Two Topics in Computer Assisted Proofs for the Problems in Fluid Dynamics

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We have been devoted for years to studying the numerical verifications of solutions to elliptic partial differential equations. Our approach is based on the combination of fixed point theorems in functional spaces and the constractive error estimations of finite element (or spectral) method. In our verification process, the interval method for finite-dimensional linear equations plays an essential role.

In this talk, we first briefly describe the basic idea of our verification method for nonlinear elliptic problems. Next, we apply the method to two important problems appeared in fluid dynamics, i.e., Rayleigh-Bénard and Kolmogorov problems. In both cases, the existence of exact solutions is verified and the usefulness of our approach have been shown.

1 The Basic Idea ([4, 5])

Suppose that the concerned elliptic problem is reformulated as the following fixed point problem of a nonlinear compact operator F in some appropriate infinite-dimensional function space X:

$$u = F(u). \tag{1}$$

Suppose also that we find a nonempty, bounded, convex, and closed subset $U \subset X$, which is referred to as a *candidate set* of solutions, satisfying

$$F(U) = \{F(u) | u \in U\} \subset U. \tag{2}$$

Then by the Schauder fixed point theorem, an infinite-dimensional version of Brouwer's theorem, there exists an element $u \in F(U)$ such that $u = F(u)$.

Let S_h be a finite-dimensional subspace of X dependent on h $(0 < h < 1)$. Let $P_h : X \longrightarrow S_h$ be the orthogonal projection operator, where the parameter

h corresponds to the degree of approximation. For example, it means the mesh size in the finite element methods or the reciprocal of the term number in the spectral approximations. We usually choose a candidate set U of the form $U = U_h \oplus U_\perp$, where $U_h \subset S_h$ and $U_\perp \subset S_h^\perp$. Here, $S_h^\perp$ stands for the orthogonal complement subspace of S_h in X. Then, the verification condition (2) can be decomposed into the two parts as follows:

$$\begin{cases} P_h F(U) \subset U_h \\ \\ (I - P_h)F(U) \subset U_\perp. \end{cases} \tag{3}$$

Since the first inclusion is in the finite-dimensional space S_h, it may be verified on computer using interval arithmetic. The second inclusion is in the infinite-dimensional space $S_h^\perp$, and will be verified by constructive error analysis of the numerical method in use. Combining verifications of both inclusions in (3) we may conclude the inclusion (2) is verified.

The set U_h consists of linear combinations of base functions in S_h with interval coefficients, and the set $U_\perp$ is constructed as a ball in $S_h^\perp$ with radius $\alpha \geq 0$. Namely, we represent U_h and $U_\perp$ by

$$U_h = \sum_{j=1}^{M} [\underline{A}_j, \overline{A}_j]\phi_j \quad \text{and} \quad U_\perp = \{\phi \in S_h^\perp \mid \ ||\phi||_{H_0^1} \leq \alpha\},$$

respectively, where $\{\phi_j\}_{j=1}^M$ is a basis of S_h. Here, $\displaystyle\sum_{j=1}^{M} [\underline{A}_j, \overline{A}_j]\phi_j$ is interepreted as the set of functions in which each element is a linear combinaion of $\{\phi_j\}_{j=1}^M$ whose coefficient of ϕ_j belongs to the corresponding interval $[\underline{A}_j, \overline{A}_j]$ for each $1 \leq j \leq M$.

Then, it can be easily seen that $P_h F(U)$ is directly computed or enclosed of the form

$$P_h F(U) \subset \sum_{j=1}^{M} [\underline{B}_j, \overline{B}_j]\phi_j$$

by solving a linear system of equations with interval right-hand side which is determined from U_h and $U_\perp$ using interval computations. Thus, the first condition in (3) is validated as the inclusion relations of corresponding coefficient intervals, that is, $[\underline{B}_j, \overline{B}_j] \subset [\underline{A}_j, \overline{A}_j]$. On the other hand, $(I - P_h)F(U)$ is not directly computable but can be numerically evaluated by the effective use of constructive *a priori* error estimates of the projection P_h. Hence, the second condition can be verified by a simple comparison of two nonnegative real numbers which correspond to the radii of balls. In the actual computation, we use some iterative methods for both part of $P_h F(U)$ and $(I - P_h)F(U)$.

In order to apply the verification method to more general problems, we usually utilize a version of Newton-like method (see e.g., [5], [6] for details) which is also considered as an extension of the interval Newton method (e.g., [1]) to the infinite-dimensional cases. We also note that, in our verification, we estimate rigorously not only the rounding error of floating point computations, but also the truncation error due to the approximation of the infinite-dimensional operator. Therefore, our method can also be applied to the guaranteed *a posteriori* error analysis for the various kinds of approximation methods for elliptic problems.

2 Heat Convection Problems Governed by the Navier-Stokes Equation

The two-dimensional (x-z) Oberbeck-Boussinesque approximations for the Rayleigh-Bénard convection are described as follows [7]:

$$\left\{\begin{array}{rcl} u_t + uu_x + wu_z & = & p_x + \mathcal{P}\Delta u, \\ w_t + uw_x + ww_z & = & p_z - \mathcal{P}\mathcal{R}\theta + \mathcal{P}\Delta w, \\ u_x + w_z & = & 0, \\ \theta_t + w + u\theta_x + w\theta_z & = & \Delta\theta, \end{array}\right. \tag{4}$$

where (u, w), p and θ denote the velocity field, pressure and temperature from a linear profile while $\mathcal{P}$ and $\mathcal{R}$ denote Prandtl and Rayleigh numbers, respectively. We consider the steady-state solution branches of (4). By using the stream function Ψ for the velocity and setting $\Theta \equiv \sqrt{\mathcal{P}\mathcal{R}}\theta$, we have the following system of equations on the domain $\{-\infty < x < \infty, 0 < z < \pi\}$.

$$\left\{\begin{array}{lcl} \mathcal{P}\Delta^2\Psi & = & \sqrt{\mathcal{P}\mathcal{R}}\Theta_x - \Psi_z\Delta\Psi_x + \Psi_x\Delta\Psi_z \\ -\Delta\Theta & = & -\sqrt{\mathcal{P}\mathcal{R}}\Psi_x + \Psi_z\Theta_x - \Psi_x\Theta_z \\ \Psi = 0 & , & \Psi_{zz} = 0, \quad \Theta = 0 \qquad on\ \ z = 0,\ \pi \end{array}\right. \tag{5}$$

We suppose the periodic boundary condition in x and the stress free boundary condition on $z = 0$ and $z = \pi$. We have numerically verified several solution branches from the trivial solution of (5) by using the spectral approximations and the constructive error estimates. Several new results which would be difficult to derive by theoretical approaches are obtained.

3 Kolmogorov's Problem of Viscous Incompressible Fluid

This is a non-selfadjoint eigenvalue problem of the linearlized stationary Navier-Stokes equation in two dimension of the following form [3]:

Find a stream function ϕ, periodic in x and y, and a number $\sigma \in R^1$ such that

$$\begin{cases} \dfrac{1}{R}\Delta^2\phi - \sin y(\Delta + I)\dfrac{\partial\phi}{\partial x} = \sigma\Delta\phi, & (x, y) \in T_\alpha \\ \displaystyle\int_{T_\alpha} \phi^2 dxdy = 1, & \end{cases} \tag{6}$$

where R is the Reynolds number, $T_\alpha \equiv [-\pi/\alpha, \pi/\alpha] \times [-\pi, \pi]$ (α: aspect ratio).

The final purpose of the computer assisted proof is the validation of a stability condition of the flow. This can be carried out by showing that a certain inequality holds for the numerically verified eigenpair (σ, ϕ). Using the Fourier-Galerkin method with explicit error estimates as in the previous problem, we have actually succeeded to verify stability results related to the aspect ratio α. Proving this result would also be very difficult by any kind of theoretical analysis up to now.

In the presentation, we will show some numerical examples of the above topics. In both examples we use the spectral method. Note that it is also possible to use the finite element approximation with constructive error estimates in stead of the spectral method.

References

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