Reliable multiprecision implementation of elementary and special functions

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In the past few years a lot of progress has been made in obtaining sharp truncation and rounding error upper bounds for continued fraction approximants. It is well-known that a wide range of functions can be approximated much more accurately by rational functions. Often these rational approximants are obtained by truncating a continued fraction expansion for the function. It is less well-known that the truncated part of a convergent continued fraction need not converge to zero, unlike the tail of a convergent Taylor series expansion.

1. Univariate functions.

Consider for instance the continued fraction expansion

$$\sum_{i=1}^{\infty} \frac{x|}{\lceil 1}$$

Its tails converge to $1/2(\sqrt{1+4x}-1)$. More remarkable is that the even-numbered tails of the convergent continued fraction

$$\sum_{i=1}^{\infty} \left(\frac{1|}{\lceil 1} + \frac{2|}{\lceil 1} \right)$$

converge to $\sqrt{2}-1$ while the odd-numbered tails converge to $\sqrt{2}$ (hence the tails do not converge), and that the tails of

$$\sum_{i=0}^{\infty} \frac{i(i+2)|}{\lceil 1}$$

converge to $+\infty$.

When carefully monitoring the behaviour of these continued fraction tails, a practical version of the oval sequence theorem [5] leads to a very sharp truncation error upper bound. The slight overestimation is of no practical importance anymore when used in an implementation.

As far as the rounding error is concerned, the traditional result given in [4], which holds when the truncated tail value is replaced by zero, can be generalized to a modified version, in which the tail value is estimated by a machine number.

Combining the latest results on truncation and rounding error estimates allows to develop a fast and reliable multiprecision implementation of some elementary functions, in the rounding modes nearest, up (the cumulative error upper bound is added), down (the cumulative error upper bound is subtracted), zero and interval.

2. Multivariate functions.

For multivariate functions a lot less is known. When constructing multivariate rational approximants there's a wealth of choice in arranging the variables and/or the information available on the function (function values, derivative values). But the number of valuable convergence theorems for these approximation processes is much less.

As of now, the most general result can be found in [1]. It was subsequently put to good use in the construction of some rational approximants for the Apfunctioopell functions

$$F_1(a,b,\beta;c;x,y) \quad F_2(a,b,\beta;c,\gamma;x,y) \qquad F_3(a,\alpha,b,\beta;c;x,y) \qquad F_4(a,b;c,\gamma;x,y)$$

which are bivariate generalizations of the univariate Gauss or ordinary hypergeometric function.

In [3] the authors resorted to the use of interval arithmetic when computing rational approximants to the bivariate Beta function, defined by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

to formulate a conjecture that was afterwards proved in [2]. In the near future a reliable multiprecision implementation of several multivariate functions is planned. Some open problems that lie ahead will be pointed out.

References.

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