On the Shape of the Limit of the Total Step Method in Interval Analysis

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Given an $n \times n$ interval matrix [C] and an interval vector $[b]$ with n components the solution set S of the interval linear system $[C]x = [b]$ is defined by

$$S := \{\, x \in \mathbb{R}^n \mid Cx = b,\ C \in [C],\ b \in [b] \,\} \ .$$

Since this set normally cannot easily be described (cf. [7], e.g.) one looks for enclosures of S by an interval vector. One of the simplest iterative methods to obtain such an enclosure is based on the Richardson splitting $[C] = I - [A]$, i.e., $[A] := I - [C] = ([\underline{a}_{ij}, \overline{a}_{ij}])$, and reads

$$[x]^{(k+1)} = [A][x]^{(k)} + [b], \quad k = 0, 1, \dots \ . \tag{1}$$

It can be found in [1] and can be regarded as the starting point of many other iterative algorithms for enclosing S, among them such well–known iterations like

$$[x]^{(k+1)} = (I - R[C])[x]^{(k)} + R[b], \quad k = 0, 1, \dots \ , \tag{2}$$

and

$$[x]_\Delta^{(k+1)} = (I - R[C])[x]_\Delta^{(k)} + R([b] - [C]\tilde{x}), \quad k = 0, 1, \dots \ , \tag{3}$$

(cf. [6], [8], e.g.) where $R \in \mathbb{R}^{n \times n}$ denotes any nonsingular preconditioning matrix, $\tilde{x}$ is an approximation of an arbitrary element of S and $[x]_\Delta^{(k)}$ is used to approximate and/or to enclose the errors $x - \tilde{x}$ for $x \in S$. Replacing $[A]$ in (1) by $I - R[C]$, $[b]$ by $R[b]$, and $R([b] - [C]\tilde{x})$, respectively, ends up with (2) and (3), respectively. Thus these latter iterations are particular cases of (1).

Assuming infinite precision the method (1) was extensively studied in the late sixties and the early seventies where the question of convergence was completely answered by a well–known theorem of Otto Mayer [5]. It turned out that $\rho(|[A]|) < 1$ is necessary and sufficient for the convergence of (1) where $|[A]| := (\max\{|\underline{a}_{ij}|, |\overline{a}_{ij}|\}) \in \mathbb{R}^{n\times n}$ denotes the absolute value of $[A]$ and $\rho(|[A]|) := \max\{|\lambda| \mid \lambda \text{ eigenvalue of } |[A]|\}$ is the spectral radius of the real matrix $|[A]|$. It is easily seen that in this case the limit $[x]^*$ of (1) satisfies $S \subseteq [x]^*$. Moreover, $S \subseteq [x]^* \subseteq [x]^{(k+1)} \subseteq x^{(k)},\ k = 0, 1, \ldots$ can be shown if $[x]^{(1)} \subseteq [x]^{(0)}$ is true. In the case $\rho(|[A]|) < 1$ this holds, e.g., if one starts with $[x]^{(0)} = (I - |[A]|)^{-1}|[b]|[-1, 1]$ or with $[x]^{(0)} = (1 - \| \, |[A]| \, \|_\infty)^{-1}|[b]|[-1, 1]$ where the absolute value $|[b]|$ of $[b]$ is defined analogously to $|[A]|$ and where $\| \cdot \|_\infty$ denotes the row sum norm.

Two questions on (1) remained open up to now: How does the limit $[x]^*$ look like and how close with respect to the well–known Hausdorff distance does it approach the interval hull $\square(S)$ of S. If, for instance, $[A] \equiv A \in \mathbb{R}^{n\times n}$, $[b] \equiv b \in \mathbb{R}^n$ one gets $[x]^* \equiv x^* = (I - A)^{-1}b$, hence $[x]^*$ is known and equals $\square(S) = S$. If, however, $[A] = \begin{pmatrix} 0 & [-1,1] \\ [-\frac{1}{2}, \frac{1}{2}] & 0 \end{pmatrix}$, $[b] \equiv b = (-1, 1)^T$ the limit $[x]^* = ([-4, 2], [-1, 3])^T$ overestimates the interval hull $\square(S) = ([-4, 0], [\frac{1}{3}, 3])^T$. Although we could not answer our two questions completely we gained some insight into the structure of $[x]^*$. We represented $[x]^*$ by means of the solution of two coupled systems of linear equations; cf. [3]. Knowing a simple element of $[x]^*$ (which one gets, e.g., when solving a real linear system $x = Ax + b$ with fixed $A \in [A],\ b \in [b]$) one often is able to determine one half of the – a priori unknown – matrix entries of these two systems. For particular classes of matrices $[A]$ and vectors $[b]$ we even were able to represent $[x]^*$ precisely by means of midpoint and radius of $[A]$ and $[b]$; cf. [2].

As a limit case of (1) with the convergence condition $\rho(|[A]|) < 1$ we consider algebraic solutions $[x]^*$ of the equation

$$[x] = [A][x] + [b]$$

with $\rho(|[A]|) = 1$ and – as a generalization thereof - with $\rho(|[A]|) > 1$. Assuming $|[A]|$ to be irreducible (cf. [9], e.g.) both cases could completely be handled in [4] concerning existence, uniqueness and shape of $[x]^*$. In the first case 'the' Perron vector of $|[A]|$, i.e. the (up to a positive factor) unique positive eigenvector of $|[A]|$ associated with the eigenvalue $\rho(|[A]|)$ plays a crucial role while in the second case $[x]^*$ necessarily is degenerate. Maybe that in the case $\rho(|[A]|) < 1$ the final structure of $[x]^*$ can be derived by means of some ideas used in the case $\rho(|[A]|) = 1$.

Matrices $[A]$ with $\rho(|[A]|) = 1$ and reducible absolute value $|[A]|$ normally involve the case $\rho(|[\tilde{A}]|) < 1$ by some of their diagonal blocs $[\tilde{A}]$ when looking at their reducible normal form as defined in [9]. If the shape of $[x]^*$ is known for $\rho(|[\tilde{A}]|) < 1$, it can be shown that due to our results this shape is also known in the case $\rho(|[A]|) \geq 1$.

References

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