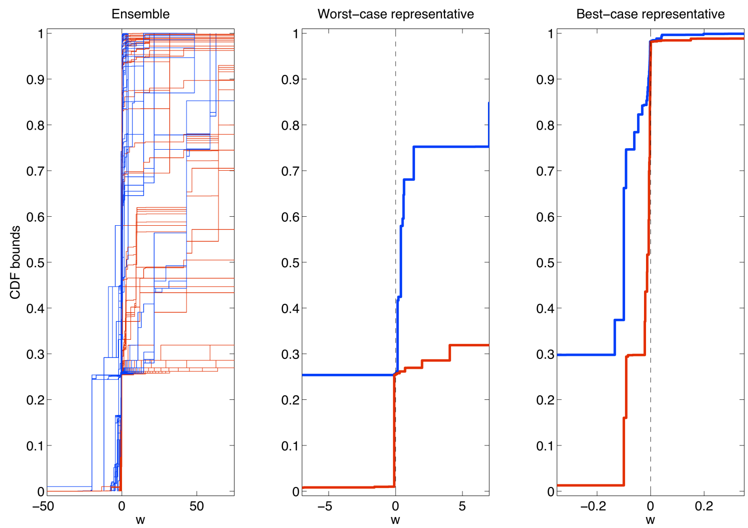

Figure 12: Ensemble of CDF bounds, worst- and best-case CDF bounds representatives for $\boldsymbol{a} = \{\boldsymbol{p}_2\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_1\}$.

These expressions result from applying Equations (26-27) to the bounding functions in (71). As before, the right-hand side of the inequalities prescribing the supersets can be found exactly in a finite number of steps. The algorithmic implementation of these ideas, which follows the same rationale of Algorithm 2, is as follows.

Algorithm 4: Let $\boldsymbol{g}(\boldsymbol{p})$ defined over $\boldsymbol{p} \in \mathcal{D}$ be a performance function. Prescribe the epistemic range $\Delta_{\boldsymbol{e}} \subseteq \mathbb{R}^e$ and the CDF of the aleatory variable $F(\boldsymbol{a}) : \Delta_{\boldsymbol{a}} \subseteq \mathbb{R}^a$, where $\Delta_{\boldsymbol{a}} \times \Delta_{\boldsymbol{e}} \subseteq \mathcal{D}$. Set $i = 1$ and $Q_1 = \{\mathcal{D}\}$. Pick a selection criterion $0 < \eta < 1$ and a convergence criterion $\epsilon > 0$.

1. Calculate the bounding functions $\underline{g}(\boldsymbol{a}, \boldsymbol{e})$ and $\overline{g}(\boldsymbol{a}, \boldsymbol{e})$ over $\mathcal{D}$.
2. Calculate the bounding interval X_i via Equations (66) or (71), and the supersets $\mathcal{E}_i^{\min}$ and $\mathcal{E}_i^{\max}$ via Equations (69-70) or Equations (74-75).
3. Denote by U a list with the elements of the partition Q_i whose projection onto the epistemic subspace is contained by $\mathcal{E}_i^{\min} \cup \mathcal{E}_i^{\max}$.
4. Denote by L a list with the elements of U where[5] $\overline{g} - \underline{g} > \eta \max_{\mathcal{R} \in U} \{\overline{g} - \underline{g}\}$.
5. If Volume $\left[\mathcal{E}_i^{\min} \cup \mathcal{E}_i^{\max}\right] < \epsilon$ stop. Otherwise, make $Q_{i+1} = (Q_i \setminus L) \cup \rho(L)$, increase i by one, and go to Step (1).

[5]The expression $E[\overline{g} - \underline{g} \,|\, \mathcal{R}] > \eta \max_{\mathcal{R} \in U} \{E[\overline{g} - \underline{g} \,|\, \mathcal{R}]\}$ can be used instead.

As before, the outer bounding interval X converges monotonically to the range of M as the partition is refined further. In regard to the operator ρ in Step 4, note that while finer partitions in the aleatory dimensions will tighten the bounding interval X, finer partitions in the epistemic dimensions will tighten the supersets $\mathcal{E}^{\min}$ and $\mathcal{E}^{\max}$.

Example 4: Given the same problem statement of Example 3, we now apply Algorithm 4 to bound the range of $E[w(e)]$ and $V[w(e)]$. To start we consider, $\boldsymbol{a} = \{\boldsymbol{p}_1\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_2\}$. The left graphs of Figure 13 show the resulting piecewise constant bounds of the mean and the variance for a fixed partition of the master domain. A logarithmic scale has been used to facilitate visualization. The bounding functions lead to Range($E[w(e)]$) $\subseteq [-0.69975, 61.0855]$ and Range($V[w(e)]$) $\subseteq [0, 11541.14248]$. The endpoints of these bounding intervals are given by the extreme values of the bounding functions. The dashed line in between the bounds is a Monte Carlo approximation with 1000 sample points. Note that by design, the offset between the Monte Carlo approximation and the upper bounding function is small when the moments are large, while the offset between the Monte Carlo approximation and the lower bounding function is small when the moments are small. Further notice the locations where the Monte Carlo approximations lay outside the bounds (e.g., see $V[w(e)]$ near $\boldsymbol{p}_2 = -1$). This outcome is the result of the approximation error in the Monte Carlo estimates. We now consider $\boldsymbol{a} = \{\boldsymbol{p}_2\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_1\}$. The corresponding results are shown to the right of Figure 13. In this case, the bounding functions lead to Range($E[w(e)]$) $\subseteq [-1.72522, 1022.75501]$ and Range($V[w(e)]$) $\subseteq [0, 1060602.13]$. While the Monte Carlo approximations capture the trends of $E[w(e)]$ and $V[w(e)]$ well in all cases, its generation and the calculation of its extrema may be complex tasks, e.g., sampling the PDF in (39), finding the global extrema of a high-order piecewise polynomial $M(\boldsymbol{e})$ over $\Delta_{\boldsymbol{e}}$.

6 Discussion

The strategies above require polynomial response metrics with known functional forms. When these forms are available, the resulting analysis is not only exempt from approximation error but also is formally verifiable [6]. Furthermore, the outer bounding interval X and the supersets $\mathcal{E}^{\min}$ and $\mathcal{E}^{\max}$ can always be made arbitrarily tight with additional computational effort, i.e., the conservatism in the output metric is reducible.

In principle, response metrics assuming known non-polynomial forms can be formally bounded with polynomials. These bounding polynomials may be defined globally or locally, e.g., $g = \exp(p)$ in $p \in [-1, 1]$ is bounded by $l = x + \alpha$ with $\alpha \leq 1$ from below and by $u = x + \beta$ with $\beta \geq \exp(1) - 1$ from above. The application of the proposed algorithms to such polynomials, i.e., the calculation of the lower bounding function $\underline{l}$ and the upper bounding function $\overline{u}$, renders formally verifiable results. However, the offset between the original response metric g and the bounding polynomials l and u introduces irreducible conservatism into the calculated metrics. This conservatism can be reduced, but not totally eliminated, by using tighter bounding polynomials, e.g., by making $\alpha = 1$ and $\beta = \exp(1) - 1$. Unfortunately, the generation of tight bounding polynomials is in itself a laborious task.

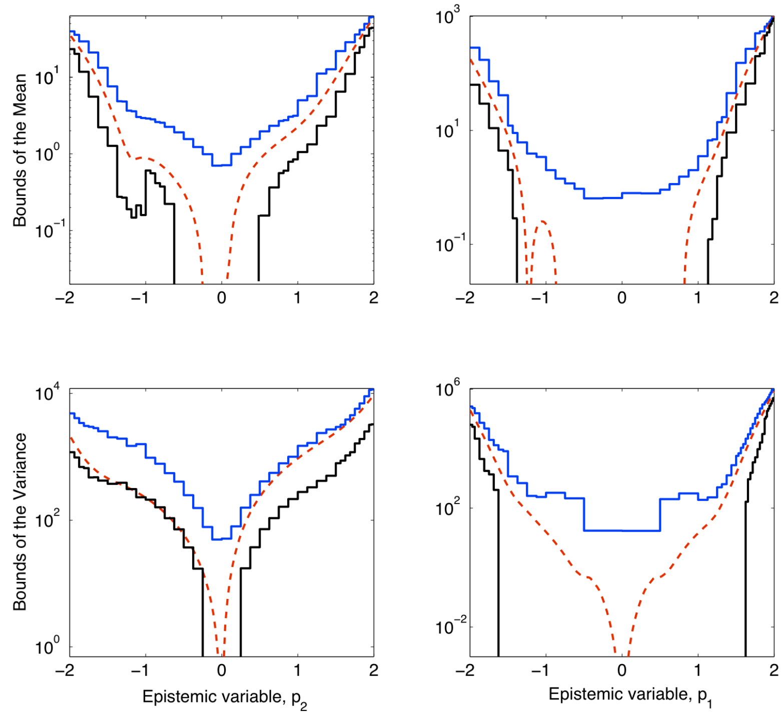

Figure 13: Upper and lower bounds of the mean (top) and variance (bottom) for $\boldsymbol{a} = \{\boldsymbol{p}_1\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_2\}$ (left) and for $\boldsymbol{a} = \{\boldsymbol{p}_2\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_1\}$ (right). Dashed lines indicate Monte Carlo approximations.

When function evaluations of the response metrics are available, $\boldsymbol{g}(\boldsymbol{p}^{(i)})$ for $i = 1, \ldots n$, polynomial surrogate models can be readily built. Surrogate models bounding such data from below and above can be constructed and used in the above UQ analyses. While this practice guarantees a rigorous UQ treatment, it will inherit the approximation error that stems from using surrogate models at points outside the data set. This approximation error, which affects the reliability and performance analyses equally, can only be managed in an ad hoc fashion by augmenting the data set.

A few remarks on computational complexity are in order. The number of Bernstein coefficients that must be calculated per hyper-rectangle grows exponentially with the number of uncertain parameters s, linearly with the number of requirement functions v, and polynomially of degree s with the degree of each $\boldsymbol{g}_i$. The former dependency restricts the applicability of the method to systems with a moderate number of uncertainties. The number of hyper-rectangles, on the other hand, increases linearly with each subdivision of the hyper-rectangles of interest. While non-overlapping portions of the master domain can be processed in parallel, the interdependencies between the response metrics and the uncertain parameters preclude using parallel computing to mitigate the computational demands of systems with large number of uncertainties. The ideas proposed however, can be used to identify unimportant uncertainties, thus, to reduce the number of uncertainties that should be considered in UQ. The following paragraph explains the rational supporting this statement.

Note that the sensitivity of the failure probability, the mean and the variance of a response metric to the uncertain parameters in $\boldsymbol{e}$ is less than the spread of the bounding interval X. Small bounding intervals indicate that $\boldsymbol{e}$ can assume any uncertainty model in $\Delta_{\boldsymbol{e}}$, even a probabilistic one, and the resulting value of the statistic will remain within the bounding set, e.g., if X is a bounding interval of the failure probability range for $\boldsymbol{a} = \{\boldsymbol{p}_1\}$ and $\boldsymbol{e} = \{\boldsymbol{p}_2\}$, where $F(\boldsymbol{p}_1)$ is supported in δ_1 and $\Delta_{\boldsymbol{e}} = \delta_1$; the failure probability corresponding to $\boldsymbol{a} = \{\boldsymbol{p}_1, \boldsymbol{p}_2\}$, where $F(\boldsymbol{p}_1)$ is supported in δ_1 and $F(\boldsymbol{p}_2)$ is supported in δ_2, will remain within X.

7 Conclusions

This article presents a unifying framework to uncertainty quantification of polynomial systems subject to both epistemic and aleatory uncertainties. The methods are applicable to piecewise polynomial functions of arbitrarily and possibly dependent aleatory variables. The Bernstein expansion approach enables the calculation of analytical bounds to moments and failures probabilities as well as the calculation of supersets of the corresponding best- and worst-case epistemic realizations. These bounds and supersets, which can be made as tight and as small as desired, are rigorous, e.g., the supersets are guaranteed to contain all global optima. The analytical nature of the approach eliminates the approximation error that characterizes techniques commonly used in UQ as well as the possibility of under predicting the range of the statistic of interest that may result from using nonlinear optimization or sampling. The computational complexity of the method restricts its applicability to problems having a moderate number of uncertain parameters.

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