Interval Arithmetic Applied to Structural Design of Uncertain Mechanical Systems

Olivier Dessombz, Fabrice Thouverez,
Jean-Pierre Laîné, and Louis Jézéquel

Laboratoire de Tribologie et Dynamique des Systèmes
École Centrale de Lyon
BP 163, 69131 Ecully Cedex, France
E-Mail: dessombz@mecasola.ec-lyon.fr

If we are interested in the static and dynamic behavior of an industrial mechanical structure, one has to consider Finite Element Modeling, which leads to matrices (such as stiffness, mass, or damping matrix). Thus, linear systems of equations are to be solved. If some of the mechanical parameters are uncertain at design stage, or are variable such as the weight of a tank, they can be modeled using the interval theory. The uncertain parameters can be geometrical ones (length, thickness, ...), or physical ones (Young's Modulus, ...). Then the matrices given by the Finite Element theory are interval matrices, and the problem is written as:

$$[A]\{x\} = \{b\} \tag{1}$$

with $[A] \in [\boldsymbol{A}]$ and $\{b\} \in \{\boldsymbol{b}\}$. Although several problems can be distinguished, as done by Chen and Ward in [1] and by Shary in [8], we will focus exclusively in this paper on the outer problem which is defined as $\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$,where $[\boldsymbol{A}]$ is an interval matrix and $\{\boldsymbol{b}\}$ an interval vector:

$$\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\}) = \{x \in \mathbb{R}^n | (\exists [A] \in [\boldsymbol{A}]), (\exists \{b\} \in \{\boldsymbol{b}\}) / [A]\{x\} = \{b\}\} \tag{2}$$

In general this set is not an interval vector. It is a non convex polyhedra. The Oettli and Prager theorem [5] give the exact solution set.

Nevertheless, this method is quite difficult to use with matrices corresponding to real physical cases in a n-dimensional problem. Most of the time, we will consider the smallest interval vector containing $\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$, which is defined as $\square\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$. In this case, this ensures that the true solution is included in the numerical solution found $\square\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$.

The existing algorithms used to solve $\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$ have been formulated for reliable computing on a numerical point of view. In an interval matrix for instance, each term can vary independently of each other in its interval, which is generally sharp.

If the interval formulation has to be adapted to mechanics, the dependence between the parameters must be taken into account. Many of the terms of the matrices are depending on the same parameters. For example if the Young's modulus varies in $\boldsymbol{E}$, the stiffness matrix can formally be written $\boldsymbol{\alpha}\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$, which is not the same as $\begin{bmatrix} \boldsymbol{\alpha}k_{11} & \boldsymbol{\alpha}k_{12} \\ \boldsymbol{\alpha}k_{21} & \boldsymbol{\alpha}k_{22} \end{bmatrix}$, that is treated as $\begin{bmatrix} \alpha_1 k_{11} & \alpha_2 k_{12} \\ \alpha_3 k_{21} & \alpha_4 k_{22} \end{bmatrix}$, with $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ varying in $\boldsymbol{\alpha}$ independently.

Moreover, the stiffness matrices are symmetric positive and definite. If all the matrices $[K] \in [\boldsymbol{K}]$ are considered, we must notice that many of them do not physically correspond to stiffness matrices. We shall then consider several problems. For a system with few independent degrees of freedom, general algorithms can be used. Nevertheless, any information on the special form of the matrices (symmetric positive and definite for a stiffness matrix, for instance) is lost, and the solution set can be widely overestimated. The problem is then formulated as:

$$[A] = [A_0] + \sum_{n=1}^{N} \boldsymbol{\epsilon_n}[A_n] \qquad \{b\} = \{b_0\} + \sum_{p=1}^{P} \boldsymbol{\beta_p}\{b_p\} \tag{3}$$

N and P are the number of parameters to be taken into account when building the matrix $[A]$ and the vector $\{b\}$. $\boldsymbol{\epsilon_n}$ and $\boldsymbol{\beta_p}$ are independent centered intervals, generally $[-1, 1]$. $[A_0]$ and $\{b_0\}$ correspond to the matrices and vector built from the mean values of the parameters.

Nevertheless, a special algorithm is required, because the solution set is not given by a combination of the bounds of the parameters. The following section is devoted to the presentation of a novel algorithm which enables to obtain a robust and including solution.

Our aim is obtaining an including solution by means of a new algorithm.

The algorithm that has been choosen is the Rump's inclusion ([4]), which relies on the fixed point theorem. In the general case, this algorithm converges rapidly, with a good accuracy (see [6]), and the convergence conditions have been studied by Rohn and Rex in [7].

In order to apply this algorithm to the mechanical formulation some adaptations are required, due to the specific differences of the mechanical problems highlighted previously.

Let us consider a system in which only one parameter is an interval. The general equation of this system is:

$$([A_0] + \alpha[A_1])\{x\} = \{b\} \qquad \alpha \in \boldsymbol{\alpha} \tag{4}$$

And the particular form of the matrix has to be taken into account (see [2]).

$$[\boldsymbol{A}] = [A_0] + \boldsymbol{\alpha}[A_1] \tag{5}$$

A method to ensure the convergence of this algorithm is also proposed (the algorithm is also based on the fixed point theorem, and the iteration matrix must be contracting). The strategy proposed is to split the interval into a partition of it, and then work on narrower intervals, on which the convergence condition will be verified.

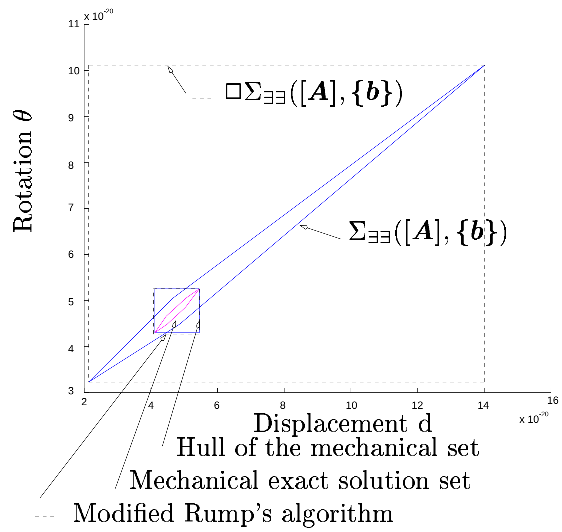

Figure 1: Solution sets for the clamped free beam. EI is uncertain ($\pm 2\%$). Numerical global problem, and reduced mechanical problem, and their respective hulls.

We can first compare the results of the general algorithm and the adapted one, to show the importance of the factorization.

The results found with the modified Rump's algorithm are often much sharper than the ones found with the classical formulation, which will be shown on a simple structure: a clamped free beam.

To test our algorithm, we have computed the result of the modified Rump's algorithm. It is illustrated on Figure 1. As we can see, it is overestimating the exact solution, but it gives a good idea of the size of the solution.

As it had been noticed in [1], a large overestimation is obtained when including the parameters in the elements of the matrices. For finite element matrices, this overestimation can become critical, and often leads to an insolvable problem. As we have shown above, even on 2×2 matrices, the overestimation can reach 10 times or more. Such an adaptation of this algorithm enables its use for industrial problems involving huge size matrices.

If static problems can be solved, which means finding solutions for linear interval systems, the algorithm can also be applied to the calculation of transfer functions.

We consider some kind of realistic structure. It is a two dimensional frame structure (Figure 2). It can model for instance the structure of a building, as in [3].

It is a 18 elements structure, with 12 degrees of freedom. Only a concentrated load on the beams is considered, applied on the DOFF 3, the torque $F = 10^3\ Nm$. The parameters of the model are the lengths of the beams $L1 = L2 = 1\ m$, the inertia $I = \pi\ 10^{-8}/4\ m^4$ and their area $S = \pi\ 10^{-4}\ m^2$. We assume that the bending rigidity E is uncertain ($E = 210 \pm 10\%\ GPa$).

We will study the two-dimensional frame structure from a dynamical point of view. In the Finite Element Model, we use a Euler Bernoulli Beam model, leading to a stiffness and a mass matrix. We suppose that there is hysteretic damping ($\eta = 2\%$) in the model. Because all of the beams are identical, some of the modes are relatively close to each other. This means that when the bending rigidity is varying, the eigenmodes can overlap each others. This behavior is illustrated in the Figure 3, where a harmonic torque is applied on node 3 of the truss. The collocated transfer function $H(3,3)$ is computed thanks to the proposed algorithm, and compared to deterministic transfer functions calculated for various values of the Young's modulus.

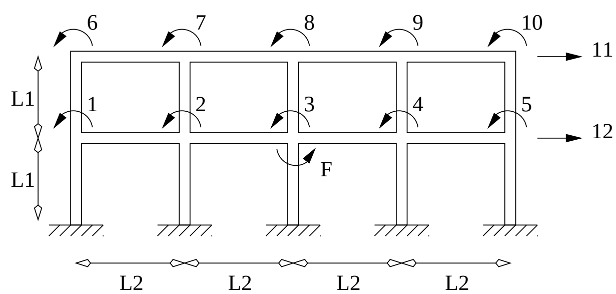

Figure 2: Two dimensional frame structure with 12 DOFF. A torque F is applied on node 3.

As we can see in Figure 3, the proposed algorithm can take overlapping eigenfrequencies into account. It leads to an envelope of the modulus of the transfer function. Some of the deterministic transfer function have been plotted, to show that the envelope found does not overestimate the real solution too much.

We will also consider a three blades wheel that is modeled with a 7 DOFF system (see Figure 4). As the 3 blades are identical in the deterministic model, the eigenfrequencies are found as multiple eigenvalues of a matrix system. If one of the blades is mistuned, then the eigenvalues are no more multiple ones, and new resonances can appear.

On Figure 5 the transfer function $H(1,3)$ is shown. Blue line represent the deterministic case for which all of the three blades are identical, and magenta lines the envelope of that transfer function, when the blade one has an uncertain Young's modulus ($E = E_0 \pm 10\%$). The envelope shows four resonance zones This is due to the mistuning phenomenon.

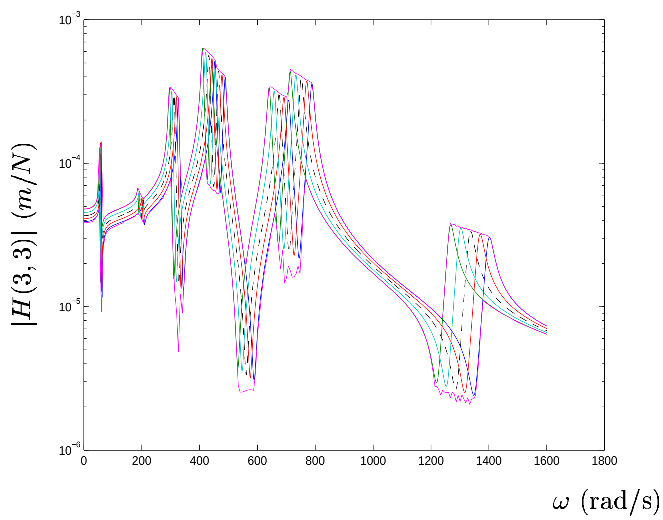

Figure 3: Modulus of the collocated transfer function $H(3,3)$ for the frame. The bending rigidity is uncertain ($E = 210 \pm 10\%$ GPa). The min and max values calculated with the modified algorithm are represented, wrapping the transfer function for several values of E.

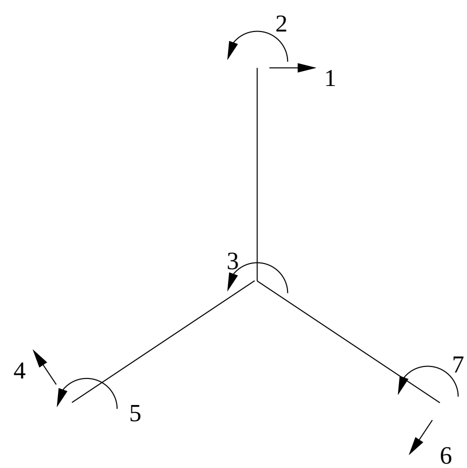

Figure 4: 3 blades wheel, and the 7 DOFF.

The modified algorithm gives accurate results, once again with the advantage of getting a robust envelope. This method can improve considerably the accuracy of prediction of the dynamic behavior of mechanical systems.

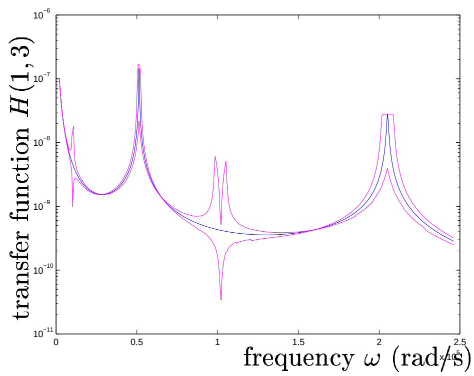

Figure 5: Modulus of the transfer function $H(1,3)$. The Young's modulus of the first blade is uncertain ($E = E_0 \pm 10\%$).

References

[1] R. Chen and A. C. Ward, "Generalizing interval matrix operations for design", *Journal of Mechanical Design*, 1997, Vol. 119, pp. 65–72.

[2] O. Dessombz, F. Thouverez, J.P. Laîné, and L. Jézéquel, "Analysis of mechanical systems using interval computations applied to finite elements methods", *Journal of Sound and Vibration*, 2001, Vol. 239, No. 5, pp. 949–968.

[3] H. U. Koyluoglu, A. S. Cakmak, and S. R. K. Nielsen, "Interval algebra to deal with pattern loading and structural uncertainties", *Journal of Engineering Mechanics*, Nov. 1995, pp. 1149–1157.

[4] S. M. Rump, "Solving algebraic problems with high accuracy", In: U. Kulisch and W. Miranker, *A New Approach to Scientific computation*, Academic Press, New York, 1983, pp. 51–120.

[5] W. Oettli and W. Prager, "Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides", *Math.*, 1964, Vol. 6, pp. 405–409.

[6] J. Rohn, *Validated solutions of linear equations*, Technical Report 620, Institute of Computer Science, Academy of Sciences of the Czech Republic, Jan. 1995.

[7] J. Rohn and G. Rex, *Enclosing solutions of linear equations*, Technical Report 666, Institute of Computer Science, Academy of Sciences of the Czech Republic, Feb. 1996.

[8] S. P. Shary, "On optimal solution of interval linear equations", *SIAM J. Numer.Anal.*, 1995, Vol. 32, No. 2, pp. 610–630.