Optimal Inclusion of the Solution Set concerning a Linear Complementarity Problem with Interval Data

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I. The Linear Complementarity Problem

Given: $A \in R^{n\times n}$ **and** $b \in R^n$.

$LCP(A,b)$: **Find** $x \in R^n$ **with**

$$\begin{aligned} b + Ax &\geq o, \\ x &\geq o, \\ (b+Ax)^T x &= 0. \end{aligned}$$

Definition:

Let $A = (a_{ij}) \in R^{n\times n}$**, then:**

1. $A \in Z^{n\times n} :\Leftrightarrow a_{ij} \leq 0, i \neq j.$

2. A **is called M-matrix**

$:\Leftrightarrow A \in Z^{n\times n}$**,** $A^{-1} \geq O$**.**

Lemma (Fan)

Let A **be an M-matrix and**

$A \leq B \in Z^{n\times n}$**,**

then B **is an M-matrix, too,**

i.e. $B^{-1} \geq O$**.**

Definition:

Let $[A] = [\underline{A}, \overline{A}] = ([a_{ij}]) = ([\underline{a}_{ij}, \overline{a}_{ij}]) \in IR^{n\times n}$**, then:**

1. $[A]$ **is an M-matrix**

$:\Leftrightarrow$ **each** $A \in [A]$ **is an M-matrix.**

2. $[A]$ **is an H-matrix,**

if $\langle [A] \rangle = (c_{ij})$ **is an M-matrix, where**

$$c_{ij} = \begin{cases} \min\{|a| : a \in [a_{ij}]\} & \textbf{if } i = j, \\ -\max\{|a| : a \in [a_{ij}]\} & \textbf{if } i \neq j. \end{cases}$$

3. $[A]$ **is called H$^+$-matrix,**

if $[A]$ **is an H-matrix and**

$0 < \underline{a}_{ii}$, $i = 1, ..., n$**.**

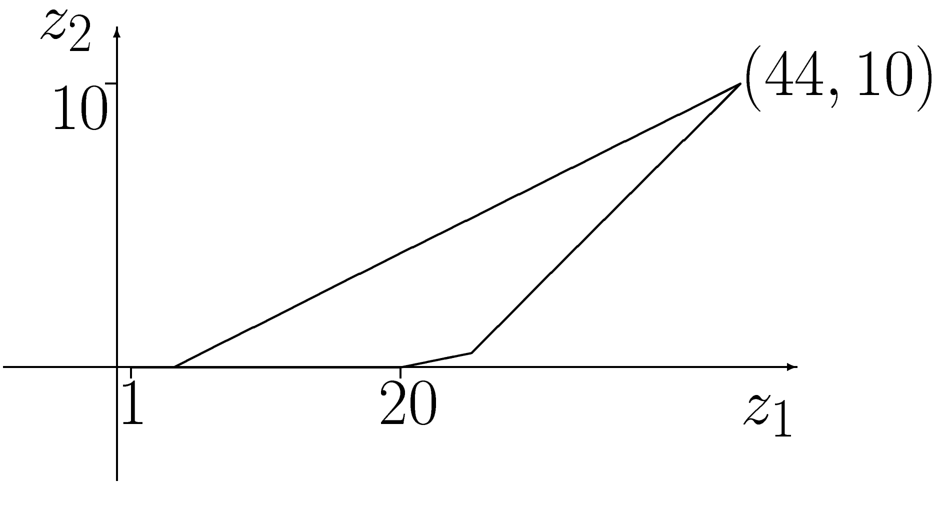
II. The Solution set $\Sigma = \Sigma([A],[b])$

$$\Sigma := \{x : x \textbf{ solves } LCP(A,b),$$
$$A \in [A],\, b \in [b]\}$$

Example:

$$[A] = \begin{pmatrix} [0.125, 1] & [-0.25, -0.2] \\ [-0.25, -0.1] & 1 \end{pmatrix},$$

$$[b] = \begin{pmatrix} [-3, -1] \\ [1, 2] \end{pmatrix}$$



$$[\inf \Sigma, \sup \Sigma] = \begin{pmatrix} [1, 44] \\ [0, 10] \end{pmatrix}$$

Theorem

Let $[A] = [\underline{A}, \overline{A}]$ **an M-matrix and** $[b] = [\underline{b}, \overline{b}]$ **arbitrary. Then:**

1. $\inf \Sigma = u$**,**

u **unique solution of** $LCP(\overline{A}, \overline{b})$

2. $\sup \Sigma = v$**,**

v **unique solution of** $LCP(\underline{A}, \underline{b})$

3. $[x^*] = [u, v]$**,**

for the fixed point $[x^*]$ **of the total step method.**

Proof of 1.:

u **is the unique solution of** $LCP(\overline{A}, \overline{b})$**; i.e.:**

$$\begin{aligned} \overline{b} + \overline{A}u &\geq o, \\ u &\geq o, \\ (\overline{b} + \overline{A}u)^T u &= 0. \end{aligned}$$

Let $A \in [A]$**,** $b \in [b]$**, let** x **be the unique solution of** $LCP(A, b)$**; i.e.:**

$$\begin{aligned} b + Ax &\geq o, \\ x &\geq o, \\ (b + Ax)^T x &= 0. \end{aligned}$$

Aim: $u \leq x$

Define

$$\Im = \{i : u_i = 0\}$$

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define $\tilde{A} = (\tilde{a}_{ij})$ **and** $\tilde{b} = (\tilde{b}_i)$ **via** $\overline{A}$, $\bar{b}$ **by**

$$\tilde{a}_{ij} := \left\{ \begin{array}{ll} \overline{a}_{ij} & \textbf{if } i \notin \Im, \\ 0 & \textbf{if } i \in \Im \textbf{ and } j \neq i, \\ \overline{a}_{ii} & \textbf{if } i \in \Im \textbf{ and } j = i, \end{array} \right\} i, j = 1, ..., n$$

$$\tilde{b}_i := \left\{ \begin{array}{ll} \bar{b}_i & \textbf{if } i \notin \Im, \\ 0 & \textbf{if } i \in \Im, \end{array} \right\} i = 1, ..., n$$

Example, let $n = 4$ **and** $\Im = \{2, 4\}$ **then**

$$\overline{A} \leq \tilde{A} = \begin{pmatrix} \overline{a}_{11} & \overline{a}_{12} & \overline{a}_{13} & \overline{a}_{14} \\ 0 & \overline{a}_{22} & 0 & 0 \\ \overline{a}_{31} & \overline{a}_{32} & \overline{a}_{33} & \overline{a}_{34} \\ 0 & 0 & 0 & \overline{a}_{44} \end{pmatrix} \in Z^{n \times n}, \quad \tilde{b} = \begin{pmatrix} \bar{b}_1 \\ 0 \\ \bar{b}_3 \\ 0 \end{pmatrix}$$

Show

$$\tilde{b} + \tilde{A}u = o \leq \tilde{b} + \tilde{A}x.$$

Then it follows $u \leq x$**.**

!! $\quad \tilde{b} + \tilde{A}u = o \leq \tilde{b} + \tilde{A}x \quad$ **!!**

$$\Im = \{2,4\},\ \tilde{A} = \begin{pmatrix} \overline{a}_{11} & \overline{a}_{12} & \overline{a}_{13} & \overline{a}_{14} \\ 0 & \overline{a}_{22} & 0 & 0 \\ \overline{a}_{31} & \overline{a}_{32} & \overline{a}_{33} & \overline{a}_{34} \\ 0 & 0 & 0 & \overline{a}_{44} \end{pmatrix}, \qquad \tilde{b} = \begin{pmatrix} \overline{b}_1 \\ 0 \\ \overline{b}_3 \\ 0 \end{pmatrix}$$

Case 1: If $i \in \Im$, then $u_i = 0$ and

$$\begin{aligned} (\tilde{b} + \tilde{A}u)_i &= 0 \\ &\leq \overline{a}_{ii} x_i \\ &= (\tilde{b} + \tilde{A}x)_i \end{aligned}$$

Case 2: If $i \notin \Im$, then $u_i > 0 \Rightarrow$

$$(\tilde{b} + \tilde{A}u)_i = (\overline{b} + \overline{A}u)_i = 0 \quad \textbf{and}$$

$$\begin{aligned} \overline{b}_i + \sum_{j=1}^{n} \overline{a}_{ij} u_j &= 0 \\ &\leq (b + Ax)_i = b_i + \sum_{j=1}^{n} a_{ij} x_j \\ &\leq \overline{b}_i + \sum_{j=1}^{n} \overline{a}_{ij} x_j \\ &= (\overline{b} + \overline{A}x)_i = (\tilde{b} + \tilde{A}x)_i \end{aligned}$$

Proof of 3.: $[u,v]=[x^*]$

$$[u,v] \stackrel{!}{=} \max\{o,[D]^{-1}([R][u,v]-[b])\} =: [c,d]$$

with $[D]=diag([a_{11}],...,[a_{nn}])$ **and**

$$[R]=-\begin{pmatrix} 0 & [a_{12}] & \cdots & [a_{1n}] \\ [a_{21}] & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & [a_{n-1n}] \\ [a_{n1}] & \cdots & [a_{nn-1}] & 0 \end{pmatrix}$$

It is $[\underline{r}_{ij},\overline{r}_{ij}]=[-\overline{a}_{ij},-\underline{a}_{ij}]$, $i\neq j$**,** $\underline{R}\geq O$

$$[c_i,d_i]=\max\{0,\frac{[(\underline{R}u-\overline{b})_i,(\overline{R}v-\underline{b})_i]}{[a_{ii}]}\} \quad i=1..n$$

1.Case: $(\overline{R}v-\underline{b})_i\leq 0$

Then $d_i=0\stackrel{!}{=}v_i$

Assume $v_i > 0$

$\Rightarrow$

$$(\underline{A}v - \underline{b})_i = 0$$

However,

$$(\overline{R}v - \underline{b})_i \leq 0 \Leftrightarrow - \sum_{j=1, j\neq i}^{n} \underline{a}_{ij} v_j - \underline{b}_i \leq 0$$

i.e.

$$\begin{aligned} 0 &\leq \sum_{j=1, j\neq i}^{n} \underline{a}_{ij} v_j + \underline{b}_i \\ &< \sum_{j=1}^{n} \underline{a}_{ij} v_j + \underline{b}_i \\ &= (\underline{A}v - \underline{b})_i \end{aligned}$$

2. Case, 3. Case similar...

Remarks: Let $[A]$ H$^+$-matrix:

v **the unique solution of** $LCP(\langle[A]\rangle, \underline{b})$**, then**

$$\Sigma([A],[b]) \subseteq [o, v].$$

If $\underline{\tilde{b}} \leq \underline{b}$

then

$$\Sigma([A],[\underline{b},\overline{b}]) \subseteq \Sigma([A],[\underline{\tilde{b}},\overline{b}]),$$

$\tilde{v}$ **the unique solution of** $LCP(\langle[A]\rangle, \underline{\tilde{b}})$**, then**

$$\Sigma([A],[b]) \subseteq [o, \tilde{v}].$$

Let $\underline{\tilde{b}} \leq o$**, then**

$$\tilde{v} = \langle[A]\rangle^{-1}(-\underline{\tilde{b}}).$$