illustrates that cancellation does not fully occur in subtraction as defined by Equation (1.4). However, a fifth operation, . *cancellation subtraction*, is appropriate in various contexts. Cancellation subtraction is defined by the equation

$$[\underline{x}, \overline{x}] \ominus [\underline{y}, \overline{y}] = [\underline{x} - \underline{y}, \overline{x} - \overline{y}]. \tag{1.10}$$

For example, if interval sums $s_j = \sum_{i=1, i\neq j}^{n} \boldsymbol{x}_i$ are required for each j, the overall sum $s = \sum_{i=1}^{n} \boldsymbol{x}_i$ can first be computed, then the individual s_j can be obtained with cancellation subtraction.

Additional elementary properties of interval arithmetic are listed in [8], [77], [83], [165], and [175].

1.1.2 Notation

Throughout, boldface will denote intervals, lower case will denote scalar quantities, and upper case will denote vectors and matrices. Brackets "$[\cdot]$" will delimit intervals while parentheses "$(\cdot)$" will delimit vectors and matrices. Underscores will denote lower bounds of intervals and overscores will denote upper bounds of intervals. Corresponding lower case letters will denote components of vectors. For example, we may have:

$$\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)^T,$$

where $\boldsymbol{x}_i = [\underline{x}_i, \overline{x}_i]$. If $\boldsymbol{X}$ is an interval vector or matrix, then $\boldsymbol{X} = [\underline{X}, \overline{X}]$, where $\underline{X}$ is the vector or matrix whose components are lower bounds of corresponding components of $\boldsymbol{X}$, and $\overline{X}$ is the vector or matrix whose components are upper bounds of corresponding components of $\boldsymbol{X}$. For example, if $\boldsymbol{X} = ([1,2],[3,4])^T$, then $\underline{X} = (1,3)^T$ and $\overline{X} = (2,4)^T$.

The symbol $\check{x}$ will denote a representative point, usually in $\boldsymbol{x}$ and often its center. Similarly, $\check{X}$ will denote a representative point for the box $\boldsymbol{X}$. The actual center, or midpoint, of an interval $\boldsymbol{x}$ will be denoted by $\mathrm{m}(\boldsymbol{x})$, and the vector or matrix whose entries are midpoints of the entries of the vector or matrix $\boldsymbol{X}$ will be denoted by $\mathrm{m}(\boldsymbol{X})$. The *magnitude* of an interval is defined as $|\boldsymbol{x}| = \max\{|\underline{x}|, |\overline{x}|\}$. The magnitude of an interval vector or matrix will be interpreted componentwise:

$$|\boldsymbol{X}| = (|\boldsymbol{x}_1|, |\boldsymbol{x}_2|, \ldots, |\boldsymbol{x}_n|)^T,$$

while the norm of an interval vector is defined as $\|\boldsymbol{X}\| = \big\| |\boldsymbol{X}| \big\|_\infty$. The *mignitude* of an interval $\boldsymbol{x}$ will be defined by $<\!\boldsymbol{x}\!> = \min_{x\in\boldsymbol{x}} |x|$.

The *width* of an interval $\boldsymbol{x}$ is denoted by $\mathrm{w}(\boldsymbol{x}) = \overline{x} - \underline{x}$. The width of an interval vector $\boldsymbol{X}$, denoted by $\mathrm{w}(\boldsymbol{X})$, is defined componentwise. We use $\mathrm{w}(\boldsymbol{X})$ in the context of $\|\mathrm{w}(\boldsymbol{X})\| = \|\mathrm{w}(\boldsymbol{X})\|_\infty$.

The set of intervals will be denoted $\mathbb{IR}$, the set of n-dimensional interval vectors, also called *boxes*, will be denoted by $\mathbb{IR}^n$, and the set of m by n matrices whose entries are intervals will be denoted by $\mathbb{IR}^{m\times n}$. The set corresponding to $\mathbb{IR}^2$, when identified with rectangles in the complex plane, will be denoted by $\mathbb{IC}$.

Arithmetic operations involving both real numbers and intervals will occur. In these, such as $[1,2]+1 = [2,3]$, the real number 1 is interpreted as a *thin interval* $[1,1]$. (Thin intervals are simply intervals of width zero.)

The topological interior of a set $\mathbf{D}$ will be denoted by $\mathrm{int}(\mathbf{D})$. Unless otherwise stated, such sets (and in particular, intervals) will be considered to be topologically closed. The boundary of a set $\mathbf{D}$ will be denoted by $\boldsymbol{\partial}\mathbf{D}$.

If f is a function defined over an interval $\boldsymbol{x}$, then $\boldsymbol{f}^u(\boldsymbol{x})$ denotes[2] the range of f over $\boldsymbol{x}$. The ranges of functions $\phi : \mathbb{R}^n \to \mathbb{R}$ and $F : \mathbb{R}^n \to \mathbb{R}^m$ are similarly denoted as $\boldsymbol{\phi}^u$ and $\boldsymbol{F}^u$, respectively. The gradient of ϕ is denoted by $\nabla\phi$, while the generic notation $F'(X)$ represents the m by n Jacobi matrix of a function $F : \mathbb{R}^n \to \mathbb{R}^m$.

Comparison of intervals will be as sets. For example $\boldsymbol{x} < \boldsymbol{y}$ will mean that every element of $\boldsymbol{x}$ is less than every element of $\boldsymbol{y}$. Comparison of interval vectors $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{IR}^n$ will be componentwise, in the sense of a relation as in [133, pp. 129–130]. For example, $\boldsymbol{X} \le \boldsymbol{Y}$ will mean that $\boldsymbol{x}_i \le \boldsymbol{y}_i$ for i between 1 and n, while $\boldsymbol{X} < \boldsymbol{Y}$ will mean $\boldsymbol{x}_i \le \boldsymbol{y}_i$ for i between 1 and n, and $\boldsymbol{x}_i \ne \boldsymbol{y}_i$ for at least one i. The convex hull of two intervals $\boldsymbol{x}$ and $\boldsymbol{y}$ is an interval that will be denoted by $\boldsymbol{x} \underline{\cup} \boldsymbol{y}$.

Often, sequences of intervals $\boldsymbol{x}_i$, as well as interval vectors and interval matrices, will be considered. This leads to

Definition 1.1 *Let* $\{\boldsymbol{x}_i\}_{i=1}^\infty = \{[\underline{x}_i, \overline{x}_i]\}_{i=1}^\infty$ *be a sequence of intervals. The sequence is said to* converge *to an interval* $\boldsymbol{x} = [\underline{x}, \overline{x}]$ *provided* $\underline{x}_i \to \underline{x}$ *and* $\overline{x}_i \to \boldsymbol{x}_i$ *in the standard topology on the real line. Similarly, a sequence of intervals is said to converge to a real number* r *provided* $\underline{x}_i \to r$ *and* $\overline{x}_i \to r$. *A sequence of vectors or matrices is said to converge to a vector or matrix if each*

[2] The notation is suggestive of "united" interval extension, a term first used by Moore.

of its components or coefficients converges to the corresponding component or coefficient of the limit vector or matrix.

For example, we may say that a set of interval matrices $\boldsymbol{A}_k$ converges to the identity matrix I.

1.1.3 Rounded Interval Arithmetic

Computations in §1.1.1 illustrated how real interval arithmetic could compute rigorous bounds on the ranges of functions, assuming infinitely precise computations. However, if such an arithmetic is implemented on the computer using, for example, equations (1.5) through (1.9), then the default rounding may result in non-rigor. For example, the range of the expression $[0.123, 0.456] + [0.0116, 0.0214]$ is $[0.1346, 0.4774]$. But this would be rounded to $[0.135, 0.477]$ with three digit decimal arithmetic and rounding to nearest, and $[0.1346, 0.4774] \not\subseteq [0.135, 0.477]$. Nonetheless, with *directed rounding,* such bounds can be computed rigorously. In particular, if instead of rounding to nearest, the lower bound of the interval is rounded down to the largest machine number less than the exact result and the upper bound is rounded up to the smallest machine number greater than the actual result, then the computed interval necessarily contains the exact range. In our example, the result would be $[0.134, 0.475]$, and $[0.1346, 0.4664] \subset [0.134, 0.475]$. This process is called *outward rounding,* and the resulting widening of the intervals is called *roundout error.*

Thus, with directed rounding, a *machine interval arithmetic* can be defined, such that the result of the four elementary operations contains the result that would be obtained with real interval arithmetic. Mathematical development of this concept is found in [147] and other works of Kulisch *et al.* For notational simplicity, a distinction between the space $\mathbb{IR}$ and the corresponding set of machine intervals will not be made explicit, but should be inferred from the context.

The IEEE binary floating point standard [227] prescribes three rounding modes: nearest, round down, and round up. Thus, rounding modes suitable for interval arithmetic are available on a wide variety of machines. However, standardized programming language support for switching of rounding modes is not available, although the Fortran 90 intrinsic NEAREST [4, p. 636] comes close. Transportable implementations of interval arithmetic thus often resort to mixed assembly language programming. Notwithstanding, if the tightest possible bounds on ranges are not required, then rounding down and rounding