Exact Bounds on Sample Variance of Interval Data

Scott Ferson[1], Lev Ginzburg[1],
Vladik Kreinovich[2], and Monica Aviles[2]

[1]Applied Biomathematics, 100 North Country Road,
Setauket, NY 11733, USA, {scott,lev}@ramas.com
[2]Computer Science Department, University of Texas at El Paso
El Paso, TX 79968, USA, {maviles,vladik}@cs.utep.edu

Abstract

We provide a feasible (quadratic time) algorithm for computing the lower bound $\underline{V}$ on the sample variance of interval data. The problem of computing the upper bound $\overline{V}$ is, in general, NP-hard. We provide a feasible algorithm that computes $\overline{V}$ for many reasonable situations.

Formulation of the problem. When we have n results $x_1, \ldots, x_n$ of repeated measurement of the same quantity, traditional statistical approach usually starts with computing their sample average

$$E = \frac{x_1 + \ldots + x_n}{n}$$

and their sample variance

$$V = \frac{(x_1 - E)^2 + \ldots + (x_n - E)^2}{n-1}$$

(or, equivalently, the sample standard deviation $\sigma = \sqrt{V}$); see, e.g., [1].

Sample variance is an unbiased estimator of the variance of the distribution from which observations are assumed to be randomly sampled. For Gaussian distribution, this estimator is a maximum likelihood estimator of the distribution variance.

In some practical situations, we only have intervals $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$ of possible values of x_i. This happens, for example, if instead of observing the actual value x_i of the random variable, we observe the value $\widetilde{x}_i$ measured by an instrument with a known upper bound Δ_i on the measurement error; then, the actual (unknown) value is within the interval $\mathbf{x}_i = [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$.

As a result, the sets of possible values of E and V are also intervals. The interval $\mathbf{E}$ for the sample average can be obtained by using straightforward interval computations, i.e., by replacing each elementary operation with numbers by the corresponding operation of interval arithmetic:

$$\mathbf{E} = \frac{\mathbf{x}_1 + \ldots + \mathbf{x}_n}{n}.$$

What is the interval $[\underline{V}, \overline{V}]$ of possible values for sample variance V?

When the intervals $\mathbf{x}_i$ intersect, then it is possible that all the actual (unknown) values $x_i \in \mathbf{x}_i$ are the same and hence, that the sample variance is 0. In other words, if the intervals have a non-empty intersection, then $\underline{V} = 0$. Conversely, if the intersection of $\mathbf{x}_i$ is empty, then V cannot be 0, hence $\underline{V} > 0$. The question is (see, e.g., [2]): What is the total set of possible values of V when the above intersection is empty?

For this problem, straightforward interval computations sometimes overestimate: E.g., for $\mathbf{x}_1 = \mathbf{x}_2 = [0,1]$, the actual $V = (x_1 - x_2)^2/2$ and hence, the actual range $\mathbf{V} = [0, 0.5]$. On the other hand, $\mathbf{E} = [0,1]$, hence

$$(\mathbf{x}_1 - \mathbf{E})^2 + (\mathbf{x}_2 - \mathbf{E})^2 = [0,2] \supset [0, 0.5].$$

Three intervals $\mathbf{x}_i$ equal to $[0,1]$ show that a centered form also does not always lead to the exact range.

The problem reformulated in statistical terms. The traditional sample variance is an unbiased estimator for the following problem: observation points x_i satisfy the equation $x_i = u - \varepsilon_i$, where u is an unknown fixed constant and the ε_i are independently and identically distributed random variables with zero expectation and unknown variance σ^2.

In our paper, we want to handle a situation in which each observation point $\widetilde{x}_i$ satisfies the condition $\widetilde{x}_i - u - \varepsilon_i \in \Delta_i \cdot [-1, 1]$, where the values Δ_i are assumed to be known. From this model, we can conclude that each $u + \varepsilon_i$ is contained in the corresponding interval $\widetilde{x}_i + \Delta_i \cdot [-1,1] = \mathbf{x}_i$. As a solution to this problem, we take the interval consisting of all the results of applying the estimator V to different values $x_1 \in \mathbf{x}_1, \ldots, x_n \in \mathbf{x}_n$.

Our first result: computing $\underline{V}$. First, we design a *feasible* algorithm for computing the exact lower bound $\underline{V}$ of the sample variance. Specifically, our algorithm is *quadratic-time*, i.e., it requires $O(n^2)$ computational steps for n interval data points $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$. We have implemented this algorithm in C++, it works really fast. The algorithm is as follows (the proof that this algorithm is correct will be provided in the full paper):

- First, we sort all $2n$ values $\underline{x}_i$, $\overline{x}_i$ into a sequence $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$. This sorting requires $O(n \cdot \log(n))$ steps.
- Second, we compute $\underline{E}$ and $\overline{E}$ and pick all "small intervals" $[x_{(k)}, x_{(k+1)}]$ that intersect with $[\underline{E}, \overline{E}]$.

- For each of remaining small intervals $[x_{(k)}, x_{(k+1)}]$, we compute the ratio $r_k = S_k/N_k$, where

$$S_k \stackrel{\text{def}}{=} \sum_{i:\underline{x}_i \ge x_{(k+1)}} \underline{x}_i + \sum_{j:\overline{x}_j \le x_{(k)}} \overline{x}_j,$$

and N_k is the total number of such i's and j's (if $N_k = 0$, we take $V'_k \stackrel{\text{def}}{=} 0$). If $r_k \in [x_{(k)}, x_{(k+1)}]$, then we compute

$$V'_k \stackrel{\text{def}}{=} \frac{1}{n-1} \cdot \left(\sum_{i:\underline{x}_i > x_{(k+1)}} (\underline{x}_i - r)^2 + \sum_{j:\overline{x}_j < x_{(k)}} (\overline{x}_j - r)^2 \right).$$

- Finally, we return the smallest of the values V'_k as $\underline{V}$.

Second result: computing $\overline{V}$ is NP-hard. Our second result is that the general problem of computing $\overline{V}$ from given intervals $\mathbf{x}_i$ is NP-hard.

Third result: a feasible algorithm that computes $\overline{V}$ in many practical situations. NP-hard means, crudely speaking, that there are no general ways for solving all particular cases of this problem (i.e., computing $\overline{V}$) in reasonable time.

However, we show that there are algorithms for computing $\overline{V}$ for many reasonable situations. For example, we propose an efficient algorithm $\mathcal{A}$ that computes $\overline{V}$ for the case when the "narrowed" intervals $[\widetilde{x}_i - \Delta_i/n, \widetilde{x}_i + \Delta_i/n]$ – where $\widetilde{x}_i = (\underline{x}_i + \overline{x}_i)/2$ is the interval's midpoint and $\Delta_i = (\underline{x}_i - \overline{x}_i)/2$ is its half-width – do not intersect with each other. We also propose, for each positive integer k, an efficient algorithm $\mathcal{A}_k$ that works whenever no more than k "narrowed" intervals can have a common point.

Acknowledgments. This work was supported in part by NASA under cooperative agreement NCC5-209 and grant NCC 2-1232, by NSF grants CDA-9522207, ERA-0112968 and 9710940 Mexico/Conacyt, by the Air Force Office of Scientific Research grant F49620-00-1-0365, and by Grant No. W-00016 from the U.S.-Czech Science and Technology Joint Fund.

The authors are greatly thankful to the anonymous referees for very useful suggestions.

References

[1] S. Rabinovich, *Measurement Errors: Theory and Practice*, American Institute of Physics, New York, 1993.

[2] G. W. Walster, "Philosophy and practicalities of interval arithmetic", In: R. E. Moore (ed.), *Reliability in Computing*, 1988, pp. 307–323.