Mamdani's Rule: a "Weird" Use of "And" as Implication Justified by Modern Logic

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***Abstract*— One of the most widely used methodologies of fuzzy control has been proposed by Mamdani in his pioneer 1974 paper. This methodology contains a step, on which implication is interpreted as "and". This step is informally (and rather convincingly) justified, but from the viewpoint of classical logic it looks really weird. In the present paper we show that if instead of classical logic, we use modern non-monotonic logics (that describe commonsense reasoning better), then this seemingly weird step can be formally justified.**

I. Introduction

A. Fuzzy Control and Mamdani's Rule

One of the main applications of fuzzy logic is fuzzy control. Fuzzy control is a methodology that enables us to translate the rules formulated by an expert in terms of natural language, into an actual control strategy, i.e., into a function that transforms the results $\vec{x} = (X_1, ..., x_n)$ of measurements into the recommended control value u. The general idea of fuzzy control goes back to [7], [1], [8]. In 1974, E. Mamdani developed these ideas into a detailed methodology that immediately lead to successful real-life applications [3, 4]. This methodology is still one of the most frequently used in fuzzy control [2, 5].

This methodology is very successful, but it has a serious methodological problem: in applying it, we at one step interpret "implies" as "and" (details will be given in a moment). In the context of fuzzy control, this interpretation appears very naturally, but still, from the viewpoint of logic, it is reasonably weird: both in classical logic and in traditional fuzzy logic, implication and conjunction are two different operations. To formulate this problem in some detail, let us recall this step of fuzzy control methodology.

Fuzzy control starts with the rules formulated by an expert. These rules are usually formulated as "if–then" rules that relate the results x_i of the measurement with the recommended control. These rules are of the type "if x is small, then u should be very small". In general, if we denote the number of the rules by r, the condition of $j-$th rule by $A_j(\vec{x})$, and the conclusion of each rule by $B_j(u)$, then the starting set of rules can be written down as a sequence of implications:

$$A_1(\vec{x}) \rightarrow B_1(u);$$
$$\ldots \tag{1}$$
$$A_r(\vec{x}) \rightarrow B_r(u).$$

For example, the above rule corresponds to $A_1 =$"small" and $B_1 =$"very small". Based on these rules, we must describe when u is a reasonable control for a given input $\vec{x}$. Let us denote the fact that u is a reasonable control for a given $\vec{x}$, by $R(\vec{x}, u)$. Mamdani uses the following natural idea to describe this new predicate R: u is a reasonable control if it has been obtained by one of the expert's rules. In other words:

- either first rules has been applied, in which case, $\vec{x}$ satisfies the property A_1, and the resulting control satisfies the property B_1;
- or, the second rule has been applied, in which case, $\vec{x}$ satisfies the property A_2, and the resulting control u satisfies the property B_2;
- etc.

Formally, this idea can be described as follows: $R(\vec{x}, u)$ is defined as

$$(A_1(\vec{x}) \& B_1(u)) \vee \ldots \vee (A_r(\vec{x}) \& B_r(u)). \tag{2}$$

Transformation from (1) to (2) constitutes the so-called *Mamdani's rule*. After we get the formula (2), the further parts of fuzzy control methodology are pretty much straightforward:

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- we interpret the properties A_j and B_j as fuzzy properties,
- use some fuzzy $\&-$ and $\vee-$operations as $\&$ and $\vee$, and then
- apply some *defuzzification procedure* to get a single value u for each $\vec{x}$.

B. A Problem With Mamdani's Rule

There is nothing especially controversial about the further steps. However, Mamdani's rule, although it is reasonably natural, makes a transformation that is by all logical standards weird: namely, it replaces implication in formula (1) by "and" in formula (2). From logical viewpoint, this is strange, because implication and conjunction are radically different logical operations. We view this discrepancy between what we actually do and what logic considers reasonable as a problem.

In principle, there are two possible approaches to this problem:

- We can consider this discrepancy as an indication that Mamdani's rule has to be replaced.
- On the other hand, we can consider it as an indication that classical logic may be not the most adequate tool to describe expert's control.

We support the second approach, because Mamdani's fuzzy control methodology is very successful, and therefore, this transformation must make sense. So, following this approach, we must find a non-classical logical formalism that is capable of describing the transition from (1) to (2).

II. Modern (Non-Monotonic) Logics: Main Idea

In solving this problem, we will use modern non-monotonic logics that have been successfully used to describe commonsense reasoning (see, e.g., [6]). The word "monotonic" comes from the following observation:

- In classical logic, if a statement S has been deduced from the axioms, and then the new axiom is added, then S remains true. In other words, if we increase the set of axioms, then the set of conclusions can only increase. In mathematical terms, the mapping from sets of axioms to sets of conclusions is *monotonic* with respect to inclusion.
- In commonsense reasoning, however, unlike classical logic, it is often reasonable to *retract* the conclusion once some additional information is known. A textbook example is: if the only thing you know about Sam is that Sam is a bird, then it is natural to conclude that Sam flies. However, if later on, it turns out that Sam is a penguin (and we know that penguins do not fly), we will retract the original conclusion. So, for commonsense reasoning, the mapping from sets of axioms into sets of conclusions is not monotonic. These logics are therefore called *non-monotonic* logics.

How can one describe non-monotonic reasoning formally? The most widely used approach is to use *minimal models* (in some reasonable sense). In classical logic, we say that a conclusion C follows from a set of axioms A if C is true in *all* models of A. In commonsense reasoning, we do not consider *all* possible models, only the simplest ones. For example, if we know that Sam is a bird, and we know nothing else, then from the purely logical viewpoint, it is quite possible that Sam is a penguin, but this is a rare and strange possibility, so we usually discard it. To give an extreme example: if a bank has been robbed, then it is logically possible that a Martian have done it, but normally, we do not consider such an option seriously (unless we are pressed to consider it by some additional information). This restriction on the class of models can be described as follows: if a model contains a statement that can be safely deleted (like "Sam is a penguin"), we do not use this model. We only consider *minimal* models, that only contain undeletable statements. Let us now describe formal definitions.

III. Definitions and the Main Result

A. Preliminary Definitions: Model and Minimal Model

Before we formulate our result, let us first recall the definitions of a model and of a minimal model.

Let us fix a set of predicate symbols $P_1, ..., P_n$ (some of them can be unary $P_i(a)$, some binary $P_j(a,b)$, etc). By a *theory* T, we will mean a set of logical statements formed by using these predicate symbols, logical connectives ($\&, \vee, \rightarrow, \neg$), and quantifiers ($\forall$ and $\exists$). For example, predicate symbols can include $<$ and $=$, and a theory contains properties of order and equality, like symmetry of $=$ ($\forall a, b(a = b \rightarrow b = a)$) and transitivity of both predicates ($\forall a, b, c((a < b\&b < c) \rightarrow a < c)$ and $\forall a, b, c((a = b\&b = c) \rightarrow a = c)$).

Let us assume that a set U is fixed. This set U will be called a *Universe*. By an *atomic statement* (or, an *atom* for short), we mean an expression of the type $P_i(a, ..., b)$, where P_i is one of the given predicate symbols, and $a, ..., b$ are elements of U. For example, U may include the set of all integers. Then, examples of atomic statements are $2 = 3$ and $3 < 4$.

For each set S of atomic statements, and for each logical formula F, we can define its truth value in S

as follows:

- If F is an atomic formula, then F is true in S iff it belongs to S; else, F is false in S.
- A formula $A \& B$ is true in S iff both A and B are true in S.
- Similar natural definitions are given for $A \vee B$, $\neg A$, and $A \rightarrow B$ to be true.
- A formula $\forall x P(x)$ is true in S iff $P(a)$ is true in S for all $a \in U$.
- A formula $\exists x P(x)$ is true in S iff $P(a)$ is true in S for some $a \in U$.

A set of atoms S is called a *model* of a theory T if all statements from T are true in S. A model S of theory T is called *a minimal model of* T if none of proper subsets of S is a model of T.

There can be several minimal models of a theory, and, to simulate commonsense reasoning properly, we must consider all of them.

In some cases, we consider *many-sorted* theories, in which variables belong to different *types*: e.g., in calculus, we have variables that run over integers, and variables that run over real numbers. For such theories, we must describe *several* universes: a separate Universe for each type of variables.

B. Let's Formulate Our Theory

In our case, we clearly have two types of variables: input variables $\vec{x}$, and a control variable u. The set of predicates is easily describable: it consists of unary variables $A_1(\vec{x}), ..., A_r(\vec{x})$, unary predicates $B_1(u), ..., B_n(u)$, and a single binary predicate $R(\vec{x}, u)$. The question is: what is our theory T? We already have formulas (1) that describe exactly what the expert has said. The tricky point is that the formulas (1) have no quantifiers over variables. A similar situation occurs very frequently: e.g., transitivity of $<$ is usually described as "if $a < b$ and $b < c$, then $a < c$". In such cases, we implicitly assume that this property is true for all a, b, and c. So, the first natural idea is to interpret each formula (1) as meaning that for all $\vec{x}$ and for all u, $A_j(\vec{x})$ implies $B_j(u)$. This interpretation, however, is somewhat misleading: when we say, e.g., that "if x is small, then u should be very small", we do not necessarily mean that for every small x, every very small control is reasonable. What we actually mean is that if x is small, then there is a reasonable control that is very small. In general, we mean a formula of the type

$$A_j(\vec{x}) \rightarrow \exists u (R(\vec{x}, u) \& B_j(u)). \qquad (3)$$

In control situations, we assume that the predicates A_j and B_j are known, and we want to find R.

C. Main Result

Definition 1. We say that a predicate B is *not identically false* if there exists an u for which $B(u)$ is true.

PROPOSITION. *Assume that predicates A_j and B_j are given, and that the predicate B_j is not identically false. Then, for every $\vec{x}$, and for every u, the following two conditions are equivalent to each other:*

- *the atomic statement $R(\vec{x}, u)$ belongs to a minimal model of the theory (3);*
- *the pair $(\vec{x}, u)$ satisfies the formula $A_j(\vec{x}) \& B_j(u)$.*

Comment. For several rules, u is a reasonable control for a given $\vec{x}$ if it is a reasonable control according to one of the rules. So, $R(\vec{x}, u)$ is equivalent to the formula (2). This result provides the desired justification of Mamdani's rule (i.e., of formula (2)) in terms of minimal logics.

D. Proof

Let's first prove that if an atomic statement $R(\vec{x}, u)$ belongs to a minimal model S, then the pair $(\vec{x}, u)$ satisfies formula $A_j(\vec{x}) \& B_j(u)$. We will prove it by reduction to a contradiction. Indeed, if this pair does not satisfy the property $A_j(\vec{x}) \& B_j(u)$, then it cannot describe the u for which $A_j(\vec{x})$ and $R(\vec{x}, u) \& B_j(u)$. So, we can delete this atomic statement from S and still get a model of (3). This possibility, however, contradicts our assumption that S is a minimal model. The contradiction proves that the pair must satisfy the property $A_j(\vec{x}) \& B_j(u)$.

Vice versa, let us assume that the pair $(\vec{x}, u)$ satisfies the formula $A_j(\vec{x}) \& B_j(u)$. Let us build a minimal model that contains the atomic statement $R(\vec{x}, u)$. For this particular $\vec{x}$, this minimal model will contain only one atomic statement: $R(\vec{x}, u)$. For every other $\vec{y}$:

- if $\vec{y}$ does not satisfy the property A_j, then we do not include any atom $R(\vec{y}, v)$ into our minimal model S.
- If $A_j(\vec{y})$ is true, then we pick any v for which $B_j(v)$ is true, and include $R(\vec{y}, v)$ into S.

It is easy to see that the resulting set S of atoms is a model of the theory (3).

Let us show that thus designed set is minimal. Indeed, in a model, (3) must be true and therefore, for every $\vec{y}$ for which $A_j(\vec{y})$ is true, $R(\vec{y}, v)$ must be true for at least one v. In our model S, it is true for exactly one v. So, if we delete an atom $R(\vec{y}, v)$ from S, there will be no v for which for this $\vec{y}$, R will be true. In other words, S will no longer be a model. Therefore, S is a minimal model. Q.E.D.

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