# Validation Methods and Fuzzy Set Theory: Theory, Algorithms and Application

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**Abstract**

The idea of validation arises naturally within the context of interval analysis where the theory and methods are well developed. Approaching fuzzy set theory through fuzzy interval analysis, we develop a validation theory for fuzzy sets. An algorithm is given and applied to an optimization problem.

**Keywords:** Validation, Interval Analysis, Fuzzy Set Theory, Possibility Theory

## 1 Introduction

This talk is an extension and amplification of [4]. Our discussion is restricted to domains of real fuzzy numbers (see [1]). Validation in the context of fuzzy set theory is defined below as bounding a fuzzy set by upper and lower functions that enclose membership functions.

**Definition 1.** *Given a fuzzy number $A$ with associated membership function $\mu_A(x)$, we say that $A$ is* enclosed *if there exist bounding functions $p_A(x)$ and $n_A(x)$ such that $n_A(x) \leq \mu_A(x) \leq p_A(x), \forall x$ and for all valid membership functions, $\mu_A(x)$, of $A$. Given a fuzzy number $A$ with any associated valid membership function $\mu_A(x)$, the fuzzy number $A$ is (sequentially)* validated *if there exists a sequence of functions $p_k(x)$ and $n_k(x)$ that enclose $A$ ($n_k(x) \leq \mu_A(x) \leq p_k(x)$ for all $k$) such that $p_k(x) \to \mu_A(x)$ and $n_k(x) \to \mu_A(x)$.*

The membership functions $n(x) = 0$ and $p(x) = 1$ are enclosures for any fuzzy number and hence validate every real fuzzy number. We seek the tightest bounding membership functions. Moreover, given an arbitrary measure $\mu$, possibility and necessity measures can be defined that bound $\mu$ associated with a given measurable set $A$ (see [3]). That is,

$$N(A) \leq \mu(A) \leq \Pi(A).$$

Of interest are the distributions that arise from these measures. Their construction is straightforward (see [2, 3]). It is the construction of such bounding distributions that lead to validation.

## 2 Constructing Enclosures of Expressions

Fuzzy arithmetic and operations are used on the bounding functions when enclosing an expression whereas those associated with probability are, for example, convolutions for multiplication, so in general, more complex. Fuzzy operations on bounding functions result in measures and distributions that enclose and validate the expression. As a result the expected value of the possibility and necessity distributions bound the expected value of the underlying distribution.

Thus, enclosures can be created for expressions of random variables with known probability measure. On the other hand, when no underlying probabilities exist or are too costly to obtain and one may obtain a fuzzy membership function, this fuzzy membership function can be used to create bounding possibility and necessity distributions. The possibility (respectively necessity) function that one obtains in this case is the upper (respectively lower) bound of all probability distributions (see [2] equations (10.4)–(10.7)). Thus, in [2], there is the view of possibility and necessity of a fuzzy set (membership function) as encoding, between them, all possible probability distributions associated with the imprecise outcome represented by the given fuzzy set. This satisfies the definition of enclosure.

Given probability distributions or membership functions, one can bound the results. The process for constructing the possibility distribution from probabilities or memberships can be found in [3] where the necessity distribution is obtained from the constructed possibility measure which in turn is used to construct the necessity distribution. An example is given.

## 3 Convergence

Two types of convergence are considered. The first is when uncertainty is reduced to certainty; i.e., when the real fuzzy numbers converge to real numbers. The second is convergence of bounding functions to a given underlying distribution.

**Theorem 1.** (Uncertainty reduction convergence) *Given a sequence of random variables $X_k$ with mean $M_{X_k}$ and standard deviation $\sigma_{X_k}$, if $M_{X_k} \to x^*$ and $\sigma_{X_k} \to 0$, then $X$ can be validated; i.e., $p_k(x) \to \mu_X^*(x)$ and $n_k(x) \to \mu_X^*(x)$, where $\mu_X^*(x) = \{1$ for $x \geq x^*$ and 0 otherwise$\}$, the cumulative distribution of a real number.*

The proof will be given.

**Theorem 2.** (Convergence of the enclosure to a given distribution) *Given an underlying distribution $\mu_X(x)$ there exists sequence of enclosures, $n_k(x)$ and $p_k(x)$, such that $p_k(x) \to \mu_X(x)$ and $n_k(x) \to \mu_X(x)$ $(n_k(x) \leq \mu_X(x) \leq p_k(x))$.*

This second approach to validation assumes that there is one underlying measure for each real fuzzy number or random variable in the expression. The idea is to partition the domain to obtain tighter bounds for the expression. We will illustrate the use of consistent possibility and necessity measures to obtain tight bounds on the estimated expected value via an example.

# 4 Application

We apply the ideas developed above to optimization under uncertainty.

# 5 Conclusions

We have extended the idea of validation to validation in the context of fuzzy set theory. Moreover, two approaches to fuzzy validation were developed and numerical examples presented.

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