Fixed Point Theorems in Logic Programming

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1 Introduction

Very often scientific branches which were thought to be completely disparate are suddenly seen to be related. This is the case for example with mathematics of which the level of sophistication applied to various sciences has changed drastically in recent years. Fixed point theory furnishes good example of a central concept with multitudes of different uses. It has always been a major theoretical tool in fields as widely apart as differential equations, topology, economics, game theory, dynamical systems (and chaos), optimal control, functional nalysis, logic programming and artificial intelligence. Moreover, more or less recently, the usefulness of the concept for applications increased enormously by the development of accurate and efficient techniques for computing fixed points, making fixed point methods a major weapon in the arsenal of the applied mathematician.

2 Elementary fixed point theorems

The theory of fixed points is concerned with the conditions which guarantee that a map $F : X \to X$ of a set X into itself admits one or more fixed points, that is, points $x \in X$ for which $F(x) = x$. The set of fixed points of F will be denoted $Fix(F)$.
In the sequel, we will discuss two majors fixed point theorems which are based on a notion of completeness. Although the spaces involved are of different nature, there is a similarity between the two theorems.

2.1 Knaster-Tarski theorem

Let $(P, \leq)$ be a partially ordered set and $M \subset P$ a non-empty subset. Recall that an upper (lower) bound for M is an element $p \in P$ with $m \leq p$ ($p \leq m$) for each $m \in M$; the least-upper (greatest-lower) bound of M will be denoted $\sup M$ (resp. $\inf M$). Of course there is, in general, no reason for $\inf M$ and $\sup M$ to exist. $(P, \leq)$ will be called complete if for every subset $M \subset P$, $\inf M$ and $\sup M$ do exist. Recall that $M \subset P$ is said to be linearly ordered if for every $m_1, m_2 \in M$ we have $m_1 \leq m_2$ or $m_2 \leq m_1$. A linearly ordered subset of P is called a chain. A map $F : P \to P$ is monotone (also called isotone or increasing) if

$F(x) \leq F(y)$ whenever $x \leq y$.

Theorem of Knaster-Tarski *Let $(P, \leq)$ be a complete ordered set and $F : P \to P$ monotone. Then $Fix(F)$ is not empty, that is F has a fixed point. Moreover, $Fix(F)$ is a complete ordered subset of P. In particular, it has the least and greatest elements.*

Note that one classical proof is based on a transfinite iteration of F. Indeed, denote m to be the least element of P. Clearly, we have $m \leq F(m)$. Since F is monotone, we will get $F^n(m) \leq F^{n+1}(m)$ for every $n = 1, 2, ...$ Set $x_n = T^n(m)$. Since P is complete then $x_\omega = \sup_n x_n$ does exist. There is no reason for x_ω to be a fixed point. Therefore, we should continue the process, that is if $(x_\alpha)_{\alpha<\beta}$ are constructed, then $x_\beta = \sup_{\alpha<\beta} x_\alpha$ if β is a limit ordinal otherwise $x_\beta = F(x_{\beta-1})$. The (x_α) defines a chain which will eventually stop, that is there exists an ordinal α_0 such that $x_{\alpha_0} = x_{\alpha_0+1} = F(x_{\alpha_0})$. It is very easy to see that x_{α_0} is the least fixed point of F. By iterating F over the greatest element of P, one will generate the greatest fixed point of F. A natural question arises about what do we know of the ordinal α_0. One easy answer has to do with the size of the set P. Not very much encouraging since P can be very big. Another partial answer is given by what we calle continuity. Indeed, the map $F : P \to P$ is called continuous is for every subset $M \subset P$, we have $\sup F(M) = F(\sup M)$. Note that if F is monotone, then $\sup F(M) \leq F(\sup M)$ but not the equality. It is well known that if $F : P \to P$ is continuous and monotone, then $\alpha_0 = \omega$ regardless of the given point $x \in P$ which initiates the iteration process, where ω is the first countable ordinal. Indeed, fix $x \in P$ and consider the iteration sequence $(F^n(x))$. Then we have

$$F(x_\omega) = F(\sup_n F^n(x)) = \sup_n F^{n+1}(x) = x_\omega$$

which shows that the fixed point will always be reached after ω-iteration.

Since whenever negation is involved in logic programs, it is to be expected that the mappings involved will no longer be monotone. As a matter of fact, we may have a nonmonotonic behavior. Recall that a map $F : P \to P$, where $(P, \leq)$ is a partially ordered set, is said to be nonmonotone (or decreasing) if for every $m_1, m_2 \in P$ we have $F(m_2) \leq F(m_1)$ whenever $m_1 \leq m_2$. Clearly, $F^2 = FoF$ will be monotone. Therefore, F^2 will have fixed point (called periodic point of F of period 2). Let m be the lowest fixed point of F^2 and M be the greatest fixed point of F^2. Then we have $F(m) = M$ and $F(M) = m$.

2.2 Banach contraction principle

The theorem of Banach is the simplest and one of the most versatile results in fixed point theory. Being based on an iteration process, it can be implemented on a computer to find the fixed point of a contractive map producing approximations of any required accuracy and, moreover, even the number of iterations needed to get specified accuracy can be determined. First, recall that a pair (M, d) is called a metric space if the map $d : M \times M \to [0, +infty)$ satisfies

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x,y) = d(y,x)$

(i) (Triangle inequality) $d(x,y) \leq d(x,z) + d(z,y)$

for every $x, y, z \in M$. d is called a distance.

Using the distance, we can give a meaning to the notion of closedness. Indeed we will say x is close to y if and only if $d(x,y)$ is small. We will say that a metric is complete if whenever a sequence (x_n) of points from M is such that x_n and x_m are close for n and m big enough (such sequence is called a Cauchy sequence), then there exists a point $x \in M$ such that x_n gets close to x when n is big. In a more correct fashion, we will say that M is complete if for every sequence (x_n) such that $\lim d(x_n, x_m) = 0$ when n, m goes to infinity, then there exists $x \in M$ such that $\lim d(x_n, x) = 0$ when n goes to infinity. Note that not all metric spaces are complete. A mapping $F : M \to M$ is called Lipschitzian if there exists a constant L such that $d(F(x), F(y)) \leq Ld(x,y)$ for every $x, y \in M$; the smallest such constant L is called the Lipschitz constant $Lip(F)$ of F. If $Lip(F) < 1$, the map F is called contractive.

Theorem (Banach contraction principle) *Let (M,d) be a complete metric space and $F : M \to M$ a contractive mapping. Then F has a unique fixed point w, that is $Fix(F) = \{w\}$. Moreover, for any $x \in M$, we have*

$$d(F^n(x), w) \leq \frac{L^n}{1-L} d(x, F(x)) \; ; \quad \textit{for } n = 1, 2, ..$$

where $L = Lip(F) < 1$. Therefore we have $\lim F^n(x) = w$ when $n \to \infty$.

It is clear from the theorem that the error of approximating w by the nth iteration when starting from a given $x \in M$ is completely determined by the contraction constant $Lip(F)$ and the initial displacement $d(x, F(x))$.

It is worth to mention here that the theorem of Banach contains two important conclusions. The first one deals with the uniqueness of the fixed point. The other one deals with the way we reach this fixed point since the iteration reaches the fixed point after ω-iteration, where ω is the first countable ordinal.

Remark. Before, we get to the study of multivalued mappings, we would like to make an elementary introduction to generalized metric spaces. The main idea behind this new structure is to replace the set of positive numbers as the set value for the distance by something more general. The reason behind is that many structures are not of continuous nature. More precisely sets which are of discrete nature will have hard time to fit into the classe of classical metric spaces. Therefore people working in this area were the pionneers in developping the appropriate extension. Let us describe what they did. Let $\mathcal{V}$ be a set with a binary operation which will be denoted $\oplus$. We will assume that $\oplus$ enjoys most of the properties that the classical addition does. In particular, we assume that there exist a zero element $0 \in \mathcal{V}$ which satisfies $u \oplus 0 = 0 \oplus u = u$ for every $u \in \mathcal{V}$. We will also assume (as for the set of positive numbers) that we have an order $\leq$ such that $0 \leq u$ and $u \oplus v \leq u' \oplus v'$ provided $u \leq u'$

and $v \leq v'$ for every $u, v, u', v' \in \mathcal{V}$. Note that the set $\mathcal{V}$ is not behaving totally like the set of positive numbers since we do not have a priori a multiplication operation defined on $\mathcal{V}$.

Definition. Let $\mathcal{V}$ be a set as described above and M be an arbitrary set. The mapping $d : M \times M \to \mathcal{V}$ is called a generalized distance if

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$

(i) (Triangle inequality) $d(x, y) \leq d(x, z) \oplus d(z, y)$

for every $x, y, z \in M$. A pair (M, d) is called a generalized metric space.

Example of $\mathcal{V}$ Because of applications to disjunctive logic programming, one interesting example of the abstract set $\mathcal{V}$ is given by the expressions $2^{-\alpha}$, where α is a countable ordinal, augmented with the element 0. An order on $\mathcal{V}$ is defined as follows: $0 \leq v$, $\forall v \in V$, and $2^{-\alpha} \leq 2^{\beta}$ whenever $\beta \leq \alpha$. The operation $\oplus$ is defined by $u \oplus v = \max(u, v)$. Note that $\mathcal{V}$ enjoys a natural multiplication property defined as: $2^{-\alpha}.2^{-\beta} = 2^{-(\alpha+\beta)}$.

2.3 The multivalued case

Multivalued mappings play a major role in studying disjunctive logic programs. First, recall that a map $F : X \to 2^X$ is called a multivalued map. A singlevalued map is a multivalued map F such that $F(x)$ is a singleton. A point $x \in X$ is a called a fixed point of a multivalued map F if $x \in F(x)$ holds. Although the fixed point theory for singlevalued maps is very rich and well developped, the multivalued case is not. Few theorems are known in topological spaces but nothing major with direct impact on the study of disjunctive logic programs. For example, it is unknown to us whether a multivalued version of Knaster-Tarski theorem is known. In what follows we will discuss this case.
Let $(P, \leq)$ be a partially ordered set. Define the relation $\prec_r$ (r for restriction) in 2^P by

$$A \prec_r B \Longleftrightarrow \forall y \in B \;\; \exists x \in A \quad x \leq y$$

it is clear that $\prec_r$ is a preorder but not an order ($\prec_r$ is not antisymetric). We can also define the extension preorder $\prec_e$ in 2^P by

$$A \prec_e B \Longleftrightarrow \forall x \in A \;\; \exists y \in B \quad x \leq y \; .$$

We say that a multivalued mapping $T : P \to 2^P$ is $\prec_r$-monotone if

$$x \leq y \Longrightarrow T(x) \prec_r T(y)$$

Remark: Recall that a subset $M \subset P$ is called antichain if for every $m_1, m_2 \in M$, we do not have $m_1 \leq m_2$ neither $m_2 \leq m_1$. Denote

$$M = \{A \in 2^P; A \text{ is an antichain } \}$$

then $(M, \prec_r)$ and $(M, \prec_e)$ are ordered sets because $\prec_r$ and $\prec_e$ are orders in M. Since the study of stable model semantics for disjunctive logic programs involves the preorder $\prec_r$, we will not discuss the preorder $\prec_e$.

Definition. Let P be an partially ordered set and $T : P \to 2^P$ a multivalued mapping. We say that the family (x_β), where β runs through the ordinals, is a decreasing T-orbit if

$$\begin{cases} x_{\beta+1} \in T(x_\beta) \\ x_{\beta+1} \leq x_\beta \end{cases}$$

Theorem (Knaster-Tarski) *Let P be a complete ordered set and T be an $\prec_r$-increasing multivalued mapping from P into 2^P such that for every $x \in P$,*

1. *$T(x)$ is not empty (i.e. $T(x) \neq \emptyset$)*
2. *For every decreasing T-orbit (x_β), there exists $x \in P$ such that $x \in T(\inf x_\beta)$ and $x \leq x_\beta$ for all β,*

Then T has a fixed point, i.e. there exists $x \in P$ such that $x \in T(x)$.

Remark: The second hypothesis of the theorem holds if $T(x)$ is finite for all $x \in P$.

Let us turn our attention to Banach contraction principle for multivalued mappings. First we should mention that after the following theorem was proved, we discovered that many multivalued versions of the theorem of Banach do exist.

Theorem (Banach contraction principle) *Let (M, d) be a complete metric space. Let $F : M \to 2^M$ be a multivalued map satisfying for every $x, y \in M$ we have*

$$\forall x_1 \in T(x) \;\; \exists y_1 \in T(y) \textit{ such that } \; d(x_1, y_1) \leq Ld(x, y)$$

where $L < 1$. Then F has a fixed point provided that $F(x)$ is closed and nonempty for every $x \in M$.

A similar result for multivalued mappings defined on a generalized metric space holds. Indeed, Let (M, d) be a generalized metric space with set-values for d to be the set $\mathcal{V}$ described in the example above. Then a multivalued map $F : M \to 2^M$ will be called 1/2-contraction if for every $x, y \in M$, and for every $z \in F(x)$, there exists $w \in F(y)$ such that $d(z, w) \leq 1/2d(x, y)$. In order to state an analogue to Banach contraction principle for generalized metric spaces, one would need to define the notion of comlpleteness. We will not give the definition here but assure the reader that this definition is similar to the one described above.

Theorem *Assume that (M, d) is a complete generalized metric space and F a multivalued mapping defined on M which is 1/2-contraction. Then F has a fixed point provided that $F(x)$ is nonempty and complete for every $x \in M$.*

3 Application to logic programming

A precise meaning or semantics must be associated with any logic program or a deductive database in order to provide its declarative specification, in a manner, which is independent of procedural considerations, context-free, and easy to manipulate, exchange and reason about. The problem of finding a suitable declarative or intended semantics is one of the most important and difficult problems in the theory of logic programming and deductive database. In this work, we will not discuss the merit of any of the semantics. Our goal is to discuss how fixed point theorems old or new can be used to answer some open questions regardless of the semantics used. For the sake of illustrating our point, we will consider the answer set semantics.

Let Lit be the set of ground literals in a first-order language $\mathcal{L}$. A rule r is an expression of the following form:

$$l_0 \mid l_1 \mid \cdots \mid l_n \leftarrow l_{n+1}, \cdots, l_m, not\ l_{m+1}, \cdots, not\ l_k$$

where $l_i \in Lit$. Set

$$Head(r) = \{l_0, l_1, \cdots, l_n\} \;\; Pos(r) = \{l_{n+1}, \cdots, l_m\} \;\text{ and }\; Neg(r) = \{l_{m+1}, \cdots, l_k\}$$

The rule r is said to be disjunctive if $n > 1$, i.e. $Head(r)$ has more than one element and nondisjunctive otherwise. An extended (disjunctinve) logic program Π is a set of (disjunctive) rules. Instead of extended logic program, we will shorten it to program. In order to define the answer set semantics of extended logic programs, let us first consider programs without *not*.

Definition. Let Π be a program (disjunctive or not) for which $Neg(r)$ is empty for every $r \in \Pi$. A subset S of Lit, i.e $S \in 2^{Lit}$, is said to be closed with respect to Π if

for every $r \in \Pi$ such that $Pos(r) \subset S$, we have $Head(r) \cap S$ is not empty.

The set $S \in 2^{Lit}$ is an answer set of Π

1. If S contains complementary literals, then $S = Lit$.

2. S does not contain strictly a closed subset, i.e. if $A \subset S$ and A is closed with respect to Π then $A = S$.

The set of answer sets of Π is denoted by $\alpha(\Pi)$. If Π is not disjunctive, then $\alpha(\Pi)$ is a singleton, i.e. Π has one answer set and if it is disjunctive then $\alpha(\Pi)$ may contain more than one element.

Now let Π be a program that may contain *not* (general case). For $S \in 2^{Lit}$, consider the program Π^S defined by the set of rules

1. If $r \in \Pi$ such that $Neg(r) \cap S$ is not empty, then $r \notin \Pi^S$

2. If $r \in \Pi$ such that $Neg(r) \cap S$ is empty, then the rule r' defined by $Head(r') = Head(r)$, $Pos(r') = Pos(r)$ and $Neg(r') = \emptyset$, belongs to Π^S.

Clearly the program Π^S does not contain *not*.

Definition.

1. The set $S \in 2^{Lit}$ is said to be an answer set of Π if $S \in \alpha(\Pi^S)$.
2. The GL-operator $T : 2^{Lit} \to 2^{Lit}$ is defined by

$$T(X) = \alpha(\Pi^X)$$

Using the GL-operator T, the set S is an answer set of Π iff $S \in T(S)$. If Π is not disjunctive then S is an answer of Π iff $S = T(S)$. In both cases, S is a fixed point of T.

Main Problem. Find conditions on Π that will insure the existence of an answer set, i.e. a fixed point of T.

In what follows, we discuss a basic property of the GL-operator.

Property. Let Π be a program. Then if $X \subset Y$, $X, Y \in 2^{Lit}$, then

$$\forall A \in \alpha(\Pi^X)\ \exists B \in \alpha(\Pi^Y) \text{ such that } B \subset A$$

We will write $\alpha(\Pi^Y) \prec_r \alpha(\Pi^X)$.

Remark. When the program Π is not disjunctive, then we have

$$X \subset Y \implies \alpha(\Pi^Y) \subset \alpha(\Pi^X)$$

in another word the GL-operator T is anti-monotonic. Therefore, T^2 is monotonic. Using the above discussion, we know that there exist a least fixed point $lfp(T^2)$ and a greatest fixed point $gfp(T^2)$ of T^2. We also know that $T(lfp(T^2)) = gfp(T^2)$ and $T(gfp(T^2)) = lfp(T^2)$. In another world, the set $\{lfp(T^2), gfp(T^2)\}$ is stable under the action of T. Using this result, Baral defined the notion of stable class semantics. It is worth to mention that there are logic programs which fail to possess stabl models while stable classes always exist. Therefore, for these programs we will not be able to consider the stable semantics. Baral was also successful to connect stable class semantics to well-founded semantics. Indeed, he proved that if Π is a normal logic program, the well-founded semantics is characterized by a particular stable class $\mathcal{C}$, i.e. a ground atom is true (resp false) in the well-founded semantics of Π iff A is true (resp. false) in all interpretations in $\mathcal{C}$. Moreover, $\mathcal{C} = \{lfp(T^2), gfp(T^2)\}$. It is amazing that the Gelfond-Lifschitz operator could catch the well-founded semantics in this way. Thereofre, a natural question to be asked is whether a similar result holds for disjunctive logic programs. The answer is still unknown. The difficulty resides in considering T^2 since T is multivalued and getting a monotonic mapping. In another world, we do not know whether a stable classes do exist for disjunctive logic programs.

3.1 Signed disjunctive programs

The notion of a signing for a program was introduced by Kunen (defined on the predicate dependency graph), who used it as a tool in his proof that two-valued and three valued completion semantics coincide on the class of strict normal programs. Gelfond and Lifschitz recasted this notion and redefined it directly on the rules of programs. Recently Turner did study extensively signed programs. In particular, he was able to give an alternative proof of Fages'result using signed programs.

Definition. We will say that Π is **signed** iff there exists $S \in 2^{Lit}$ such that for every $r \in \Pi$ we have

1. If $Neg(r) \cap S$ is empty, then $Head(r) \subset S$ and $Pos(r) \subset S$. Denote by R_S the set of such rules.
2. If $Neg(r) \cap S$ is not empty, then $Head(r) \cap S = \emptyset$, $Pos(r) \cap S = \emptyset$ and $Neg(r) \subset S$. Denote by R_{S^c} the set of such rules.

The set S is called a signing for Π.

To a signed program Π we can associate naturally two subprograms Π_1 defined by the rules R_S and Π_2 defined by the rules R_{S^c}.

For nondisjunctive signed programs, Gelfond and Lifschitz proved the existence of answer sets. It is still unknown if disjunctive signed programs have answer sets. For a more restrictive class, we do have a positive answer.

Definition. A signed program Π is said to be **semi-disjunctive** if there exists a signing S such that Π_1 is not disjunctive.

Turner proved that such programs have an answer set. The proof given by Turner relies on the study of the covers of Π. Our proof is based on the generalization of Knaster-Tarski fixed point theorem to multivalued mappings. Therefore, our approach has the advantage of applying a more general fixed point result which has nothing to do with a specific class of programs. Therefore, we may expect the the fixed point theorem to be applied to a more general class of programs.