Continuity Notions for Multi-Valued Mappings with Possibly Disconnected Images*

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Abstract

We discuss properties of continuity notions for multi-valued mappings which allow disconnected images, but still have a useful zero property. The starting point is the notion of c-continuity introduced by the second author in the book *Interval Methods for Systems of Equations* to study enclosure properties of interval Newton operators. It was claimed in that book that c-continuity possesses a zero property. However, we provide a counterexample. Two other continuity notions are introduced and examined, and applied in a logical context.

Keywords: Multi-valued mappings, continuity, fixed point theorem, Miranda theorem, witness mapping **AMS subject classifications:** 54C60, 26E25, 54H25

1 Introduction

The book *Interval Methods for Systems of Equations* ([22]) sets out to prove a set-valued version of Miranda's well-known intermediate value theorem for systems of equations, fundamental for interval analysis, by means of induction over the dimension of the system [22, Theorem 5.3.7.]. Unfortunately, as pointed out by Alexandre Goldsztejn, the proof contained a gap (for more detail see below page 92). The present paper shows that the gap cannot be filled by giving a counterexample (Example 3.2 below) and provides new concepts and theory that justify a set-valued intermediate value theorem.

A **multi-valued mapping** (sometimes called **set-valued map**) F from X to Y (in symbols, $F : X \multimap Y$) maps a point in X to a *nonempty* subset of Y. For a multi-valued mapping $F : X \multimap X$, a **fixed point** is a point $x^* \in X$ with $x^* \in F(x^*)$, and for a multi-valued mapping $F : X \multimap Y$ a **zero** is a point $x^* \in X$ with $0 \in F(x^*)$.

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Our goal is to generalize the following two well-known theorems for single-valued mappings to multi-valued mappings with possibly disconnected images:

Fixed-Point Theorem (Brouwer [5]) Let S be an closed, convex and nonempty subset of $\mathbb{R}^n$, let $f : S \to S$ be a continuous single-valued mapping. Then f has a fixed point, i.e., a point x^* with $f(x^*) = x^*$.

We use the following notation: $\mathbb{IR}$ is the set of all closed intervals in $\mathbb{R}$, and $\mathbb{IR}^n$ the set of all n-dimensional **boxes** $\boldsymbol{x}$, i.e., Cartesian products of n closed intervals; $\boldsymbol{x}_i := [\underline{\mathrm{x}}_i, \overline{\mathrm{x}}_i]$.

Multivariate Intermediate Value Theorem (Miranda [21], Poincaré [24]) Let $\boldsymbol{x} \in \mathbb{IR}^n$, let $f : \boldsymbol{x} \to \boldsymbol{x}$ be a continuous single-valued mapping such that for $i = 1, \dots, n$,

$$\begin{aligned} f_i(x) &\leq 0 \quad \text{for all } x \in \boldsymbol{x} \text{ with } x_i = \underline{\boldsymbol{x}}_i, \\ f_i(x) &\geq 0 \quad \text{for all } x \in \boldsymbol{x} \text{ with } x_i = \overline{\boldsymbol{x}}_i. \end{aligned}$$

Then f has a zero, i.e., there exists a point $x^* \in \boldsymbol{x}$ such that $f(x^*) = 0$.

We introduce some common continuity notions for multi-valued mappings:

If for a multi-valued mapping $F : X \multimap Y$ there exists a continuous single-valued mapping $f : X \to Y$ with $f(x) \in F(x)$ for every $x \in X$, we say that f is a **selection** of F and that F is **selectionable**. If we apply the fixed-point theorem or the multivariate intermediate value theorem to f, we obtain a fixed-point or a zero property for selectionable multi-valued mappings, see [16, 19, 20, 23]. If for a multi-valued mapping $F : X \multimap Y$ and every $\varepsilon > 0$ there is a continuous single-valued mapping $f : X \to Y$ with $\operatorname{graph}(f) \subseteq \mathcal{O}_\varepsilon(\operatorname{graph} F)$, we say that F is **approxable**. There exist various fixed point theorems for approxable multi-valued mappings, see [12].

We say that $F : X \multimap Y$ is **upper semicontinuous** (in short **u.s.c.**) at $x_0 \in X$ if for any neighborhood $\mathcal{N}(F(x_0))$ of $F(x_0)$ there exists a neighborhood $\mathcal{N}(x_0)$ of x_0 such that $F(\mathcal{N}(x_0)) \subseteq \mathcal{N}(F(x_0))$. We say that F is u.s.c. if F is u.s.c. at every $x_0 \in X$.

We say that $F : X \multimap Y$ is **lower semicontinuous** (in short **l.s.c.**) at x_0 if for any sequence $(x_n)_{n\in\mathbb{N}}$ converging to x_0, and for any point $y_0 \in F(x_0)$ there exists a sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \in F(x_n)$ such that $(y_n)_{n\in\mathbb{N}}$ converges to y_0. We say that F is l.s.c. if F is l.s.c. at every $x_0 \in X$.

In view of fixed-point properties, we cannot obtain strong results for mere u.s.c. or l.s.c. multi-valued mappings, unless we make assumptions to the topological properties of the sets $F(x)$ (see [1, 3, 6, 18]). For example, if for the u.s.c. multi-valued mapping $F : X \multimap X$ the set $F(x)$ is convex for all $x \in X$, we have the well known fixed point theorem of Kakutani, see [17, 10, 29]. If for all $x \in X$ the set $F(x)$ is **acyclic**, i.e., it has the same homology groups as the one point space (and hence $F(x)$ is connected for every x) we obtain fixed-pint properties via a theorem by Eilenberg-Montgomery [9, 2, 13]. For details about homology theory see [15] and [28]. There exist certain fixed point properties for multi-valued mappings with non-acyclic images, see [4, 7, 8]. A recent overview of the fixed-point theory for multi-valued mappings gives Gôrniewicz in [12], which also contains a current bibliography of publications concerning the fixed point theory of multi-valued mappings.

However, if the set $F(x)$ is disconnected for at least one x, none of the fixed point theorems above apply in general. Not even for a multi-valued mapping as simple as $F : [-1, 1] \multimap [-1, 1]$ defined by $F(x) := \{y \mid 3y^3 - y = x\}$ (see Figure 1) we can obtain a zero or a fixed point from the theorems above, although $x^* = 0$ is obviously both a zero and a fixed-point of F.

In this paper, we are concerned with continuity notions which are not defined by

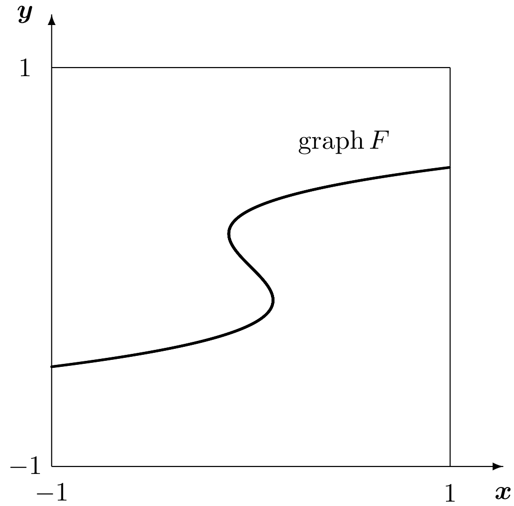

Figure 1: The graph of F

a topological property of the image of each point, but by a property of the graph of the multi-valued mapping. In particular, we are only interested in continuity notions which allow disconnected images. Furthermore, for all continuity notions discussed here, we demand that if f is a continuous single-valued mapping, then the multi-valued mapping defined by $F(x) := \{f(x)\}$ is continuous in the multi-valued sense. Hence all continuity notions considered here are generalizations of the usual continuity of single-valued mappings. We remind that by the definition of a multi-valued mapping, $F(x)$ is nonempty for all x.

Overview. The discussion of the continuity notions in Section 1 is guided by properties of multi-valued mappings which were suggested by Alexandre Goldsztejn and the second author. These are:

(SETFP)	a multi-valued version of the fixed point theorem,
(SETIV)	a multi-valued version of the multivariate intermediate value theorem,
(REDOM)	the preservation of continuity on a restricted domain,
(ENDOM)	the preservation of continuity on an enlarged domain,
(PROD)	the preservation of continuity on the Cartesian product,
(PROJ)	the preservation of continuity under projection,
(COMP)	the preservation of continuity under composition of multi-valued mappings, and
(GEN)	generalization of single-valued mapping.

The multivalued version of the multivariate intermediate value theorem is a concept by the second author in [22], the other properties are ideas of Goldsztejn in [11]. After introducing precise versions of these properties, we show that some of these properties imply others.

The second author in [22] introduced a continuity notion for multi-valued mappings, called ‘c-continuity’. A multi-valued mapping $F : \boldsymbol{x} \multimap \boldsymbol{y}$ is called c-continuous if for every continuum C in $\boldsymbol{x}$ and $\tau, \tau' \in C$, the set graph$(F|C)$ connects $\{\tau\} \times \boldsymbol{y}$

with $\{\tau'\} \times \boldsymbol{y}$. In Section 2, we show that c-continuity has two of the properties above, namely (REDOM) and (PROJ), but fails to satisfy the others. Most important, (SETIV) does not hold, although it was stated by Neumaier in [22]. We give a counterexample.

In Section 3, we introduce a continuity notion called 'is-continuity' which has many desirable properties, but others are still open questions. A multi-valued mapping $F : \boldsymbol{x} \multimap \boldsymbol{y}$ is called is-continuous if there exists a continuous single-valued mapping $f : \boldsymbol{x} \times \boldsymbol{y} \to \mathbb{R}^{\dim(\boldsymbol{y})}$ such that $f^{-1}(0)$ is a subset of graph F, plus additional assumptions to prevent that $f^{-1}(0)$ is empty.

Section 4: We introduce a continuity notion by Goldsztejn called ma-continuity. Multi-valued mappings are ma-continuous if their graph can be approximated by a sequence of smooth manifolds with boundary. Goldsztejn proved a generalization of (SETFP) in [11].

Section 5: Ratschan [26] applied the second author's (false) multi-valued version of the multivariate intermediate value theorem for c-continuous multi-valued mappings. Unfortunately, this makes the central theorem in [26] false too. (For a counterexample, see [27].) We introduce and prove a modified version of Ratschan's theorem: instead of continuous single-valued mappings, we use is-continuous multi-valued mappings.

Table 1: Summary

	SETFP	SETIV	REDOM	ENDOM	PROD	PROJ	COMP	GEN
ma-continuity	yes			*	*	*	*	yes
c-continuity	no	no	yes	no	no	yes	no	yes
is-continuity	yes	**	yes	yes	yes			yes

* These are conjectured to be true by Goldsztejn [11].

** We prove a weaker version of SETIV for is-continuous mappings.

Notation. In general, we use lower-case letters $f, g, h, \ldots$ for single-valued mappings and upper-case letter $F, G, H, \ldots$ for multi-valued mappings. Lower-case Greek letters $\varphi, \psi, \ldots$ usually denote paths. We use the upper-case letters X, Y, Z for spaces (usually subsets of $\mathbb{R}^n$, if not asserted otherwise). For a multi-valued mapping $F : X \multimap Y$ let $\operatorname{graph} F := \{(x, y) \mid x \in X, y \in F(x)\}$, and for C a **continuum** in X, i.e., a compact and connected subset of X, let $\operatorname{graph}(F|C) := \{(x, y) \mid x \in C, y \in F(x)\}$. For a space X, we write $\operatorname{dist}(x, y)$ for the **distance** of the points x and y. For a set $A \subseteq X$, $\operatorname{dist}(x, A) := \inf\{\operatorname{dist}(x, y) \mid y \in A\}$ is the distance of x and A.

Other often used symbols are: $B_r(x) := \{y \in X \mid \|x - y\| < r\}$, the open ball with center x and radius r, $\mathcal{O}_\varepsilon(A) := \bigcup_{a \in A} B_\varepsilon(a)$, an ε-neighborhood of the set A, $\operatorname{proj}_X A$ is the projection of the set A to the space X, and for the multi-valued mapping $F : X \multimap Y$, the set $F_i(x)$ is the projection of the set $F(x)$ to the ith component of Y.

2 Properties

Since we deal with different continuity notions, we will formulate the properties generally for ω-continuity.

We introduce some properties desirable for a continuity notion ω for multi-valued mappings.

(SETFP) A parametric version of the fixed-point theorem: Let $F : \boldsymbol{x} \times \boldsymbol{y} \multimap \boldsymbol{y}$ be ω-continuous. Then the multi-valued mapping $H : \boldsymbol{x} \multimap \boldsymbol{y}$ defined by

$$H(x) := \{y \in \boldsymbol{y} \mid y \in F(x, y)\}$$

is ω-continuous.

(SETIV) This is a multi-valued version of the multivariate intermediate value theorem: Let $F : \boldsymbol{x} \times \boldsymbol{y} \multimap \boldsymbol{z}$ with $\dim \boldsymbol{y} = \dim \boldsymbol{z}$ be a ω-continuous multi-valued mapping, and let

$$\begin{array}{rl} \sup(F_i(x, y)) \leq 0 & \text{if } y_i = \underline{\boldsymbol{y}}_i, \\ \inf(F_i(x, y)) \geq 0 & \text{if } y_i = \overline{\boldsymbol{y}}_i. \end{array}$$

Then the multi-valued mapping $H : \boldsymbol{x} \multimap \boldsymbol{y}$ defined by

$$H(x) := \{y \in \boldsymbol{y} \mid 0 \in F(x, y)\}$$

is ω-continuous.

(REDOM) The preservation of ω-continuity on a smaller domain: If $F : \boldsymbol{x} \multimap \boldsymbol{y}$ is ω-continuous on a continuum $C \subseteq \boldsymbol{x}$, then F is also ω-continuous on any continuum contained in C.

(ENDOM) The preservation of ω-continuity on a larger domain: Let $F : \boldsymbol{x} \multimap \boldsymbol{z}$ be ω-continuous. Then the multi-valued mapping $H : \boldsymbol{x} \times \boldsymbol{y} \multimap \boldsymbol{z}$ defined by

$$H(x, y) := F(x)$$

is ω-continuous.

(PROD) The preservation of ω-continuity on the Cartesian product of two multi-valued mappings: Let $F : \boldsymbol{x} \multimap \boldsymbol{u}$ and $G : \boldsymbol{y} \multimap \boldsymbol{v}$ be two ω-continuous multi-valued mappings. Then the multi-valued mapping $H : \boldsymbol{x} \times \boldsymbol{y} \multimap \boldsymbol{u} \times \boldsymbol{v}$ defined by

$$H(x, y) := F(x) \times G(y)$$

is ω-continuous.

(PROJ) The preservation of ω-continuity with respect to arbitrary projections: Let $F : \boldsymbol{x} \multimap \boldsymbol{y}$ be a ω-continuous multi-valued mapping. Then for any sequence $I = \{i_1, \ldots, i_k\}$ of integers $1 \leq i_1 < \ldots < i_k \leq \dim \boldsymbol{y}$, the multi-valued mapping $H : \boldsymbol{x} \multimap \boldsymbol{y}_{i_1} \times \ldots \times \boldsymbol{y}_{i_k}$ defined by

$$H(x) := \operatorname*{proj}_{\boldsymbol{y}_I}(F(x))$$

is ω-continuous.

(COMP) The preservation of ω-continuity with respect to composition of multi-valued mappings. Let $F : \boldsymbol{x} \multimap \boldsymbol{y}$ and $G : \boldsymbol{y} \multimap \boldsymbol{z}$ be two ω-continuous multi-valued mappings. Then the multi-valued mapping $H : \boldsymbol{x} \multimap \boldsymbol{z}$ defined by

$$H(x) = (G \circ F)(x) := \bigcup_{y \in F(x)} G(y)$$

is ω-continuous.

(GEN) A continuous single-valued mapping regarded as multi-valued mapping is ω-continuous: Let $f : \boldsymbol{x} \to \boldsymbol{y}$ a continuous single-valued mapping, then the multi-valued mapping $F : \boldsymbol{x} \multimap \boldsymbol{y}$ defined by

$$F(x) := \{f(x)\}$$

is ω-continuous.

Some implications

Without further assumptions on a continuity notions ω for multi-valued mappings, some properties imply others. In the following list of implications, we do not claim completeness. Proposition 2.1 is due to Goldsztejn. Let $\boldsymbol{x} \in \mathbb{IR}^n$, $\boldsymbol{y} \in \mathbb{IR}^m$ for $n, m \geq 1$.

Proposition 2.1 *The property (COMP) together with (GEN) implies (ENDOM).*

Proof: We assume an ω-continuous multi-valued mapping $F : \boldsymbol{x} \multimap \boldsymbol{z}$, and we define a multi-valued mapping $G : \boldsymbol{x} \times \boldsymbol{y} \multimap \boldsymbol{x}$ by $G(x, y) := \{x\}$. Then by (GEN) G is ω-continuous. We apply (COMP) and obtain that $H : \boldsymbol{x} \times \boldsymbol{y} \multimap \boldsymbol{z}$ defined by

$$\begin{array}{rcl} H(x,y) & := & (F \circ G)(x,y) \\ & = & \bigcup_{z \in G(x,y)} F(z) \\ & = & F(x), \end{array}$$

is ω-continuous. $\square$

Proposition 2.2 *The property (COMP) together with (GEN) implies (PROJ).*

Proof: We assume an ω-continuous multi-valued mapping $F : \boldsymbol{x} \multimap \boldsymbol{y}$. For some sequence $I = \{i_1, \ldots, i_k\}$ of integers $1 \leq i_1 < \ldots < i_k \leq \dim \boldsymbol{y}$, define the multi-valued mapping $G : \boldsymbol{y} \multimap \boldsymbol{y}_{i_1} \times \ldots \times \boldsymbol{y}_{i_k}$ by

$$G(y) := \operatorname*{proj}_{\boldsymbol{y}_I}(y).$$

Then G is the multivalued version of a continuous single-valued mapping and by (GEN) it is ω-continuous. We apply (COMP) and obtain that

$$\begin{array}{rl} H(x) & := (G \circ F)(x) \\ & = \bigcup_{y \in F(x)} G(y) \\ & = \operatorname{proj}_{\boldsymbol{y}_I}(F(x)) \end{array}$$

is ω-continuous. $\square$

Proposition 2.3 *The properties (PROD) together with (PROJ) and (GEN) imply (ENDOM).*

Proof: We assume an ω-continuous multi-valued mapping $F : \boldsymbol{x} \multimap \boldsymbol{z}$, and define the multi-valued mapping $G : \boldsymbol{y} \multimap \{0\}$ by $G(y) := \{0\}$. Then by (GEN) G is ω-continuous. Now we apply (PROD) to F and G and obtain an ω-continuous multi-valued mapping $\widehat{H} : \boldsymbol{x} \times \boldsymbol{y} \multimap \boldsymbol{z} \times \{0\}$ defined by

$$\widehat{H}(x, y) := F(x) \times \{0\}.$$

Now we apply (PROJ) and obtain that the multi-valued mapping $H : \boldsymbol{x} \times \boldsymbol{y} \multimap \boldsymbol{z}$ defined by

$$\begin{aligned} H(x, y) &= \mathrm{proj}_{\boldsymbol{z}}(F(x) \times \{0\}) \\ &= F(x), \end{aligned}$$

is ω-continuous, which completes the proof. □

2.1 ma-Continuous Multi-Valued Mappings

Many of the properties discussed here were introduced by Goldsztejn in [11]. The same paper also introduces and discusses a continuity notion called ma-continuity:

Definition 2.1 *Let $\boldsymbol{x} \in \mathbb{IR}^n$. A multi-valued mapping $F : \boldsymbol{x} \multimap \mathbb{R}^m$ is called* **ma-continuous** *(*manifold approximated*) if there exists*

- *a closed ball $\mathbf{d} \subseteq \mathbb{R}^n$ such that $\boldsymbol{x} \subseteq \mathrm{int}\mathbf{d}$,*
- *a sequence $(M_k)_{k\in\mathbb{N}}$ of C^∞ compact n-manifolds with boundary such that ∂M_k is homeomorphic to S^{n-1}, where S^{n-1} denotes the $n-1$-dimensional sphere, and*
- *a sequence $(g_k : M_k \to \mathbb{R}^{n+m})_{k\in\mathbb{N}}$ of C^∞ maps such that g_k restricted to ∂M_k is a C^∞ diffeomorphism between ∂M_k and $\partial\mathbf{d} \times \{0\}$,*

such that for any sequence $(x_k)_{k\in\mathbb{N}}$ of points in M_k,

- *the sequence $(g_k(x_k))_{k\in\mathbb{N}}$ is bounded, and*
- *if $g_k(x_k) \in \boldsymbol{x} \times \mathbb{R}^n$ for every $k \in \mathbb{N}$, then every accumulation point (x^*, y^*) of the sequence $(g_k(x_k))_{k\in\mathbb{N}}$ satisfies $y^* \in F(x^*)$.*

Goldsztejn proves the properties (GEN) (Proposition 4.1 in [11]) and and a generalization of (SETFP) (Theorem 3.1 in [11]).

3 c-Continuous Multi-Valued Mappings

The concept of c-continuity was introduced by Neumaier in [22]. It was used there in a context of interval arithmetic. We show that c-continuous multi-valued mappings have the properties (REDOM) and (PROJ) but do not have the properties (SETFP), (SETIV), (ENDOM), (PROD), (COMP) and (GEN). As pointed out before, the negative answer to the property (SETIV) is the most important one because in [22], c-continuous multi-valued mappings were claimed to possess the property (SETIV). Note that for single-valued mappings, there exists a notion related to the idea of c-continuity called 'connectivity maps', see Hamilton [14] and Jordan & Nadler [16].

Definition 3.1 *Let $A, B \subseteq \boldsymbol{x}$. A continuum $C \subseteq \boldsymbol{x}$* **connects** *the sets A with B if the sets $A \cap C$ and $B \cap C$ are both nonempty.*

Definition 3.2 *We say that the multi-valued mapping $F : \boldsymbol{x} \multimap \boldsymbol{y}$ is* **c-continuous** *(*connection continuous*) on a set $E \subseteq \boldsymbol{x}$ if for every continuum $C \subseteq E$ and any two points $\tau, \tau' \in C$, the set* $\mathrm{graph}(F|C)$ *contains a continuum $K \subseteq \boldsymbol{x} \times \boldsymbol{y}$ such that K connects the sets $\{\tau\} \times \boldsymbol{y}$ with $\{\tau'\} \times \boldsymbol{y}$.*

The next proposition gives a sufficient condition for a multi-valued mapping to be c-continuous.

Proposition 3.1 *Let $\boldsymbol{x} \in \mathbb{IR}^n$ and let $\boldsymbol{y} \in \mathbb{IR}^m$. Let $F : \boldsymbol{x} \multimap \boldsymbol{y}$ be a multi-valued mapping such that* $\mathrm{graph}(F|C)$ *is closed for every continuum $C \subseteq \boldsymbol{x}$, and $F(x)$ is connected for every $x \in \boldsymbol{x}$. Then F is c-continuous.*

Proof: Choose an arbitrary continuum $C \subseteq \boldsymbol{x}$ and $\tau, \tau' \in C$. Then $\mathrm{graph}(F|C)$ is closed by assumption. Suppose that $\mathrm{graph}(F|C)$ does not connect $\tau \times \mathbb{R}^m$ with $\tau' \times \mathbb{R}^m$. Since $\mathrm{graph}(F|C)$ is closed, there exist closed, disjoint sets $A_1, A_2 \subseteq \boldsymbol{x} \times \boldsymbol{y}$ such that $A_1 \cup A_2 = \mathrm{graph}(F|C)$. Since for every $x \in C$ the set $F(x)$ is connected by assumption, the set $\{x\} \times F(x)$ is either contained in A_1 or in A_2. Let

$$B_1 := \{x \in C \mid x \times F(x) \subseteq A_1\} \quad \text{and} \quad B_2 := \{x \in C \mid x \times F(x) \subseteq A_2\}.$$

Then B_1 and B_2 are disjoint sets that cover C. Since B_i is the projection of A_i to $\boldsymbol{x}$ for $i = 1, 2$, the sets B_1 and B_2 are also closed, hence C is not connected, a contradiction. □

(SETFP) Not every c-continuous multi-valued mapping satisfies the property (SETFP), as the following example shows.

Corollary 3.1 *Let $\boldsymbol{x} = [-1, 1]$, let $\boldsymbol{y} := [-1, 1] \times [-1, 1]$. For $r > 0$ and $\delta < \frac{r}{2}$ we define the multi-valued mapping $G : \boldsymbol{x} \times \boldsymbol{y} \multimap \boldsymbol{y}$ by*

$$G(x, y) := \overline{B_r}(0) \setminus B_\delta(y).$$

Then G is c-continuous.

Proof: Note that $G(x, y) \subseteq \boldsymbol{y}$ is connected and nonempty for all $(x, y) \in \boldsymbol{x} \times \boldsymbol{y}$ because $\delta < \frac{r}{2}$. To apply Proposition 3.1 it remains to show that $\mathrm{graph}(G|C)$ is closed for every continuum C.

$$\begin{aligned} \mathrm{graph}(G) &= \{(x, y) \times G(x, y) \mid x \in \boldsymbol{x}, y \in \boldsymbol{y}\} \\ &= \{(x, y) \times \overline{B_r}(0) \mid x \in \boldsymbol{x}, y \in \boldsymbol{y}\} \setminus \{(x, y) \times B_\delta(y) \mid x \in \boldsymbol{x}, y \in \boldsymbol{y}\}. \end{aligned}$$

We define $R := \{(x, y) \times \overline{B_r}(0) \mid x \in \boldsymbol{x}, y \in \boldsymbol{y}\}$ and $S := \{(x, y) \times B_\delta(y) \mid x \in \boldsymbol{x}, y \in \boldsymbol{y}\}$, hence $\mathrm{graph}(G) = R \setminus S$. Since $R = \boldsymbol{x} \times \boldsymbol{y} \times \overline{B_r}(0)$, the set R is closed. Via the homeomorphism $h : (x, y, z) \mapsto (x, y, z - y)$, S is homeomorphic to $\boldsymbol{x} \times \boldsymbol{y} \times B_\delta(0)$, and hence S is open, and $R \setminus S = \mathrm{graph}(G)$ is closed. The set $\mathrm{graph}(G|C)$ can be written as an intersection of closed sets $\mathrm{graph}(G) \cap C \times \boldsymbol{y}$ and is therefore closed. □

Example 3.1 *Let G be defined as in Corollary 3.1. By definition of H,*

$$\begin{aligned} H(x) &= \{y \in \boldsymbol{y} \mid y \in G(x, y)\} \\ &= \emptyset \end{aligned}$$

and H is not c-continuous, it is not even a multi-valued mapping.

(SETIV) In Neumaier [22], c-continuous multi-valued mappings were claimed to possess the property (SETIV) (Theorem 5.3.7) We give a counterexample.

Corollary 3.2 *Let $\boldsymbol{x} = [-1,1]$, let $\boldsymbol{y} := [-1,1] \times [-1,1]$. For $r > 0$ and $\delta < \frac{r}{2}$ we define the multi-valued mapping $G : \boldsymbol{x} \times \boldsymbol{y} \multimap \boldsymbol{z}$ by*

$$G(x,y) := \overline{B_r}(x) \setminus B_\delta(0).$$

Then G is c-continuous.

Proof: As in Corollary 3.1, $G(x,y)$ is connected and nonempty for all $x \in \boldsymbol{x}, y \in \boldsymbol{y}$. Again, it only remains to show that graph$(G|C)$ is closed for every continuum C.

$$\begin{aligned}\mathrm{graph}(G) &= \{(x,y) \times G(x,y) \mid x \in \boldsymbol{x}, y \in \boldsymbol{y}\}\\ &= \{(x,y) \times \overline{B_r}(x) \mid x \in \boldsymbol{x}, y \in \boldsymbol{y}\} \setminus \{(x,y) \times B_\delta(0) \mid x \in \boldsymbol{x}, y \in \boldsymbol{y}\}.\end{aligned}$$

We define $R := \{(x,y) \times \overline{B_r}(x) \mid x \in \boldsymbol{x}, y \in \boldsymbol{y}\}$ and $S := \{(x,y) \times B_\delta(0) \mid x \in \boldsymbol{x}, y \in \boldsymbol{y}\}$, hence graph$(G) = R \setminus S$. Via the homeomorphism $h : (x,y,z) \mapsto (x,y,z-y)$, R is homeomorphic to $\boldsymbol{x} \times \boldsymbol{y} \times \overline{B_r}(0)$, hence closed, and $S = \boldsymbol{x} \times \boldsymbol{y} \times B_\delta(0)$ is open. Hence $R \setminus S = \mathrm{graph}(G)$ is closed. Since $\mathrm{graph}(G|C) = \mathrm{graph}(G) \cap C \times \boldsymbol{y}$ is closed, Proposition 3.1 applies. □

Example 3.2 *Let G be defined as in Corollary 3.2, and choose $0 < r < 1$ such that*

$$\begin{aligned}&\sup(G_i(x,y)) \le 0 \quad \text{if } y_i = \underline{\boldsymbol{y}}_i,\\ &\inf(G_i(x,y)) \ge 0 \quad \text{if } y_i = \overline{\boldsymbol{y}}_i.\end{aligned}$$

By definition of H,

$$\begin{aligned}H(x) &= \{0 \in \boldsymbol{y} \mid y \in G(x,y)\}\\ &= \emptyset\end{aligned}$$

and H is not c-continuous.

The counterexample shows that the proof given by the second author in [22] cannot be correct. In fact, the stated equivalence of the term in line 15 with the term in line 16 on page 197 in [22] is incorrect. However, the error has only consequences for the following three theorems in [22], since the rest of the book only uses the well-known theorem of Leray & Schauder. Other parts than Section 5.3 of [22] are not concerned by the error.

(REDOM)

Proposition 3.2 *Every c-continuous multi-valued mapping has the property (REDOM).*

Proof: This property follows immediately from the definition of c-continuity. □

(ENDOM) Not every c-continuous multi-valued mapping satisfies the property (ENDOM). We give a counterexample, but first need two Lemmas by Neumaier [22] and a proposition that proves a certain multi-valued mapping to be c-continuous.

Lemma 3.1 *(*NEUMAIER *[22, Lemma 5.3.1.(iii)])*
Let A, B be closed subsets of $E \subseteq \mathbb{R}^n$. If E is compact and $C_l (l = 0, 1, 2, \ldots)$ is an infinite sequence of subsets of E such that each C_l connects A with B then the set C of all accumulation points of all sequences $t^l (l = 0, 1, 2, \ldots)$ with $t^l \in C_l$ for $l \geq 0$ connects A with B.

Lemma 3.2 *(*NEUMAIER *[22, Lemma 5.3.2.])*
Let a, b be closed intervals and let Σ be a closed subset of $a \times b$. If for every continuous mapping $\varphi : [0, 1] \to a \times b$ with $\mathrm{proj}_b(\varphi(0)) \in \underline{b}, \mathrm{proj}_b(\varphi(1)) \in \overline{b}$ there is a number $s \in [0, 1]$ such that $\varphi(s) \in \Sigma$ then Σ connects $\{\underline{a}\} \times b$ with $\{\overline{a}\} \times b$.

Proposition 3.3 *Let $\boldsymbol{x} = [-2, 2] \times [-2, 2], \boldsymbol{z} = [-2, 2]$. Let $\varphi : [0, 1] \to \boldsymbol{x} \times \boldsymbol{z}$ be defined by*

$$\varphi(t) := (\cos(3\pi t), \cos(2\pi t), 2t - 1).$$

For $0 < \varepsilon < \frac{1}{2}$ define the set $\Gamma \subseteq \boldsymbol{x} \times \boldsymbol{z}$ by

$$\Gamma := \boldsymbol{x} \times \boldsymbol{z} \ \setminus \ \mathcal{O}_\varepsilon(\mathrm{Im}\, \varphi).$$

Now let the multi-valued mapping $F : \boldsymbol{x} \multimap \boldsymbol{z}$ be defined by

$$F(x) := \mathrm{proj}_{\boldsymbol{z}}(\{x\} \times \boldsymbol{z} \ \cap \ \Gamma).$$

Then F is c-continuous.

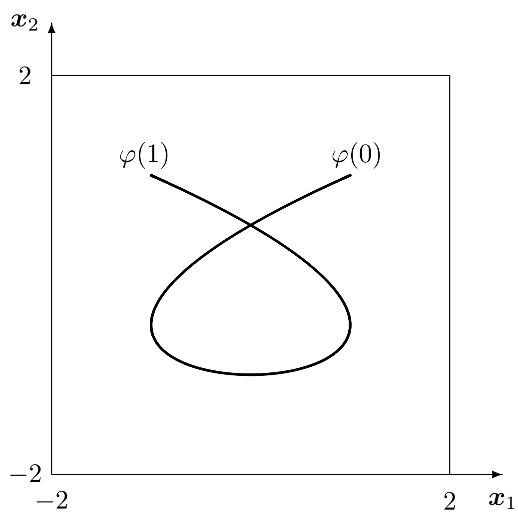

Figure 2: The projection of $\mathrm{Im}\, \varphi$ to $\boldsymbol{x}$

Proof:
(i) We first show that there exists no path α connecting $\boldsymbol{x} \times \underline{\boldsymbol{z}}$ with $\boldsymbol{x} \times \overline{\boldsymbol{z}}$, and $\mathrm{Im}\, \alpha \cap \mathrm{graph}\, F = \emptyset$ such that $\mathrm{proj}_{\boldsymbol{x}}(\mathrm{Im}\, \alpha)$ is not homeomorphic to $[0, 1]$:
Suppose that there exists such a path $\alpha : [0, 1] \to \boldsymbol{x} \times \boldsymbol{z}$. Then $\mathrm{Im}\, \alpha \subseteq \mathcal{O}_\varepsilon(\mathrm{Im}\, \varphi)$,

and for ε small enough, the projection of $\operatorname{Im}\alpha$ to $\boldsymbol{x}$ has at least one double-point (cf. Figure 2). Hence $\operatorname{proj}_{\boldsymbol{x}}(\operatorname{Im}\alpha)$ is not homeomorphic to $[0,1]$.

(ii) Next, by assuming that F is not c-continuous on the image of a path, we construct a path that contradicts (i):
Suppose that there exists a path $\beta : [0,1] \to \boldsymbol{x}$ and two points $\tau, \tau' \in \operatorname{Im}\beta$ such that $\operatorname{graph}(F|\operatorname{Im}\beta)$ does not connect $\{\tau\} \times \boldsymbol{z}$ with $\{\tau'\} \times \boldsymbol{z}$. W.l.o.g., the path β is double point-free, and hence the mapping $hom : [0,1] \times \boldsymbol{z} \to \operatorname{Im}\beta \times \boldsymbol{z}$ defined by $(t,z) \mapsto (\beta(t), z)$ is a homeomorphism. Since the homeomorphic image of a continuum is a continuum, the set $hom(\operatorname{graph}(F|\operatorname{Im}\beta)) \subseteq [0,1] \times \boldsymbol{z}$ does not connect $\{0\} \times \boldsymbol{z}$ with $\{1\} \times \boldsymbol{z}$ by assumption. Therefore, Lemma 3.2 applies ($\operatorname{graph} F = \Gamma$ and Γ is closed) and there exists a path $\gamma : [0,1] \to [0,1] \times \boldsymbol{z}$ connecting $[0,1] \times \underline{\boldsymbol{z}}$ with $[0,1] \times \overline{\boldsymbol{z}}$. But then the path $\delta := hom^{-1}(\gamma) : [0,1] \to \operatorname{Im}\beta \times \boldsymbol{z}$ is a path connecting $\boldsymbol{x} \times \underline{\boldsymbol{z}}$ with $\boldsymbol{x} \times \overline{\boldsymbol{z}}$, and $\operatorname{Im}\delta \cap \operatorname{graph} F = \emptyset$. Consequently, by (i), the set $\operatorname{proj}_{\boldsymbol{x}}(\operatorname{Im}\delta)$ cannot be homeomorphic to $[0,1]$ and neither can its superset $\operatorname{Im}\beta$, a contradiction.
Hence for every path $\beta : [0,1] \to \boldsymbol{x}$ connecting $\tau, \tau' \in \operatorname{Im}\beta$, the set $\operatorname{graph}(F|\operatorname{Im}\beta)$ connects $\{\tau\} \times \boldsymbol{z}$ with $\{\tau'\} \times \boldsymbol{z}$.

(iii) Since F is c-continuous on the image of any path, we derive that F is c-continuous:
Suppose that there exists a continuum $C \subseteq \boldsymbol{x}$ and $\tau, \tau' \in C$ such that $\operatorname{graph}(F|C)$ does not connect $\{\tau\} \times \boldsymbol{z}$ with $\{\tau'\} \times \boldsymbol{z}$. Let $C_\varepsilon := \mathcal{O}_\varepsilon(C)$ hence $(C_{1/n})_{n\in\mathbb{N}}$ is a sequence of open, connected sets such that $\tau, \tau' \in C_{1/n}$ for all $n \in \mathbb{N}$. Since every $C_{1/n}$ is also path-connected, $\operatorname{graph}(F|C_{1/n})$ connects $\{\tau\} \times \boldsymbol{z}$ with $\{\tau'\} \times \boldsymbol{z}$ by (ii). Since the sets $\{\tau\} \times \boldsymbol{z}$ and $\{\tau'\} \times \boldsymbol{z}$ are closed, we Lemma 3.1 applies for the sequence $(\operatorname{graph}(F|C_{1/n}))_{n\in\mathbb{N}}$. We obtain that the set of all accumulation points connects $\{\tau\} \times \boldsymbol{z}$ with $\{\tau'\} \times \boldsymbol{z}$. Furthermore, this set is a subset of $\operatorname{graph}(F|C)$, a contradiction to the assumption that $\operatorname{graph}(F|C)$ does not connect $\{\tau\} \times \boldsymbol{z}$ with $\{\tau'\} \times \boldsymbol{z}$. Hence no such C exists and F is c-continuous. $\square$

Now we give the counterexample to (ENDOM).
In 3.3 we saw that each continuum in the projection of the 'hole' in $\operatorname{graph}(F)$ to the (two dimensional) domain of F has a double-point. The idea is to enlarge the domain of the c-continuous multi-valued mapping F to three dimensions, such that such a double-point can be avoided, proving that this 'enlarged' multi-valued mapping is not c-continuous.

Example 3.3 *Let $\boldsymbol{x}, \boldsymbol{z}$ and $F : \boldsymbol{x} \multimap \boldsymbol{z}$ be the multi-valued mapping defined in 3.3, let $\boldsymbol{y} = [0,1]$. Then the multi-valued mapping $H : \boldsymbol{x} \times \boldsymbol{y} \multimap \boldsymbol{z}$ defined by*

$$H(x,y) := F(x)$$

is not c-continuous.

Proof: Let $\psi : [0,1] \to \boldsymbol{x} \times \boldsymbol{y}$ be defined by

$$\psi(s) := (\cos(3\pi s), \cos(2\pi s), s).$$

Let $C := \operatorname{Im}\psi \subseteq \boldsymbol{x} \times \boldsymbol{y}$ and $\tau = \psi(0), \tau' = \psi(1)$ points in C.
We show that the set $D := \{(\psi(s), 2s-1) \mid s \in [0,1]\} \subseteq \boldsymbol{x} \times \boldsymbol{y} \times \boldsymbol{z}$ does not intersect $\operatorname{graph}(H|C)$:
For an arbitrary $s \in [0,1]$,

$$\begin{aligned} H(\psi(s)) &= F(\cos(3\pi s), \cos(2\pi s)) \\ &= \operatorname{proj}_{\boldsymbol{z}}(\cos(3\pi s), \cos(2\pi s) \times \boldsymbol{z} \cap \Gamma) \\ &= \boldsymbol{z} \setminus \operatorname{proj}_{\boldsymbol{z}}(\mathcal{O}_\varepsilon(2s-1)). \end{aligned}$$

Hence for all $s \in [0,1]$, the point $(\psi(s), 2s-1)$ is not contained in $H(\psi(s))$, and consequently $D \cap \mathrm{graph}(H|C) = \emptyset$.
Since ψ is injective, $hom : [0,1] \times \boldsymbol{z}$ defined by $hom(s,z) = (\psi(s), z)$ is a homeomorphism. The set $hom(D)$ connects $[0,1] \times \underline{\boldsymbol{z}}$ with $[0,1] \times \overline{\boldsymbol{z}}$ without intersecting $hom(\mathrm{graph}(H|C))$, hence $hom(\mathrm{graph}(H|C))$ cannot connect $\{0\} \times \boldsymbol{z}$ with $\{1\} \times \boldsymbol{z}$. Therefore $\mathrm{graph}(H|C)$ does not connect $\{\tau\} \times \boldsymbol{z}$ with $\{\tau'\} \times \boldsymbol{z}$, and H is not c-continuous. $\square$

(PROD)

Corollary 3.3 *Not every c-continuous multi-valued mapping has the property (PROD).*

Proof: By Proposition 2.3, (PROD) together with (PROJ) would imply (ENDOM). $\square$

(PROJ)

Proposition 3.4 *(*NEUMAIER *[22, Corollary 5.3.6])*
All c-continuous multi-valued mappings have the property (PROJ).

(GEN)

Proposition 3.5 *(*NEUMAIER *[22, Proposition 5.3.3]) All c-continuous multi-valued mappings have the property (GEN).*

(COMP)

Corollary 3.4 *Not every c-continuous multi-valued mapping has the property (COMP).*

Proof: By Proposition 2.1, (COMP) together with (GEN) would imply (ENDOM). $\square$

4 is-Continuous Multi-Valued Mappings

We introduce a continuity notion called is-continuity. A multi-valued mapping is is-continuous if there exists a continuous single-valued mapping f such that the set of the zeros of f is a subset of graph F, plus assumptions to make sure that the set of the zeros of f is not empty. (We remind that if $F(x)$ would be empty for some x, then F would not be a multi-valued mapping.) We show that is-continuous multi-valued mappings have the properties (SETFP), (REDOM), (ENDOM), (PROD) and (GEN). Instead of (SETIV), which remains an open question, we prove the weaker version (SETIV'). The properties (PROJ) and (COMP) also remain open questions.

Definition 4.1 *Let $X \subseteq \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{IR}^m$. A multi-valued mapping $F : X \multimap \boldsymbol{y}$ is called* **is-continuous** *(*implicit selectionable*) on a continuum $C \subseteq X$ if there exists a continuous single-valued mapping $f : C \times \boldsymbol{y} \to \mathbb{R}^m$, called an* **implicit selection***, such that*

$$\begin{aligned} f_i(x,y) \geq 0 \quad &\text{for} \quad y_i = \overline{\boldsymbol{y}}_i, x \in C, \\ f_i(x,y) \leq 0 \quad &\text{for} \quad y_i = \underline{\boldsymbol{y}}_i, x \in C \end{aligned}$$

and

$$f^{-1}(0) \subseteq \mathrm{graph}(F|C).$$

If $X = \boldsymbol{x}$ is a box and if $F : \boldsymbol{x} \multimap \boldsymbol{y}$ is is-continuous on the continuum $C = \boldsymbol{x}$, we simply say that F is is-continuous.

Remark 4.1 *For an is-continuous multi-valued mapping $F : X \multimap \boldsymbol{y}$ and for a fixed $x \in X$ we can apply the single-valued multivariate intermediate value theorem [21] to the continuous single-valued mapping f, and obtain that the set $\{y \in \boldsymbol{y} \mid f(x,y) = 0\}$ is nonempty. This implies that $F(x)$ is nonempty for every $x \in X$.*

(SETFP)

Theorem 4.1 *Every is-continuous multi-valued mapping has the property (SETFP).*

Proof: Let $f : \boldsymbol{x} \times \boldsymbol{y} \times \boldsymbol{y} \to \mathbb{R}^m$ be the implicit selection for the given multi-valued mapping $F : \boldsymbol{x} \times \boldsymbol{y} \multimap \boldsymbol{y}$. We define the single-valued mapping $h : \boldsymbol{x} \times \boldsymbol{y} \to \mathbb{R}^m$ by

$$h(x,y) := f(x,y,y).$$

The mapping h is continuous because f is continuous, and

$$\begin{array}{ll} h_i(x,y) = f_i(x,y,y) \geq 0 & \text{for} \quad y_i = \overline{\boldsymbol{y}}_i, \\ h_i(x,y) = f_i(x,y,y) \leq 0 & \text{for} \quad y_i = \underline{\boldsymbol{y}}_i. \end{array}$$

Hence it only remains to show that $h^{-1}(0) \subseteq \operatorname{graph} H$:

If $h(x,y) = 0$ then by definition $f(x,y,y) = 0$ and hence $y \in F(x,y)$ and $y \in H(x)$. □

(SETIV') Let $F : \boldsymbol{x} \times \boldsymbol{y} \multimap \boldsymbol{z}$ with $\dim \boldsymbol{y} = \dim \boldsymbol{z}$ be an is-continuous multi-valued mapping, and let

$$\begin{array}{ll} \sup(F_i(x,y)) < 0 & \text{if } y_i = \underline{\boldsymbol{y}}_i, \\ \inf(F_i(x,y)) > 0 & \text{if } y_i = \overline{\boldsymbol{y}}_i. \end{array}$$

Then the multi-valued mapping $H : \boldsymbol{x} \multimap \boldsymbol{y}$ defined by

$$H(x) := \{y \in \boldsymbol{y} \mid 0 \in F(x,y)\}$$

is is-continuous.

Proof: Let $f : \boldsymbol{x} \times \boldsymbol{y} \times \boldsymbol{z} \to \mathbb{R}^{\dim \boldsymbol{z}}$ be an implicit selection of F. Since $f^{-1}(0) \subseteq \operatorname{graph}(F)$ and f is continuous, there is an $\varepsilon > 0$ such that $f(x,y,z) \neq 0$ if for all i either $y_i - \underline{\boldsymbol{y}}_i < \varepsilon$ and $z_i \leq 0$ or $\overline{\boldsymbol{y}}_i - y_i < \varepsilon$ and $z_i \geq 0$.

We define a the linear mapping $\operatorname{tr} : \boldsymbol{y} \to \boldsymbol{z}$ by $\operatorname{tr}(y)_i := \underline{\boldsymbol{z}}_i + \frac{y_i - \underline{\boldsymbol{y}}_i}{\overline{\boldsymbol{y}}_i - \underline{\boldsymbol{y}}_i}(\overline{\boldsymbol{z}}_i - \underline{\boldsymbol{z}}_i)$.

We define $\boldsymbol{y}_i^\varepsilon := [\underline{\boldsymbol{y}}_i + \varepsilon, \overline{\boldsymbol{y}}_i - \varepsilon]$, and we define $z^\varepsilon = \operatorname{tr}(y^\varepsilon)$.

For $z \in \boldsymbol{z} \setminus 0$ we define λ_z to be the positive number with $\lambda_z \cdot z \in \partial \boldsymbol{z}$ and λ_z^ε to be the positive number with $\lambda_z^\varepsilon \cdot z \in \partial \boldsymbol{z}^\varepsilon$. Note that such numbers exist since $0 \in \boldsymbol{z}^\varepsilon$, and furthermore $\lambda_z > \lambda_z^\varepsilon$ for all z. We define the mapping $l : \boldsymbol{z} \setminus \boldsymbol{z}^\varepsilon \to \boldsymbol{z} \setminus 0$ by $l(z) := \frac{\lambda - \lambda^\varepsilon \lambda}{\lambda - \lambda^\varepsilon} z$ (see illustration in Figure 3).

Now we can explicitly define an implicit selection $h : \boldsymbol{x} \times \boldsymbol{y} \to \mathbb{R}^{\dim \boldsymbol{y}}$ for H by:

$$h(x,y) \quad = \begin{cases} f(x,y,0) & \text{for } y \in \boldsymbol{y}^\varepsilon, \\ f(x,y,l(\operatorname{tr}(y))) & \text{otherwise.} \end{cases}$$

(i) Since f, tr and l are continuous, $f(x,y,l(\operatorname{tr}(y)))$ is continuous. Assume a sequence $(x,y)_n$ that converges to (x^*,y^*) with $\operatorname{tr}(y^*) \in \partial \boldsymbol{z}^\varepsilon$. Then $\lambda^\varepsilon_{\operatorname{tr}(y_n)} \to 1$ and $l(\operatorname{tr}(y_n)) \to 0$, hence h is continuous.

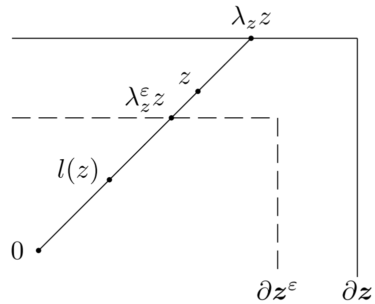

Figure 3: Illustration of the mapping l

(ii) For y with $y_i = \underline{\boldsymbol{y}}_i$, $\mathrm{tr}(y)_i = \underline{\boldsymbol{z}}_i$, and $h(x, y) \leq 0$. Analogously $h(x, y) \geq 0$ for $y_i = \overline{\boldsymbol{y}}_i$.

(iii) For $y \in \boldsymbol{y}^\varepsilon$, $h(x, y) = 0$ implies $f(x, y, 0) = 0$. If $y \notin \boldsymbol{y}^\varepsilon$, then either $y_i - \underline{\boldsymbol{y}}_i < \varepsilon$ or $\overline{\boldsymbol{y}}_i - y_i < \varepsilon$ for some i. In the first case, $\mathrm{tr}(y)_i < 0$, hence also $l(\mathrm{tr}(y))_i < 0$, and therefore $f(x, y, l(\mathrm{tr}(y))) \neq 0$, analogously for the second case. Therefore $h(x, y) = 0$ is only possible if $f(x, y, 0) = 0$, hence $0 \in F(x, y)$, and $h^{-1}(0) \subseteq \mathrm{graph}(H)$. $\square$

Note that the continuity of h is not guaranteed for the limit $\varepsilon \to 0$. Therefore we cannot prove (SETIV) as a limit of (SETIV').

(REDOM)

Proposition 4.1 *Every is-continuous multi-valued mapping has the property (REDOM)*

Proof: Let $f : C \times \boldsymbol{y} \to \mathbb{R}^m$ be an implicit selection of F. Let C' be a continuum contained in C. Then the single-valued mapping $h : C' \times \boldsymbol{y} \to \mathbb{R}^m$ defined by $h := f_{|C' \times \boldsymbol{y}}$ (i.e., the restriction of f to the set $C' \times \boldsymbol{y}$) satisfies:

$$\begin{aligned} h_i(x, y) = f_i(x, y) \geq 0 \quad &\text{for} \quad y_i \in \overline{\boldsymbol{y}}_i, x \in C' \\ h_i(x, y) = f_i(x, y) \leq 0 \quad &\text{for} \quad y_i \in \underline{\boldsymbol{y}}_i, x \in C' \end{aligned}$$

and

$$\begin{aligned} h^{-1}(0) \quad &= f^{-1}(0) \ \cap \ C' \times \boldsymbol{y} \\ &\subseteq \mathrm{graph}(F|C) \ \cap \ C' \times \boldsymbol{y} \\ &= \mathrm{graph}(F|C'). \end{aligned}$$

Obviously h is continuous, hence F is is-continuous on the continuum $C' \subseteq C$. $\square$

(ENDOM)

Proposition 4.2 *Every is-continuous multi-valued mapping has the property (ENDOM)*

Proof: Let $\boldsymbol{x} \in \mathbb{IR}^n$, $\boldsymbol{y} \in \mathbb{IR}^m$, $\boldsymbol{z} \in \mathbb{IR}^k$ and let $F : \boldsymbol{x} \multimap \boldsymbol{z}$ be is-continuous. Let $f : \boldsymbol{x} \times \boldsymbol{z} \multimap \mathbb{R}^k$ be an implicit selection of F. We define the single-valued mapping $h : \boldsymbol{x} \times \boldsymbol{y} \times \boldsymbol{z} \to \mathbb{R}^k$ by

$$h(x, y, z) := f(x, z).$$

Then h is obviously continuous and satisfies

$$h_i(x,y,z) = f_i(x,z) \geq 0 \quad \text{for} \quad z_i \in \overline{\boldsymbol{z}}_i,$$
$$h_i(x,y,z) = f_i(x,z) \leq 0 \quad \text{for} \quad z_i \in \underline{\boldsymbol{z}}_i.$$

It remains to show that $h^{-1}(0) \subseteq \operatorname{graph} H$:

If $h(x,y,z) = 0$ then $f(x,z) = 0$ and hence $z \in F(x)$ and $z \in H(x,y)$. $\square$

(PROD)

Proposition 4.3 *Every is-continuous multi-valued mapping has the property (PROD)*

Proof: For $\dim(\boldsymbol{u}) = k$ and $\dim(\boldsymbol{v}) = l$ there exist by assumption single-valued mappings $f : \boldsymbol{x} \times \boldsymbol{u} \to \mathbb{R}^k$ and $g : \boldsymbol{y} \times \boldsymbol{v} \to \mathbb{R}^l$ with $f^{-1}(0) \subseteq \operatorname{graph} F$ and $g^{-1}(0) \subseteq \operatorname{graph} G$. We define a single-valued mapping $h : \boldsymbol{x} \times \boldsymbol{y} \times \boldsymbol{u} \times \boldsymbol{v} \to \mathbb{R}^k \times \mathbb{R}^l$ by

$$h(x,y,u,v) := f(x,u) \times g(y,v).$$

Then h is continuous since f and g are both continuous. By definition, for $i = 1,\dots,k$ we have

$$h_i(x,y,u,v) = f_i(x,u), \qquad (u,v)_i = u_i, \qquad (\overline{\boldsymbol{u} \times \boldsymbol{v}})_i = \overline{\boldsymbol{u}}_i,$$

and for $i = k+1,\dots,k+l$ we have

$$h_i(x,y,u,v) = g_{i-k}(y,v), \qquad (u,v)_i = v_{i-k}, \qquad (\overline{\boldsymbol{u} \times \boldsymbol{v}})_i = \overline{\boldsymbol{v}}_{i-k}.$$

Hence $h_i(x,y,u,v) \geq 0$ for $(u,v)_i \in (\overline{\boldsymbol{u} \times \boldsymbol{v}})_i$ because

$$h_i(x,y,u,v) \quad = \begin{cases} f_i(x,u) \geq 0 & \text{for } u_i \in \overline{\boldsymbol{u}}_i, \\ g_{i-k}(y,v) \geq 0 & \text{for } v_{i-k} \in \overline{\boldsymbol{v}}_{i-k}. \end{cases}$$

Similarly, we obtain that $h_i(x,y,u,v) \leq 0$ for $(u,v)_i \in (\underline{\boldsymbol{u} \times \boldsymbol{v}})_i$.

Again, it only remains to show that $h^{-1}(0) \subseteq \operatorname{graph} H$:

From $h(x,y,u,v) = 0$ we infer $f(x,u) = 0$ and $g(y,v) = 0$ and furthermore $u \in F(x)$ and $v \in G(y)$ and finally $(u,v) \in H(x,y)$. $\square$

(GEN)

Proposition 4.4 *Every is-continuous multi-valued mapping has the property (GEN).*

Proof: Let $m := \dim \boldsymbol{y}$. Define the single-valued mapping $g : X \times \boldsymbol{y} \to \mathbb{R}^m$ by

$$g(x,y) := y - f(x).$$

For $y_i \in \overline{\boldsymbol{y}}_i$ we have $f_i(x) \leq y_i$ and hence $g(x,y) = y_i - f_i(x) \geq 0$, and
for $y_i \in \underline{\boldsymbol{y}}_i$ we have $f_i(x) \geq y_i$ and hence $g(x,y) = y_i - f_i(x) \leq 0$.
Obviously, $g(x,y) = 0$ if and only if $y \in F(x) := \{f(x)\}$, and hence $g^{-1}(0) = \operatorname{graph} F$.
Therefore, F is is-continuous. $\square$

The properties (SETIV), (PROJ) and (COMP) are open questions.

We give an additional proposition which will be applied in Section 5.

Proposition 4.5 *Let $F : \boldsymbol{x} \times \boldsymbol{z} \multimap \boldsymbol{y}$ and $G : \boldsymbol{y} \times \boldsymbol{z} \multimap \boldsymbol{x}$ be is-continuous multi-valued mappings. Then the multi-valued mapping $H : \boldsymbol{z} \multimap \boldsymbol{x} \times \boldsymbol{y}$ defined by*

$$H(z) := \operatorname{graph}(F|\{z\}) \cap \operatorname{graph}(G|\{z\})$$

is is-continuous.

Proof: Let $f : \boldsymbol{x} \times \boldsymbol{z} \times \boldsymbol{y} \to \mathbb{R}^{\dim \boldsymbol{y}}$ and $g : \boldsymbol{y} \times \boldsymbol{z} \times \boldsymbol{x} \to \mathbb{R}^{\dim \boldsymbol{x}}$ be the implicit selections for F and G respectively. We define the mapping $h : \boldsymbol{z} \times \boldsymbol{x} \times \boldsymbol{y} \to \mathbb{R}^{\dim \boldsymbol{x} + \dim \boldsymbol{y}}$ by

$$h(z, x, y) = (f(x, z, y), g(y, z, x))$$

and show that h is an implicit selection for H.

Obviously, h is continuous, and for $(x, y) \in \overline{\boldsymbol{x} \times \boldsymbol{y}}_i$ we obtain

$$h_i(z, x, y) \quad = \begin{cases} f_i(x, z, y) \geq 0 & \text{if } 1 \leq i \leq \dim \boldsymbol{y}, \\ g_i(y, z, x) \geq 0 & \text{if } \dim \boldsymbol{y} < i \leq \dim \boldsymbol{x} + \dim \boldsymbol{y} \end{cases}$$

and analogous we obtain that $h_i \leq 0$ for $z \in \underline{\boldsymbol{x} \times \boldsymbol{y}}_i$.

Hence it remains to show that $h^{-1}(0) \subseteq \operatorname{graph} H$:
$h(z, x, y) = 0$ implies both $f(x, z, y) = 0$ and $g(y, z, x) = 0$. Hence $y \in F(x, z)$ and $x \in G(y, z)$ and therefore $(x, y) \in H(z)$. □

5 An Application in Logic

This section is concerned with the article Ratschan [26], which uses Theorem 5.3.7 in [22] (that is (SETIV) for c-continuous multi-valued mappings) which is, as we saw in Chapter 2, not valid in general. We give a theorem analogous to the central theorem in [26], but we use is-continuous multi-valued mappings instead of c-continuous ones.

Ratschan's articles [25] and [26] deal with **constraints**, i.e., formulas in first-order predicate language.

Let φ be a constraint with variable $x = (x_1, \ldots, x_n)$ and depending on parameters $z = (z_1, \ldots, z_k)$. Let the possible values of x and z be restricted to boxes $\boldsymbol{x} \in \mathbb{IR}^n$ and $\boldsymbol{z} \in \mathbb{IR}^k$ respectively. We assume that for every parameter z there exists a point x such that φ is true, but not necessarily vice versa.

A **witness function** for a constraint φ is a mapping f such that φ is true for every point in $\operatorname{Im} f$. The central statement in [26] is that if there exists a witness function for the constraint φ_1 and one for the constraint φ_2, then there exists a witness function for the constraint $\varphi_1 \wedge \varphi_2$. We reformulate Ratschan's theorem using is-continuous multi-valued mappings and give a proof.

Definition 5.1 *Given a constraint φ with variables in $\boldsymbol{x}$ and parameters in $\boldsymbol{y}$, we say that $F : \boldsymbol{y} \multimap \boldsymbol{x}$ is an* **is-continuous witness multimapping** *for φ if F is is-continuous, and φ is true for all $(x, y) \in \operatorname{graph} F$.*

Theorem 5.1 *If for the constraints φ_1 with variables in $\boldsymbol{x}$ and parameters in $\boldsymbol{y} \times \boldsymbol{z}$ and for φ_2 with variables in $\boldsymbol{y}$ and parameters in $\boldsymbol{x} \times \boldsymbol{z}$ there exist is-continuous witness multimappings, then there also exists an is-continuous multimapping for the constraint $\varphi_1 \wedge \varphi_2$ with variables in $\boldsymbol{x} \times \boldsymbol{y}$ and parameters in $\boldsymbol{z}$.*

Proof: Given witness multimappings F for φ_1 and G for φ_2, we apply Proposition 4.5 and obtain an is-continuous multi-valued mapping $H : \boldsymbol{z} \multimap \boldsymbol{x} \times \boldsymbol{y}$. Since by definition, $H(z) = \operatorname{graph}(F|\{z\}) \cap \operatorname{graph}(G|\{z\})$, for all $(x, y, z) \in \operatorname{graph}(H)$, both $(x, y, z) \in \operatorname{graph}(F)$ and $(x, y, z) \in \operatorname{graph}(G)$ hold. Hence φ_1 and φ_2 are true for points in $\operatorname{graph} H$. Hence H is a witness multimap for $\varphi_1 \wedge \varphi_2$. □

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