Enclosing Solutions of Singular Systems

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Overview

1. Introduction
2. Single systems
3. Interval systems

1. Introduction

Given

$$[C] \in \mathbf{IR}^{n\times n},\ [b] \in \mathbf{IR}^n; \quad [A] := I - [C].$$

Consider

$$Cx = b, \quad b \in [b], \quad C \in [C] \text{ possibly singular.}$$

Wanted:

Enclosures for $\quad x \quad$ and

$$S := \{\, x \in \mathbf{R}^n \mid (I - A)x = b,\ A \in [A],\ b \in [b] \,\}$$

Total step method (Richardson splitting)

$$\begin{aligned} x^{k+1} &= Ax^k + b, & k = 0, 1, \ldots \quad (1)\\ [x]^{k+1} &= [A][x]^k + [b], & k = 0, 1, \ldots \quad (2) \end{aligned}$$

Remark

The iterations

$$[x]^{k+1} = (I - R[C])[x]^k + R[b]$$

and

$$[x]_\triangle^{k+1} = (I - R[C])[x]_\triangle^k + R([b] - [C]\tilde{x})$$

$$k = 0, 1, \ldots,$$

with $[x]_\triangle^k = [x]^k - \tilde{x},\ R \in \mathbf{R}^{n\times n},\ \tilde{x} \in \mathbf{R}^n$, also have the above–mentioned form.

Problem

Give criteria such that

$$\forall\, [x]^0 \in \mathbf{IR}^n \;\exists\, [x]^*([x]^0) \in \mathbf{IR}^n \;\; :$$

$$[x]^*([x]^0) = \lim_{k\to\infty} [x]^k.$$

Theorem 1 (O. Mayer, 1971)

$$\left(\exists_1\, [x]^* \in \mathbf{IR}^n \;\forall\, [x]^0 \in \mathbf{IR}^n \;:\; [x]^* = \lim_{k\to\infty} [x]^k\right)$$

$$\iff \quad \rho(|[A]|) < 1.$$

In this case we have $S \subseteq [x]^*$.

Theorem 2

If $[x]^*([x]^0)$ exists then it is a fixed point of (2) and for each $A \in [A]$, $b \in [b]$ it contains at least one solution of $(I - A)x = b$.

2. Single systems

Example 1

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$C = I - A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{is regular,}$$

$$\rho(A) = \frac{1}{\sqrt{2}} < 1, \quad \rho(|A|) = 1, \quad x^* = (2, 0)^T.$$

$$\boxed{\exists_1\, x^* \ \forall x^0 \in \mathbf{R}^n \ : \ x^* = \lim_{k\to\infty} x^k}$$

Example 2

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$C = I - A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{is singular,}$$

$$\rho(A) = \rho(|A|) = 1,$$

$Cx = b$ is solvable, i.e., it is consistent, with

$$S = \{ x \mid x = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ t \in \mathbf{R} \}.$$

$$x^0 = \alpha \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \alpha, \beta \in \mathbf{R},$$

$$\Rightarrow \quad x^k = x^0 + (1-\alpha) b = x^*(x^0), \quad k = 1, 2, \ldots .$$

$$\boxed{\forall x^0 \in \mathbf{R}^n \ \exists\, x^*(x^0) \in \mathbf{R}^n : \ x^*(x^0) = \lim_{k \to \infty} x^k}$$

Definition 1 (Drazin inverse)

Let

$$J = S^{-1}AS = \begin{pmatrix} \hat{J}_0 & O \\ O & \hat{J}_r \end{pmatrix}$$

be the Jordan normal form of $A \in \mathbf{R}^{n\times n}$, where $\hat{J}_0$ and $\hat{J}_r$, respectively, are square block diagonal matrices whose diagonal blocks are just the singular Jordan blocks of J, and the non–singular ones, respectively. Then

$$A^D := S \begin{pmatrix} O & O \\ O & (\hat{J}_r)^{-1} \end{pmatrix} S^{-1}.$$

is called Drazin inverse of A.

Definition 2 (Semi-convergence)

$A \in \mathbf{R}^{n\times n}$ is called semi-convergent if and only if $A^\infty := \lim_{k\to\infty} A^k$ exists.

Theorem 3

Let $A \in \mathbf{R}^{n\times n}$ be semi-convergent. Then

a) $A^\infty = O \Longleftrightarrow \rho(A) < 1$ (A convergent)

b) $A^\infty \neq O \Longleftrightarrow \rho(A) = 1$ and every eigenvalue λ of A with $|\lambda| = 1$ is one and has only linear elementary divisors.

Theorem 4

a) x^* exists if and only if $\rho(A) < 1$.

b) Let $(I - A)x = b$ be consistent. Then $x^*(x^0)$ exists if and only if A is semi-convergent.
In this case

$$x^*(x^0) = (I-A)^D b + \{I-(I-A)(I-A)^D\}x^0.$$

3. Interval systems

Definition 3

Let $[A] = ([a]_{ij}) \in \mathbf{IR}^{n\times n}$.

$$
\begin{array}{rcl}
J_{\text{sym}} & := & \{j \mid \exists\, i \in \{1,\ldots,n\} : [a]_{ij} = -[a]_{ij} \neq 0\} \\
e^{(j)} & := & j\text{–th column of } I. \\
[B] & := & [A],\ \text{with } j\text{–th columns replaced by} \\
 & & e^{(j)} \text{ for all } j \in J_{\text{sym}}. \\
\hat{A} & \in & [B] \text{ such that } |\hat{A}| = |[B]|. \\
 & & (\hat{A} \text{ is unique!}) \\
v & := & \text{any fixed Perron vector of } |[A]|. \\
 & & \text{(will exist whenever it occurs!)}
\end{array}
$$

Theorem 5 (Existence)

Let $|[A]|$ be irreducible with $\rho(|[A]|) = 1$.

Then (2) has a fixed point if and only if $[b] \equiv b \in \mathbf{R}^n$ and

$$x = \hat{A}x + b \tag{3}$$

is solvable.

In this case there is a solution $\breve{x}$ such that

$$\breve{x}_i = 0 \text{ for all } i \in J_{\text{sym}}. \tag{4}$$

Theorem 6 (Representation)

Let $|[A]|$ be irreducible with $\rho(|[A]|) = 1$ and let (2) have a fixed point. Let $\breve{x}$ as in Theorem 5 and define

$$t_{\text{min}} := \max \left\{ 0,\ \frac{r([a]_{ij})}{|\breve{a}_{ij}|} \cdot \frac{|\breve{x}_j|}{v_j},\ \frac{|\breve{x}_j|}{v_j} \ \middle|\ 1 \le i,j \le n,\ \breve{a}_{ij} \neq 0,\ r([a]_{ij}) \neq 0 \right\}. \quad (5)$$

If $t \geq t_{\text{min}}$ then $[x]_t^* := \breve{x} + t[-v, v]$ is a fixed point of (2).

Conversely, if $[x]^*$ is a fixed point of (2) then $[x]^* = \breve{x}^* + t[-v, v]$, $t \geq t_{\text{min}}$, where $\breve{x}^*$ solves (3), (4) and replaces $\breve{x}$ in (5).

Theorem 7 ('Uniqueness')

Let $|[A]|$ be irreducible with $\rho(|[A]|) = 1$.

a) If $J_{\mathsf{sym}} \neq \emptyset$ then (3), (4) either have no solution or a unique one.

b) If $J_{\mathsf{sym}} = \emptyset$ then $\hat{A} \in [A]$ with $|\hat{A}| = |[A]|$, (4) is trivially true and one of the following mutually excluding cases occurs:

i) $\rho(\hat{A}) < 1$, whence (3) has a unique solution.

ii) $\rho(\hat{A}) = 1$ and $\hat{A} \neq D^{-1}|[A]|D$ for every D with $|D| = I$, whence (3) has a unique solution.

iii) $\rho(\hat{A}) = 1$ and $\hat{A} = D^{-1}|[A]|D$ for D with $|D| = I$. Here, (3) has no solution if and only if b cannot be represented as linear combination of the column vectors of $I - \hat{A}$. Otherwise it has infinitely many solutions. They are given by

$$\breve{x}^* = \breve{x} + sD^{-1}v,$$

where $\breve{x}$ is any fixed particular solution of (3) and s is any real number.

Example 3

$$[A] = \begin{pmatrix} \frac{1}{2} & [0, \frac{1}{2}] \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad [b] \equiv b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$[x]^*$ is a fixed point of (2) if and only if

$$[x]^* = \breve{x} + sv + tv[-1, 1],$$

$$t \geq t_{\text{min}} = |1 - s|, \quad s \in \mathbf{R},$$

with $\breve{x} = b$, $v = (1, 1)^T$.

Remark

$$\begin{cases} \widehat{A} &= \overline{A} \\ \widehat{A}v &= v \\ \breve{x} &= \widehat{A}\breve{x} + b \\ \breve{x} + sv &= \widehat{A}(\breve{x} + sv) + b \end{cases}$$

Theorem 8

Let $|[A]|$ be irreducible and let (2) have a fixed point.

a) $[x]^*([x]^0)$ exists if and only if the following two properties hold:

(i) $|[A]|$ is semi-convergent;

(ii) if $\rho(|[A]|) = 1$ and if there is a unique matrix $\dot{A} \in [A]$ such that $|\dot{A}| = |[A]|$ then $\dot{A} \neq -D|[A]|D$, $|D| = I$.

b) If $[x]^*([x]^0)$ exists then

$$S([x]^0) := \Big\{(I-A)^D b + \{I-(I-A)(I-A)^D\}x^0 \mid A \in [A],\ x^0 \in [x]^0 \Big\}$$

satisfies $S([x]^0) \subseteq [x]^*([x]^0)$.

Example 4 $(I - [A]$ regular$)$

$$
\begin{aligned}
[A] &= \begin{pmatrix} \frac{1}{2} & [-\frac{1}{2}, 0] \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = v \\
\check{A} &= \underline{A} \\
\rho(|[A]|) &= \rho(|\check{A}|) = 1 \quad \rho(\check{A}) = 1/\sqrt{2} < 1 \\
S &= \left\{ \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;\middle|\; 2 \le \beta \le 4 \right\} \\
\square S &= \begin{pmatrix} [0,2] \\ [2,4] \end{pmatrix} \\
[x]^*([x]^0) &= (0,2)^T + t[-v, v], \quad t \ge t_{\min} = 2 \\
[x]^*([x]^0) &\supseteq [x]^*_{\min} := \begin{pmatrix} [-2,2] \\ [0,4] \end{pmatrix} \supseteq \square S
\end{aligned}
$$

Matlab/Intlab program for Example 4

```
function f = iteration(x0,n)
k = 0;
['k = ', num2str(k)]
x = x0
diam(x0)
y = x0 + 1; % avoids crash of loop for k = 0
A = infsup([1/2,-1/2;1/2,1/2],[1/2,0;1/2,1/2]);
b = intval([1;1]);
while k < n & x ~= y
   k = k+1;
   ['k = ', num2str(k)]
   y = x;
   x = A * y + b
   diam(x)/2
end
if x == y
   'fixed point'
elseif in(x,y)
   'superset of fixed points'
end
f = x;
```

Example 5 $\quad (I - [A]$ singular)

$$[A] = \begin{pmatrix} \frac{1}{2} & [0, \frac{1}{2}] \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\dot{A} = \overline{A}$$

$$\rho(|[A]|) = \rho(|\dot{A}|) = \rho(\dot{A}) = 1$$

Case 1: $\quad$ (1) with $A \in [A],\ \rho(A) < 1$

$$x^* = (2, 0)^T$$

Case 2: $\quad$ (1) with $A = \overline{A}$, i.e., $\rho(A) = 1$

$$x^*(x^0) = b + \overline{A}x^0 \subseteq [y]^* := b + \overline{A}[x]^0$$

for all $x^0 \in [x]^0$.

$$\square([y]^*, x^*) \subseteq [x]^*([x]^0)$$

with equality for all starting vectors of the table.

$$x^{k+1} = Ax^k + b, \quad x^* = Ax^* + b$$

$$\Rightarrow x^k - x^* = A^k(x^0 - x^*), \quad k = 0, 1, \ldots$$

$$x^*(x^0) = \lim_{k\to\infty} x^k \text{ always exists } \Leftrightarrow A^\infty \text{ exists}$$

Definition 4

$$[A]^0 := I, \;\; [A]^{k+1} := [A]^k \cdot [A]$$
$${}^0[A] := I, \;\; {}^{k+1}[A] := [A] \cdot {}^k[A]$$

Theorem 9 (M. 1985)

Let $|[A]|$ be irreducible with $r([A]) \neq O$.

a) $[A]^\infty := \lim_{k\to\infty} [A]^k$ exists if and only if ${}^\infty[A] := \lim_{k\to\infty} {}^k[A]$ exists.

b) $[A]^\infty := \lim_{k\to\infty} [A]^k$ exists if and only if the following two properties hold:

(i) $|[A]|$ is semi-convergent;
(ii) if $\rho(|[A]|) = 1$ and if there is a unique matrix $\dot{A} \in [A]$ such that $|\dot{A}| = |[A]|$ then $\dot{A} \neq -D|[A]|D$, $|D| = I$.