Verified Estimation of Taylor Coefficients and Taylor Remainder Series of Analytic Functions

Markus Neher
Institut für Angewandte Mathematik, Universität Karlsruhe
D-76128 Karlsruhe, Germany
markus.neher@math.uni-karlsruhe.de

1 Introduction

Let $y = \sum_{j=0}^{\infty} b_j z^j$ be the analytic solution of a problem $F(y) = 0$ that depends on analytic functions $f_1, \ldots, f_n$ and that can be solved by recurrent computation of the Taylor coefficients b_j of y. Only a finite number of the b_j can be calculated in practical computations. Then a finite sum $\sum_{j=0}^{p} b_j z^j$ only yields an approximation to $y(z)$.

In some cases, however, it is possible to determine a geometric series (or some derivative of a geometric series) that serves as a bound for the remainder series of y, provided that the remainder series of the functions $f_1, \ldots, f_n$ can also be estimated by geometric series (or derivatives of geometric series). In [4], such an error analysis was used for the validated solution of linear ODEs.

A prerequisite of this general method is the computation of bounds for the Taylor coefficients of arbitrary order of a given analytic function f. The subject of this talk is the validated solution of the latter problem.

2 Estimates for Taylor Coefficients

In the following, let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be analytic in B and bounded on C, where B is the complex disc $\{z : |z| < r\}$ and C the circle $\{z : |z| = r\}$, for some $r > 0$. A well known bound for the Taylor coefficients of f is Cauchy's estimate $M(r)$:

$$|a_j| \;\le\; \frac{M(r)}{r^j}, \quad M(r) := \max_{|z|=r} |f(z)|, \quad j \in \mathbb{N}_0.$$

Unfortunately, Cauchy's estimate is sometimes very pessimistic. To obtain better bounds, two modifications of Cauchy's estimate were proposed in [5]. The first uses a Taylor polynomial to approximate f:

Theorem 1 *Let f be analytic in B and bounded on C. Furthermore, let $T_l(z)$ denote the Taylor polynomial of order l to f. Then*

$$|a_j| \;\le\; \frac{N(r,l)}{r^j} \quad \text{for } j > l, \quad \text{where} \quad N(r,l) := \max_{|z|=r} |f(z) - T_l(z)| \,.$$

Cauchy's estimate can also be improved using the derivatives of f:

Theorem 2 *Let f be analytic in B and let $f^{(m)}$ (the m–th derivative of f) be bounded on C. Furthermore, let $P(j,m) := (j+1)\cdots(j+m)$, $P(j,0) := 1$ for $m \in \mathbb{N}$, $j \in \mathbb{N}_0$. Then*

$$|a_j| \;\le\; \frac{U(r,m) r^m}{P(j-m,m) r^j} \quad \text{for } j \ge m, \quad \text{where} \quad U(r,m) := \max_{|z|=r} |f^{(m)}(z)| \,.$$

3 Estimates for Taylor Remainder Series

The estimation of the remainder series $R_p := \sum_{j=p+1}^{\infty} a_j z^j$ is obtained by addition of the above estimates of the Taylor coefficients. Using Theorem 1, at some point z with $|z| = \omega r$, $\omega \in (0,1)$, for arbitrary $p \ge l$ we have

$$|R_p(z)| \;\le\; \sum_{j=p+1}^{\infty} N(r,l)\,\omega^j \;=\; N(r,l)\,\frac{\omega^{p+1}}{1-\omega},$$

whereas for $p \ge m-1$ we have

$$|R_p(z)| \;\le\; \sum_{j=p+1}^{\infty} \frac{U(r,m)\,r^m}{P(j-m,m)} \omega^j$$

by Theorem 2. For $p = m-1$ the latter sum is derived by repeated integration of $\frac{1}{1-\omega}$. For $p \gg m$ the resulting formula suffers from severe cancellation. In this case, the estimate

$$R_p \;\le\; \frac{U(r,m)\,r^m}{P(p+1-m,m)} \frac{\omega^{p+1}}{1-\omega}$$

is better suited for practical calculations.

In the situation that was mentioned in the introduction, it is usually not known in advance which order p of the Taylor polynomial T_p is required for sufficient accuracy of the approximation of the unknown solution y. Because it is very expensive to recalculate the bounds N or U for different values of l or m, respectively, it is better to guess sufficiently large values for l or m a priori and to calculate N or U only once. Only if $p = l$ or $p = m$ are not sufficient for a good approximation of y, then p is augmented iteratively, but R_p is still calculated for the same values of l or m (note that $R_p \to 0$ for $p \to \infty$ independently of l or m).

4 Implementation

Using interval arithmetic [1] on the computer, the validated computation of the above estimates is possible for analytic compositions of rational functions and of those complex standard functions that are available on the computer (like e^z, $\sin z$, $\mathrm{Log}\, z, \ldots$). The real and imaginary parts of many standard functions can be expressed as compositions of real standard functions. Such compositions have been utilized in [2] for the construction of complex interval standard functions that enclose the respective ranges over complex intervals. These inclusion functions and well known methods for rigorous global optimization [3] are used in the practical calculation of the estimates $M(r)$, $N(r,l)$, or $U(r,m)$.

5 ACETAF Software

The above algorithms for the computation of guaranteed upper bounds for the Taylor coefficients and remainder series of analytic functions have been implemented in a C–XSC program called ACETAF. The program also contains routines for the validated automatic computation of derivatives of complex analytic functions and for the check of analyticity of user–defined functions in circles in the complex plane.

ACETAF is distributed under the terms of the GNU General Public License and is available at the following site:

`http://www.uni-karlsruhe.de/~Markus.Neher/acetaf.html`

References

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