Efficient and Reliable Multiprecision implementation of Elementary and Special Functions

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Problem statement

1. Efficient multiprecision implementation of elementary functions

2. Approximation of some special multivariate functions

- obtaining an appropriate approximation is very time-consuming
- most complex useful model is a rational approximant

e.g.

$$\left|\frac{\exp(x) - \mathtt{EXP}(x)}{\exp(x)}\right| \leq 2^{-52}$$

Mathematical technique

- step I: argument reduction

$$
\begin{aligned}
\cos(z) &= \cos(2\pi - z) \\
\cos(z) &= -\cos(\pi - z) \\
\cos(z) &= \sin(\pi/2 - z) \\
\sin(z) &= \sin(\pi - z) \\
\sin(z) &= -\sin(2\pi - z) \\
\sin(12532.14) &= \sin(3.47)!!
\end{aligned}
$$

$$
\begin{aligned}
\exp(x) &= \exp(k \ln 2 + r) \\
& |r| \leq \frac{\ln 2}{2} \\
& \ln 2 \approx \mathtt{LN2} \in \mathbb{F}_{106} \\
& r = (\mathtt{hi} - \mathtt{lo})(1 + \epsilon) \\
& |\epsilon| \leq 2^{-106}
\end{aligned}
$$

- step II: mathematical model

$$E(r) \approx \exp(r)$$

bound truncation error $|\exp(r) - E(r)|$

$$\begin{aligned}
f(r^2) &= r\,\frac{\exp(r)+1}{\exp(r)-1} \\
\exp(r) &= \frac{f(r^2)+r}{f(r^2)-r} \\
f(r^2) &\approx p(r^2) = 2 + \sum_{i=1}^{5} p_i\, r^{2i} \\
p_i &\in \mathbb{F}_0(2, 53, -1022, 1023) \\
|(f-p)(r^2)| &\leq 2^{-59} \qquad 0 \leq r \leq \frac{\ln 2}{2}
\end{aligned}$$

$$\begin{aligned}
\exp(r) &\approx E(r) = \frac{p(r^2)+r}{p(r^2)-r} \\
p_1 &= \texttt{3FC5}\ldots\texttt{53E} \qquad \text{16 hex digits} \\
&\approx 1.66\ldots 66019037 \times 10^{-1}
\end{aligned}$$

- step III: arithmetic implementation

$$\texttt{EXP}(r) := E(r)$$

bound rounding error $|\texttt{EXP}(r) - E(r)|$

$$\begin{aligned} E(r) &= 1 + r + \frac{r \times q(r)}{2 - q(r)} \qquad |r| \le \frac{\ln 2}{2} \\ q(r) &= r - (p_1 r^2 + p_2 r^4 + p_3 r^6 + p_4 r^8 + p_5 r^{10}) \end{aligned}$$

$$\begin{aligned} s &= \texttt{hi} \ominus \texttt{lo} \\ t &= s \otimes s \\ q &= s \ominus t \otimes (p_1 \oplus t \otimes (\\ &\qquad p_2 \oplus t \otimes (p_3 \oplus t \otimes (p_4 \oplus t \otimes p_5)))) \end{aligned}$$

$$\texttt{EXP}(r) = 1 \ominus ((\texttt{lo} \ominus (s \otimes q) \oslash (2.0 \ominus q)) \ominus \texttt{hi})$$

$$\left| \frac{2^k \times \texttt{EXP}(r) - \exp(x)}{\exp(x)} \right| \le 2^{-52}$$

- step IV: verification

$$x_i = -1 + i \times 2^{-18} \qquad i = 0, \ldots, 2^{19}$$

$$\left| \frac{\exp(x_i) - \texttt{EXP}(x_i)}{\exp(x_i)} \right|$$

$$\beta = 2, t = 53 \Rightarrow \frac{1}{2}\mathrm{ULP} = 2^{-53}$$

Multiprecision implementation

reliability:

$$\left|\frac{func(x) - F(x)}{func(x)}\right| \le \beta^{-t+1} = 1\text{ULP}(t)$$

base β: $\beta = 2^k, 10^k$

precision t: $t \gg 64$

- determine mathematical model $f(x)$ for the function $func(x)$ such that the truncation error satisfies:

$$\left|\frac{func(x) - f(x)}{func(x)}\right| \le \frac{1}{2}\beta^{-t+1}$$

- determine working precision s such that the rounding error on the implementation $F(x)$ satisfies:

$$\left|\frac{f(x) - F(x)}{func(x)}\right| \le 1/2\beta^{-t+1}$$

directed roundings:

$$x \text{ machine number}$$

$$\underline{F}(x) \leq func(x)$$

$$func(x) \leq \overline{F}(x)$$

$$\underline{F}(x) \leq F(x) \leq \overline{F}(x)$$

Elementary and special functions

univariate elementary/hypergeometric functions:

exp	gon	inv	hyp
	$\sin(x)$	$\arcsin(x)$	$\sinh(x)$
	$\cos(x)$	$\arccos(x)$	$\cosh(x)$
	$\tan(x)$	$\arctan(x)$	$\tanh(x)$
	$\cot(x)$	$\mathrm{arccot}(x)$	$\coth(x)$
e^x		$\log(x)$	
10^x		$\log(10, x)$	
2^x		$\log(2, x)$	

$$(a)_i := \begin{cases} a(a+1)(a+2)\ldots(a+i-1) & i \geq 1 \\ 1 & i = 0 \end{cases}$$

$${}_2F_1(a, b; c; z) := \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i i!} z^i$$

- exponential function

$$
\begin{aligned}
e^z &= \frac{1\,|}{|\,1} - \frac{z\,|}{|\,1} + \sum_{n=3}^{\infty} \frac{a_n\,|}{|\,1} \\
a_{2n} &= \frac{-z}{2(2n-1)}, \quad a_{2n+1} = \frac{z}{2(2n-1)} \\
\lim_{n\to\infty} a_n &= 0
\end{aligned}
$$

- natural logarithm

$$
\begin{aligned}
\log(1+z) &= \frac{z\,|}{|\,1} + \sum_{n=2}^{\infty} \frac{a_n\,|}{|\,1} \\
a_{2n} &= \frac{nz}{2(2n-1)}, \quad a_{2n+1} = \frac{nz}{2(2n+1)} \\
\lim_{n\to\infty} a_n &= \frac{1}{4} z > 0 \text{ when } 0 < z
\end{aligned}
$$

- tangent function

$$
\begin{aligned}
\tan(z) &= \frac{z\,|}{|\,1} + \sum_{n=2}^{\infty} \frac{a_n\,|}{|\,1} \\
a_{n+1} &= \frac{-z^2}{4n^2-1} \\
\lim_{n\to\infty} a_n &= 0
\end{aligned}
$$

- arctangent function

$$
\begin{aligned}
\arctan(z) &= \frac{z\,|}{|\,1} + \sum_{n=2}^{\infty} \frac{a_n\,|}{|\,1} \\
a_{n+1} &= \frac{n^2 z^2}{4n^2-1} \\
\lim_{n\to\infty} a_n &= \frac{1}{4} z^2
\end{aligned}
$$

univariate special functions:

- incomplete gamma function

$$
\begin{aligned}
\gamma(a,z) &= \frac{e^z z^a|}{|1+z-a} + \sum_{n=2}^{\infty} \frac{a_n|}{|1} \\
& \qquad\qquad 0 \le z \le 1 \\
a_{n+1} &= \frac{-n(n-a)}{(2n-1+z-a)(2n+1+z-a)} \\
\lim_{n\to\infty} a_n &= -\frac{1}{4}
\end{aligned}
$$

- error function

$$
\begin{aligned}
\mathrm{erfc}(z) &= \frac{e^{-z^2}|}{|2z} + \sum_{n=2}^{\infty} \frac{a_n|}{|1} \\
a_{n+1} &= \frac{n}{2z^2} \\
\lim_{n\to\infty} a_n &= \infty
\end{aligned}
$$

multivariate special functions:

- Beta function

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

meromorphic with poles at $x = -k$ and $y = -k$ and zeroes at $y = -x - k$ for $k = 0, 1, 2, \ldots$

- Appell functions

$$F_1(a,b,b';c;x,y) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b)_i(b')_j x^i y^j}{(c)_{i+j} i! j!}$$

$$F_2(a,b,b';c,c';x,y) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b)_i(b')_j x^i y^j}{(c)_i(c')_j i! j!}$$

$$F_3(a,a',b,b';c;x,y) = \sum_{i,j=0}^{\infty} \frac{(a)_i(a')_j(b)_i(b')_j x^i y^j}{(c)_{i+j} i! j!}$$

$$F_4(a,b;c,c';x,y) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b)_{i+j} x^i y^j}{(c)_i(c')_j i! j!}$$

Truncation error univariate case

$z^* \in \mathbb{R}$:

$$func(z^*) = \sum_{n=1}^{\infty} \frac{a_n \rvert}{\lvert 1} \qquad a_n \equiv a_n(z^*)$$

e.g. $\log \sqrt{2}$

$$f_N(w) = \sum_{n=1}^{N-1} \frac{a_n \rvert}{\lvert 1} + \frac{a_N \rvert}{\lvert 1 + w}$$

$$t_N = \sum_{n=N+1}^{\infty} \frac{a_n \rvert}{\lvert 1}$$

e.g. $\lim_{N \to \infty} t_N = \left(\sqrt[4]{2} - 1 \right) / 2 \approx 0.095$

$$f_N^{(k)}(w) = \sum_{n=k+1}^{N-1} \frac{a_n \rvert}{\lvert 1} + \frac{a_N \rvert}{\lvert 1 + w}$$

sequence of value sets $\{V_n\}_{n=1}^{\infty}$ for $func(z^*)$:

$$s_n(V_n) := \frac{a_n}{1 + V_n} \subseteq V_{n-1}$$

e.g. $w \in V_N$

$f_N^{(k)}(w) \in V_k$

$func(z^*) \in V_1$

sequence of convergence sets $\{E_n\}_{n=1}^{\infty}$ for $func(z^*)$:

$$\forall n \geq 1 : a_n \in E_n \Longrightarrow \sum_{n=1}^{\infty} \frac{a_n \rvert}{\lvert 1} \text{ convergent}$$

Theorem 1 [Lorentzen & Ruscheweyh, 1992]:

Let $C > -1/2$ and $0 < R < 1 + C$ be given. Then

$$E = \{a \in \mathbb{R} : |a(1+C) - C((1+C)^2 - R^2)| +$$
$$+R|a| \leq R((1+C)^2 - R^2)\}$$

is a convergence set for $\sum_{n=1}^{\infty} \frac{a_n|}{|1}$ and

$$V = \{w \in \mathbb{R} : |C - w| < R\}$$

is a value set for E.

$$w \in V \Rightarrow \begin{cases} |func - f_N(w)| \leq 2R\dfrac{|C| + R}{1 + C - R} M^{N-1} \\ M = \max\{|\dfrac{w}{1+w}| : w \in \overline{V}\} \end{cases}$$

Remark: E above equals $[p, q]$ with

$$\begin{aligned} p &= \begin{cases} (C - \text{sign}(C)R)(1 + C + R) & \text{if} \quad R \leq |C| \\ (C - \text{sign}(C)R)(1 + C - R) & \text{if} \quad R > |C| \end{cases} \\ q &= (C + \text{sign}(C)R)(1 + C - R) \end{aligned}$$

Example 1

$$\log(1+z^*) = \frac{z^*|}{|1} + \sum_{n=2}^{\infty} \frac{a_n|}{|1}$$

$$a_{2n} = \frac{nz^*}{2(2n-1)}, \quad a_{2n+1} = \frac{nz^*}{2(2n+1)}$$

For $z^* = \sqrt{2} - 1$:

$$\begin{aligned} p &= z^*/6 \qquad q = z^* \\ R &= 5z^*/12 \quad C = \tfrac{\sqrt{36+84z^*+25z^{*2}}-6}{12} = 0.2221 \\ M &= 0.282988 \end{aligned}$$

$$\left|\log(\sqrt{2}) - f_N(w)\right| \leq \frac{5z^*}{12} \frac{C+R}{1+C-R} M^{N-1} \leq 1/2\beta^{-t+1}$$

$$\Downarrow$$

$$\beta = 2,\, t = 53 : \; N \geq 29$$

Theorem 2 [Lorentzen & Waadeland, 1992]:

Let $0 < R_n < 1 + C_n$, $|C_{n-1}|R_n < |1 + C_n|R_{n-1}$. Then

$$V_n = \{w \in \mathbb{R} : |C_n - w| < R_n\}$$

is a sequence of value sets for

$$E_n = \{a \in \mathbb{R} : |a(1+C_n) - C_{n-1}((1+C_n)^2 - R_n^2)| +$$

$$+R_n|a| \leq R_{n-1}((1 + C_n)^2 - R_n^2)\}$$

$$w \in V_N \Rightarrow \begin{cases} |func - f_N(w)| \leq 2R_N \dfrac{|C_0| + R_0}{1 + C_N - R_N} \\ \qquad\qquad\qquad\qquad \times \displaystyle\prod_{k=1}^{N-1} M_k \\ M_k = \max\{|\dfrac{w}{1+w}| : w \in \overline{V}_k\} \end{cases}$$

Remark: E_n above equals $[p_n, q_n]$ with

$$p_n = \begin{cases} (|C_{n-1}| - R_{n-1})(1 + C_n + R_n), & R_{n-1} \leq |C_{n-1}| \\ (|C_{n-1}| - R_{n-1})(1 + C_n - R_n), & R_{n-1} > |C_{n-1}| \end{cases}$$

$$q_n = (|C_{n-1}| + R_{n-1})(1 + C_n - R_n)$$

Example 2

$$\log(1+z^*) = \frac{z^*|}{|1} + \sum_{n=2}^{\infty} \frac{a_n|}{|1}$$

$$a_{2n} = \frac{nz^*}{2(2n-1)}, \quad a_{2n+1} = \frac{nz^*}{2(2n+1)}$$

$$\lim_{n\to\infty} a_n = \frac{1}{4} z^*$$

For $z^* = \sqrt{2} - 1$:

$$C_n \equiv C = \tfrac{\sqrt{1+z^*}-1}{2} \text{ from } C = \tfrac{z^*/4}{1+C}$$

$$R_n = \tfrac{D-\sqrt{D^2-4d_{n+1}}}{2}$$

with $d_n = \max\limits_{m \geq n} |a_m - \frac{z^*}{4}|$ and $D = 1$.

$$\left|\log(\sqrt{2}) - f_N(w)\right| \leq \frac{1}{2}\beta^{-t+1}$$

$$\Downarrow$$

$$\beta = 2,\, t = 53 :\; N \geq 17$$

Multiprecision case

- For $\log(\sqrt{2})$ or $z^* = 2^{\frac{1}{2}} - 1$:

t \ N	$\vert func - f_N(w)\vert$	Thm 1	Thm 2
53	14	29	17
113	31	62	34
256	71	140	74
512	143	281	147

- For $\log(2^{1/4})$ or $z^* = 2^{\frac{1}{4}} - 1$:

t \ N	$\vert func - f_N(w)\vert$	Thm 2
53	11	11
113	24	24
256	55	57
512	111	115

- For $\log(2^{1/8})$ or $z^* = 2^{\frac{1}{8}} - 1$:

t \ N	$\lvert func - f_N(w) \rvert$	Thm 2
53	9	9
113	20	21
256	39	46
512	91	94

Numerical results multivariate case

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$\begin{aligned}
B(x,y) &\approx r(x,y) \\
r(x,y) &= \frac{\sum_{k=0}^{n} a_{i_k j_k} x^{i_k} y^{j_k}}{b_{00} + b_{10}x + b_{01}y + b_{11}xy} \\
\{(i_k, j_k)\} &= \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2), \\
&\qquad (3,0),\ldots,(0,3),\ldots,(s,0),\ldots,(0,s)\} \\
n+1 &= \frac{(s+1)(s+2)}{2} \\
(x,y) &= (-0.75, -0.75) \\
B(x,y) &= 9.88839829\ldots
\end{aligned}$$

n \ m	0	2	3
95	6.272469539927058	6.382241356832849	6.252199337861037
96	6.272481219879103	6.373969995330925	6.252199326007094
97	6.272487831740126	6.382244442674818	6.266577260911057
98	6.272494443601187	6.410072833517622	6.300404767334740
99	6.272506123553238	6.469560908765764	6.369481476489413
100	6.272544497731672	6.597917327186983	9.722877909033864
101	6.272810203348119	9.737548612905988	9.895775024125525
102	6.277863394645692	9.847326401470260	9.312348514619972
103	6.757612224375486	5.373819134560423	9.420269191090856
104	6.517174482717973	9.879269550748257	9.920113846171462
105	6.306954585440170	9.801348477243014	9.793178514188524
106	6.726741855333899	7.021564358750562	6.826017824698794
107	6.729671095099285	6.906974546767763	6.761484996863231
108	6.729784277460005	6.852286196295363	6.727815545486309
109	6.729796844258718	6.824588041048912	6.710913104188903
110	6.729799835419403	6.812657490354739	6.705716580531475
111	6.729801157749264	6.812657738724513	6.710913100069901
112	6.729802171934143	6.824588676978205	6.727815664029961
113	6.729803494264011	6.852286709821294	6.761485208089148
114	6.729806485424670	6.906972882457715	6.826017206347385
115	6.729819052223378	7.021547253788186	9.792927727303981
116	6.729932234584101	9.799275262541185	9.891548307005404
117	6.732861474349487	9.864013647323537	9.553039471914062
118	7.152648744243218	5.949314052499588	9.619286021637370
119	6.942428846965417	9.882493034609045	9.905525832249440

$$s = 13 \quad \rightarrow \quad n = 104$$

$$s = 14 \quad \rightarrow \quad n = 119$$

n \ m	2	3
100	$[6.5979172, 6.5979175]$	$[9.722877904, 9.722877914]$
101	$[9.737548608, 9.737548618]$	$[9.89577500, 9.89577505]$
102	$[9.84732638, 9.84732643]$	$[9.3123483, 9.3123487]$
103	$[5.373819134, 5.373819135]$	$[9.4202690, 9.4202694]$
104	$[9.87926953, 9.87926957]$	$[9.92011383, 9.92011387]$
105	$[9.801348474, 9.801348481]$	$[9.793178510, 9.793178518]$
106	$[7.0215641, 7.0215646]$	$[6.826016, 6.826020]$
107	$[6.906973, 6.906976]$	$[6.76147, 6.76150]$
108	$[6.852282, 6.852290]$	$[6.72778, 6.72785]$
109	$[6.82458, 6.82460]$	$[6.7108, 6.7110]$
110	$[6.812654, 6.812661]$	$[6.7056, 6.7059]$

$\rightarrow$ rounding error under control

$\rightarrow$ highly varying truncation error selection of optimal subsequence,

i.e. optimal numerator form

Theoretical results multivariate case

Appell functions $\qquad F_i(x,y) \qquad i=1,\ldots,4$

$$F_i(z,0)=F_i(0,z)={}_2F_1(z)$$

homogeneous Padé approximant:

$$\begin{aligned} A_k(x,y) &= \sum_{i+j=nm+k} a_{ij}x^iy^j & k&=0,\ldots,n \\ B_k(x,y) &= \sum_{i+j=nm+k} b_{ij}x^iy^j & k&=0,\ldots,m \end{aligned}$$

$$r(x,y) = \frac{\sum_{k=0}^{n} A_k(x,y)}{\sum_{k=0}^{m} B_k(x,y)}$$

normal Padé approximant: unique over all n, m

normal Padé table: all $[n/m]^f(z)$ are normal

- univariate Padé table for the hypergeometric series ${}_2F_1(a, 1; c; z)$ with $c > a > 0$ is normal
- generalized to the homogeneous Padé table for the Appell functions $F_1, \ldots, F_4$
- univariate Padé approximants $[n/n-k]$ for ${}_2F_1(a, 1; c; z)$ with $a, c, c-a \notin \mathbb{N}^-$ are normal
- generalized to the homogeneous Padé table for the Appell functions $F_1, \ldots, F_4$
- uniform convergence and convergence in measure of certain multivariate Padé approximants for the Beta function
- explicit formula for denominator of $[n/n-k]$ for ${}_2F_1(a, 1; c; z)$ with $n \geq k \geq -1$
- looking for explicit formulae for multivariate Padé denominators

Rounding error univariate case

$$
\begin{aligned}
f_N(w) &= \sum_{k=1}^{N-1} \frac{a_k|}{|1} + \frac{a_N|}{|w} \\
&\Downarrow \\
+ &\to \oplus \\
\div &\to \oslash \\
\hat{a}_k &= a_k(1+\alpha_k) \\
& \qquad\qquad \alpha_k, \omega : \text{ relative data error} \\
\hat{w} &= w(1+\omega)
\end{aligned}
$$

backward recurrence algorithm:

$$
\begin{aligned}
G_{N+1}^{(N)} &= w \\
G_k^{(N)} &= \frac{a_k}{1+G_{k+1}^{(N)}} \qquad k = N, N-1, \ldots, 1 \\
f_N(w) &= G_1^{(N)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{G}_{N+1}^{(N)} &= \hat{w} = w(1+\omega) \\
\hat{G}k^{(N)} &= \hat{a}_k \oslash \left(1 \oplus \hat{G}_{k+1}^{(N)}\right) \qquad k = N, \ldots, 1 \\
&= \frac{\hat{a}_k}{1 + \hat{G}_{k+1}^{(N)}}(1+\gamma_k)
\end{aligned}
$$

γ_k : relative rounding error

$$
\hat{G}_1^{(N)} = G_1^{(N)}(1+\epsilon_1^{(N)})
$$

accumulated error $|\epsilon_1^{(N)}| \leq$?

$$
\begin{aligned}
\hat{G}_{N+1}^{(N)} &= G_{N+1}^{(N)}(1+\epsilon_{N+1}^{(N)})
\end{aligned}
$$

$$
\begin{aligned}
\hat{G}_k^{(N)} &= \frac{\hat{a}_k}{1+\hat{G}_{k+1}^{(N)}}(1+\gamma_k) \\
&= \frac{a_k(1+\alpha_k)}{1+G_{k+1}^{(N)}(1+\epsilon_{k+1}^{(N)})}(1+\gamma_k) \\
&= G_k^{(N)}(1+\epsilon_k^{(N)})
\end{aligned}
$$

notation:

$$
g_k^{(N)} = G_{k+1}^{(N)}/(1+G_{k+1}^{(N)}) \qquad k=1,\ldots,N
$$

$$
\begin{aligned}
\epsilon_{N+1}^{(N)} &= \omega \\
\epsilon_k^{(N)} &= \frac{\alpha_k+\gamma_k+\alpha_k\gamma_k-g_k^{(N)}\epsilon_{k+1}^{(N)}}{1+g_k^{(N)}\epsilon_{k+1}^{(N)}}
\end{aligned}
$$

Theorem 3 [Jones & Thron, 1974]
[Bonan & al., 2001]

Let $G_{N+1}^{(N)} = w$ be a machine number and let for

$k = 1, \ldots, N$

$$|\alpha_k| \leq \alpha \mathrm{ULP}(s) \quad |\gamma_k| \leq \gamma \mathrm{ULP}(s) \quad \alpha + \gamma \geq 1$$

$$|g_k^{(N)}| \leq M$$

Under trivial conditions for β and s:

$$|\epsilon_1^{(N)}| \leq \frac{1}{2}(1 + 2\alpha + 2\gamma)\mathrm{ULP}(s) \sum_{j=0}^{N-1} M^j$$

key : determination of upperbound M for $g_k^{(N)}$!

Lemma [Jones & Thron, 1974]:
Let $\{V_n\}_{n=1}^{\infty}$ be a sequence of value sets for $\sum\limits_{n=1}^{\infty} \left\lceil \frac{a_n}{1} \right|$. If $G_{N+1}^{(N)} = w \in V_N$, then

$$
\begin{aligned}
|g_k^{(N)}| &= \left| \frac{G_{k+1}^{(N)}}{1 + G_{k+1}^{(N)}} \right| \\
&\leq M_k = \max\{|\frac{w}{1+w}| : w \in \overline{V}_k\} \\
&\leq M = \max_{k=1,\ldots,N} M_k
\end{aligned}
$$

Theorem 4 [Verdonk & al., 2001]
Let $0 < R_n < 1 + C_n$, $|C_{n-1}|R_n < |1 + C_n|R_{n-1}$.
Then

$$V_n = \{w \in \mathbb{R} : |C_n - w| < R_n\}$$

is a sequence of value sets for

$$E_n = \{a \in \mathbb{R} : |a(1+C_n) - C_{n-1}((1+C_n)^2 - R_n^2)| +$$

$$+R_n|a| \leq R_{n-1}((1 + C_n)^2 - R_n^2)\}$$

Let α, β and s be as in Theorem 3. If $w \in V_N$ is a machine number, then

$$\begin{cases} |func - f_N(w)| \leq 2R_N \dfrac{|C_0| + R_0}{1 + C_N - R_N} \displaystyle\prod_{k=1}^{N-1} M_k \\ \left|\dfrac{f_N(w) - F}{f_N(w)}\right| \leq \dfrac{1}{2}(1 + 2\alpha + 2\gamma)\mathrm{ULP}(s)\times \\ \qquad\qquad\qquad\qquad \displaystyle\sum_{j=0}^{N-1} \prod_{k=0}^{j-1} M_{k+1} \\ M_k = \max\{|\dfrac{w}{1 + w}| : w \in \overline{V}_k\} \end{cases}$$