Examples of Application of Interval Analysis to Process Simulation and Control

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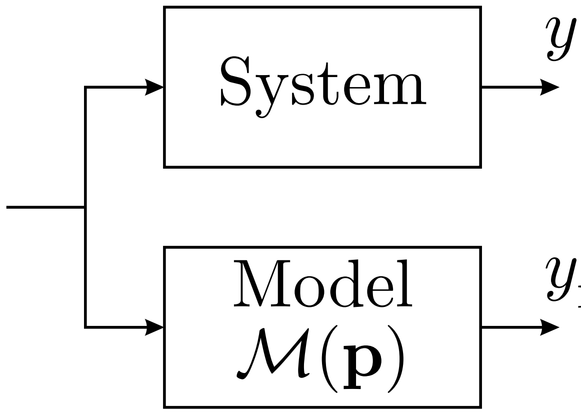
joint work with I. Braems, L. Jaulin and E. Walter

Dagstuhl – January 19–24, 2003

- Parameter estimation
- State estimation
- Robust stability analysis
- Robust controller design
- Path planning
- ...

1 Parameter estimation



$\mathbf{y}$: vector of experimental data

$\mathbf{p}$: vector of unknown, constant parameters

$\mathbf{y}_m(\mathbf{p})$: vector of model output

Parameter estimation :

Determination of $\widehat{\mathbf{p}}$ from $\mathbf{y}$.

1.1 Problem formulation

1. Minimisation of a cost function, *e.g.*,

$$\widehat{\mathbf{p}} = \arg\min_{\mathbf{p}} j\left(\mathbf{p}\right) = \left(\mathbf{y} - \mathbf{y}_{\mathrm{m}}\left(\mathbf{p}\right)\right)^{\mathrm{T}}\left(\mathbf{y} - \mathbf{y}_{\mathrm{m}}\left(\mathbf{p}\right)\right)$$

   - Local search : Gauss-Newton, Levenberg-Marquardt…
   - Random search : simulated annealing, genetic algorithms…
   - Global guaranteed search : Hansen's algorithm…

2. Bounding techniques

1.2 Parameter bounding

Experimental data : $y(t_i)$,

t_i, $i = 1 \ldots, N$, known measurement times

$[\varepsilon_i] = [\underline{\varepsilon}_i, \overline{\varepsilon}_i]$, $i = 1, \ldots, N$, known acceptable errors

$\mathbf{p} \in \mathcal{P}_0$ deemed acceptable if for all $i = 1, \ldots, N$,

$$\underline{\varepsilon}_i \leqslant y(t_i) - y_{\mathrm{m}}(\mathbf{p}, t_i) \leqslant \overline{\varepsilon}_i.$$

Bounded-error parameter estimation

Characterize $\mathcal{S} = \{\mathbf{p} \in \mathcal{P}_0 \mid y(t_i) - y_{\mathrm{m}}(\mathbf{p}, t_i) \in [\underline{\varepsilon}_i, \overline{\varepsilon}_i],\ i = 1, \ldots, N\}$

- When $y_{\mathrm{m}}(\mathbf{p}, t_i)$ is linear in $\mathbf{p}$
  - exact description by polytopes
    (Walter and Piet-Lahanier, 1989...)
  - outer approximation by ellipsoids, polytopes...
    (Schweppe, 1973 ; Fogel ang Huang, 1982...)
- When $y_{\mathrm{m}}(\mathbf{p}, t_i)$ is non-linear in $\mathbf{p}$
  - outer approximation by polytopes, ellipsoids...
    (Norton, 1987 ; Clément and Gentil, 1988 ; Cerone, 1991...)
  - approximate but guaranteed enclosure of $\mathcal{S}$ by SIVIA
    (Moore, 1992 ; Jaulin and Walter 1993)

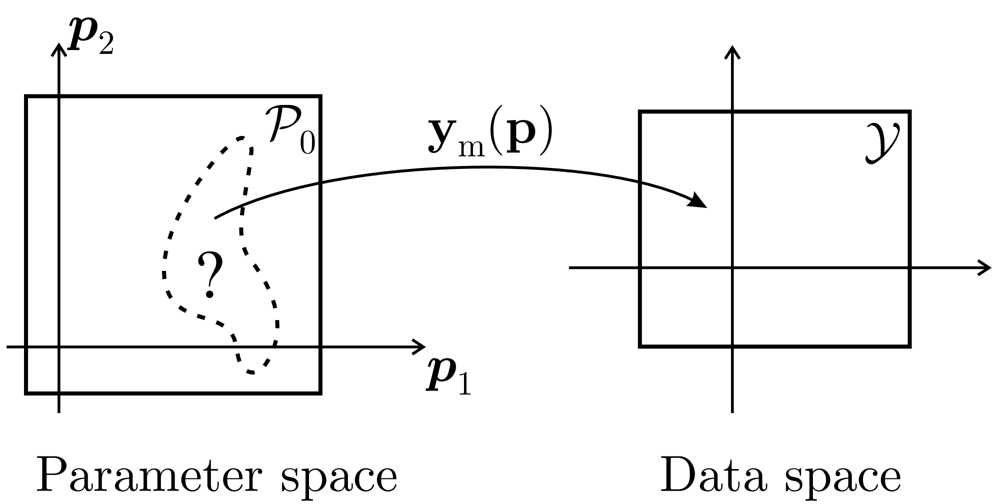
1.3 Sivia

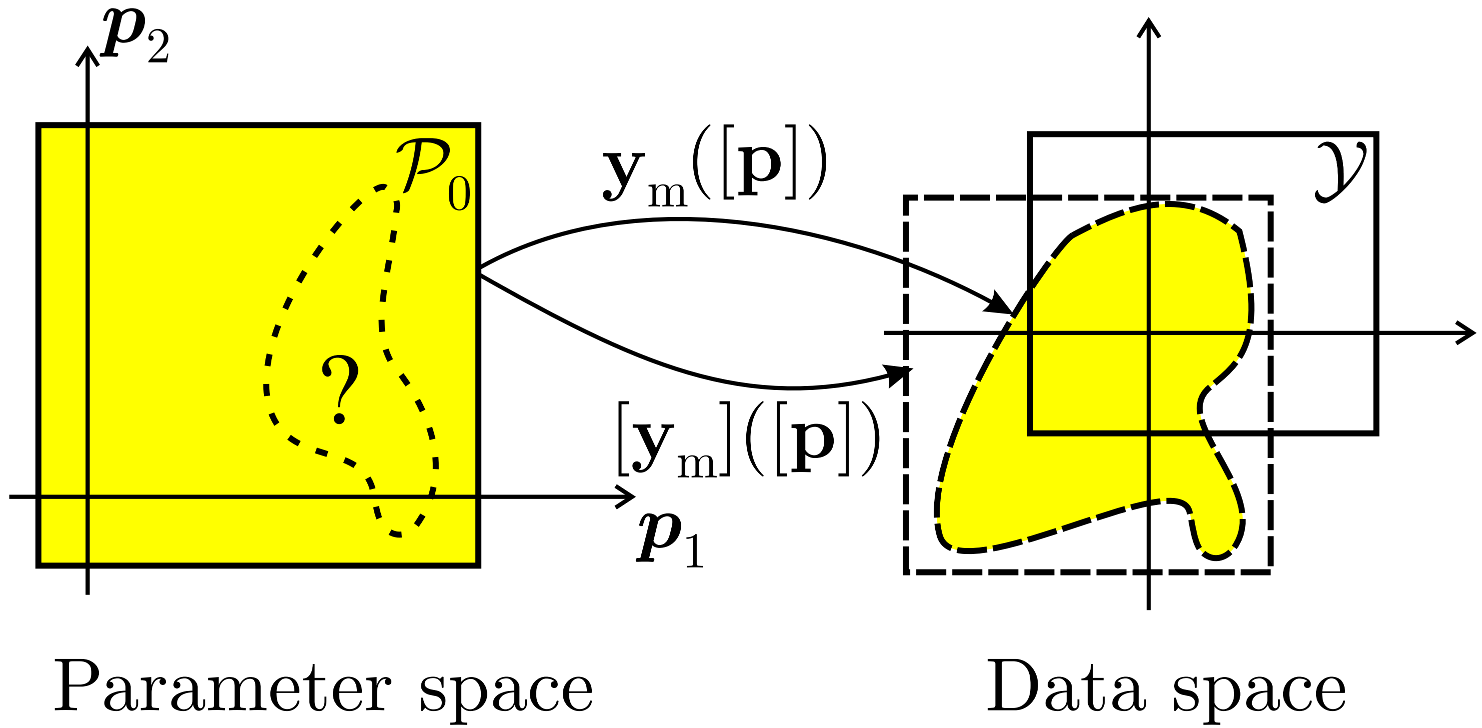
Set to be characterized

$$\begin{aligned}\mathcal{S} &= \{\mathbf{p} \in \mathcal{P}_0 \mid y(t_i) - y_{\mathrm{m}}(\mathbf{p}, t_i) \in [\underline{\varepsilon}_i, \overline{\varepsilon}_i], i = 1, \dots, N\} \\ &= \{\mathbf{p} \in \mathcal{P}_0 \mid \mathbf{y}_{\mathrm{m}}(\mathbf{p}) \subset \mathcal{Y}\},\end{aligned}$$

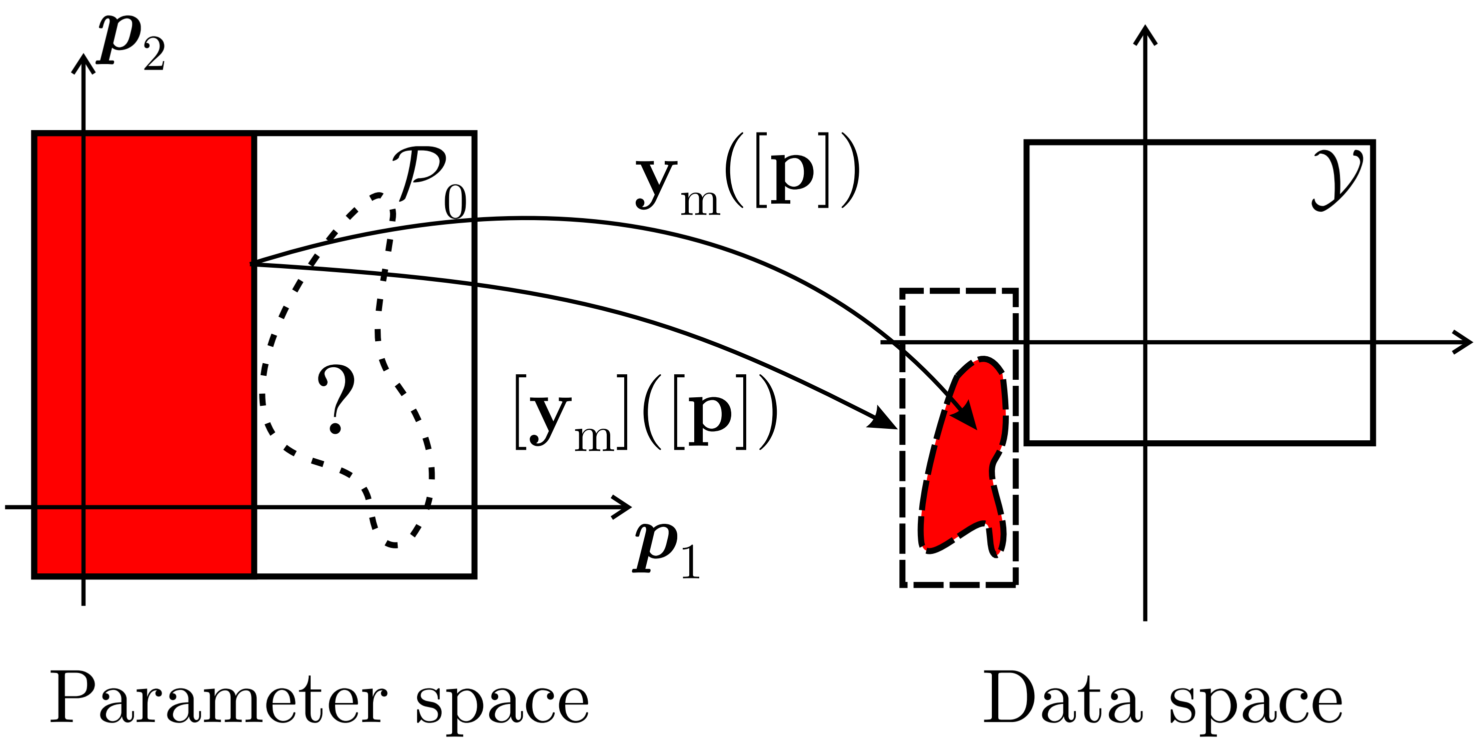
with

$$\mathcal{Y} = [y(t_1) - \overline{\varepsilon}_1, y(t_1) - \underline{\varepsilon}_1] \times \cdots \times [y(t_N) - \overline{\varepsilon}_N, y(t_N) - \underline{\varepsilon}_N]$$

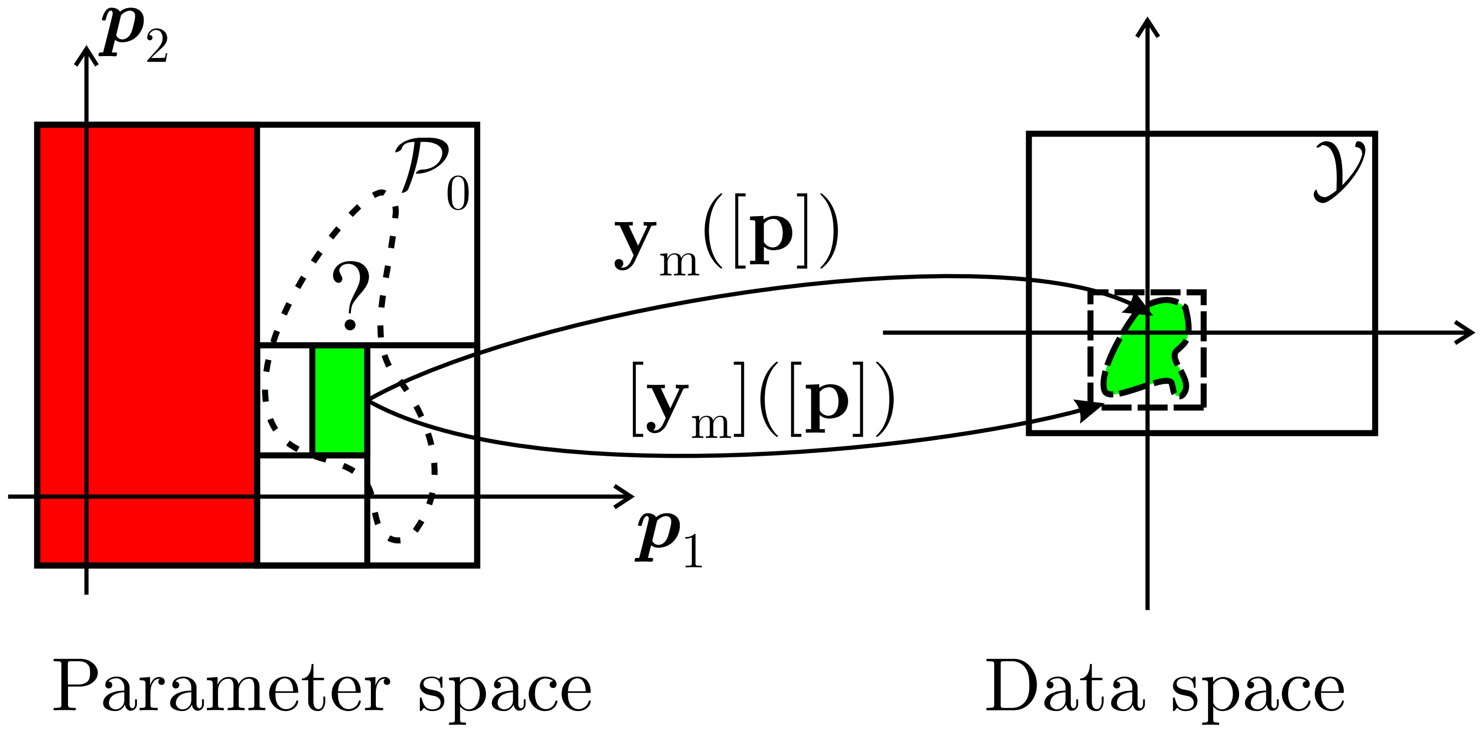




Yellow box is undetermined



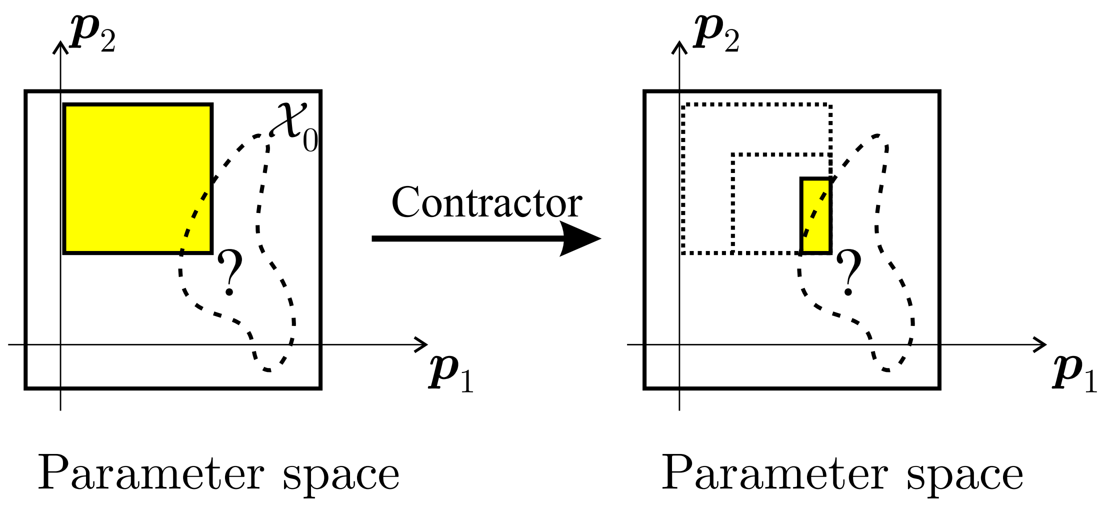
Red box proven to be outside $\mathcal{S}$



Green box proven to be included in $\mathcal{S}$

1.4 Sivia with contractors

Reduce the size of undetermined boxes without any bisection

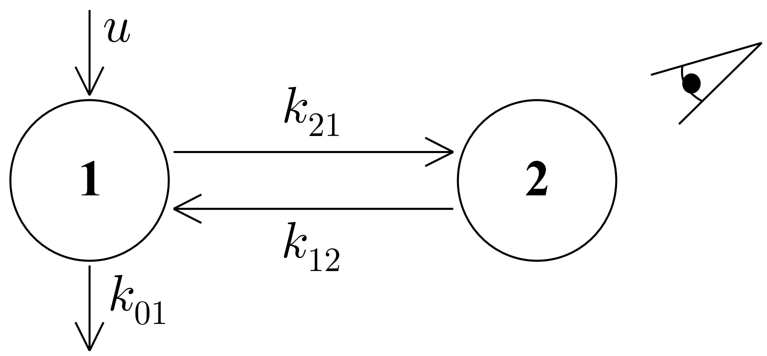


Contractors based on

- interval constraint propagation
- linear programming
- parallel linearization
- ...

1.5 Example

Estimation of the parameters of a compartmental model



State equation

$$\begin{cases} x_1' = -\left(k_{01} + k_{21}\right) x_1 + k_{12} x_2 \\ x_2' = k_{21} x_1 - k_{12} x_2 \end{cases} \quad \text{with} \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 0 \end{cases}$$

Observation equation

$$y(t_i) = x_2(t_i) + b(t_i)\,,\ i = 1, \ldots, 16$$

Model

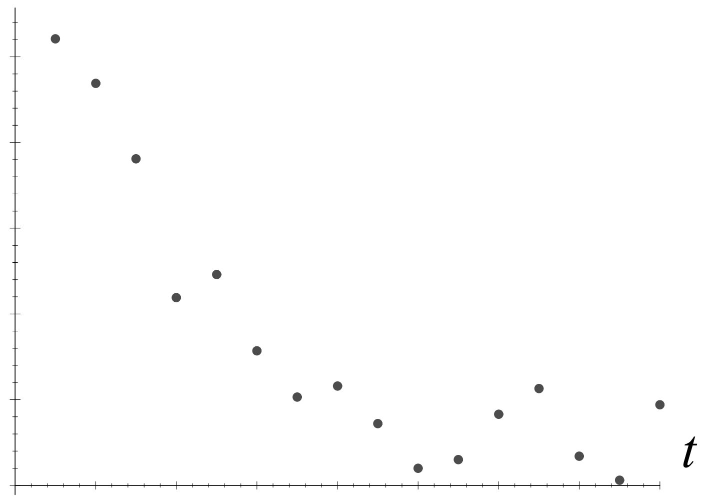
$$y_{\mathrm{m}}(\mathbf{p}, t_i) = p_1 \left(\exp(p_2 t_i) - \exp(p_3 t_i)\right), \quad i = 1, \ldots, 16,$$

where the macroparameters

$$\mathbf{p} = (p_1, p_2, p_3)^{\mathrm{T}}$$

depends on the microparameters

$$(k_{01}, k_{12}, k_{21}).$$



Simulated
noisy experimental data

Macroparameter estimation with

$$\underline{\varepsilon}_i = -0.09,\ \overline{\varepsilon}_i = 0.09,\ i = 1, \ldots, 16$$

Results

	SIVIA	SIVIA + ICP	ICP only
Comp. time (s)	8	6.2	0.63
Bounding box	$[0.49, 1.06]$ $[-0.293, -0.141]$ $[-5, -1.054]$	$[0.49, 1.06]$ $[-0.293, -0.141]$ $[-5, -1.054]$	$[0.52, 0.98]$ $[-0.282, -0.156]$ $[-5, -1.167]$

1.6 Limitations

To test $[\mathbf{p}]$, SIVIA evaluates $[y_{\mathrm{m}}]([\mathbf{p}], t_i)$, $i = 1, \ldots, 16$:

– explicit expression of $y_{\mathrm{m}}(\mathbf{p}, t_i)$ required

– if available, can be complicated, *e.g.*, here

$$y_{\mathrm{m}}(\mathbf{p}, t_i) = \frac{k_{21}}{\sqrt{(k_{01} - k_{12} + k_{21}) + 4k_{12}k_{21}}} \times \left(\exp\left(- \left((k_{01} + k_{12} + k_{21}) + \sqrt{(k_{01} - k_{12} + k_{21}) + 4k_{12}k_{21}} \right) \frac{t_i}{2} \right) - \exp\left(- \left((k_{01} + k_{12} + k_{21}) - \sqrt{(k_{01} - k_{12} + k_{21}) + 4k_{12}k_{21}} \right) \frac{t_i}{2} \right) \right)$$

$\hookrightarrow$ multiple occurences

$\hookrightarrow$ non-minimal inclusion functions

1.7 Alternative approach

Guaranteed numerical integration of state equation

State equation

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \mathbf{f}\left(\mathbf{x}, \mathbf{p}^*, \mathbf{w}, \mathbf{u}\right),$$

where

$\mathbf{w}$ state perturbation assumed bounded,

$\mathbf{u}$: known input.

Observation equation

$$\mathbf{y}\left(t_i\right) = \mathbf{h}\left(\mathbf{x}\left(\mathbf{p}^*, t_i\right)\right) + \mathbf{v}\left(t_i\right), \; i = 1, \ldots, N,$$

where

$\mathbf{v}$ measurement noise assumed bounded.

Example of model output

$$\mathbf{y}_{\mathrm{m}}\left(\mathbf{p}, t_i\right) = \mathbf{h}\left(\mathbf{x}\left(\mathbf{p}, t_i\right)\right), \; i = 1, \ldots, N.$$

Efficiency of Sivia depends on quality of enclosure of $\mathbf{y}_{\mathrm{m}}\left([\mathbf{p}], t_i\right)$

Particularly difficult when integrating with large $[\mathbf{p}]$

- important wrapping effect
- pessimism introduced

For general models, guaranteed numerical integration not adapted.

But can be used for cooperative systems.

1.8 Parameter estimation for cooperative systems

Definition 1 *(Smith, 94) The dynamical system*

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, t),$$

where $\mathbf{f}(\mathbf{x}, t)$ *is continuous and differentiable is cooperative on a domain* $\mathcal{D}$ *if*

$$\frac{\partial f_i}{\partial x_j} \geqslant 0, \ \textit{for any } i \neq j, \ t \geqslant 0 \ \textit{and } \mathbf{x} \in \mathcal{D}.$$

Theorem 1 *(Smith, 94) Consider the system*

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \mathbf{f}\left(\mathbf{x}, \mathbf{p}, \mathbf{w}, \mathbf{u}\right).$$

If there exists a pair of cooperative systems

$$\left\{ \begin{array}{l} \mathbf{x}' = \underline{\mathbf{f}}\left(\mathbf{x},t\right) \\ \mathbf{x}' = \overline{\mathbf{f}}\left(\mathbf{x},t\right) \end{array} \right.$$

satisfying

– $\underline{\mathbf{f}}\left(\mathbf{x},t\right) \leqslant \mathbf{f}\left(\mathbf{x}, \mathbf{p}, \mathbf{w}, \mathbf{u}\right) \leqslant \overline{\mathbf{f}}\left(\mathbf{x},t\right)$,
for any $\mathbf{p} \in \left[\underline{\mathbf{p}}, \overline{\mathbf{p}}\right]$, $\mathbf{w} \in \left[\underline{\mathbf{w}}\left(t\right), \overline{\mathbf{w}}\left(t\right)\right]$, $t \geqslant 0$ *and* $\mathbf{x} \in \mathcal{D}$,

– $\underline{\mathbf{x}}_0 \leqslant \mathbf{x}\left(0\right) \leqslant \overline{\mathbf{x}}_0$,

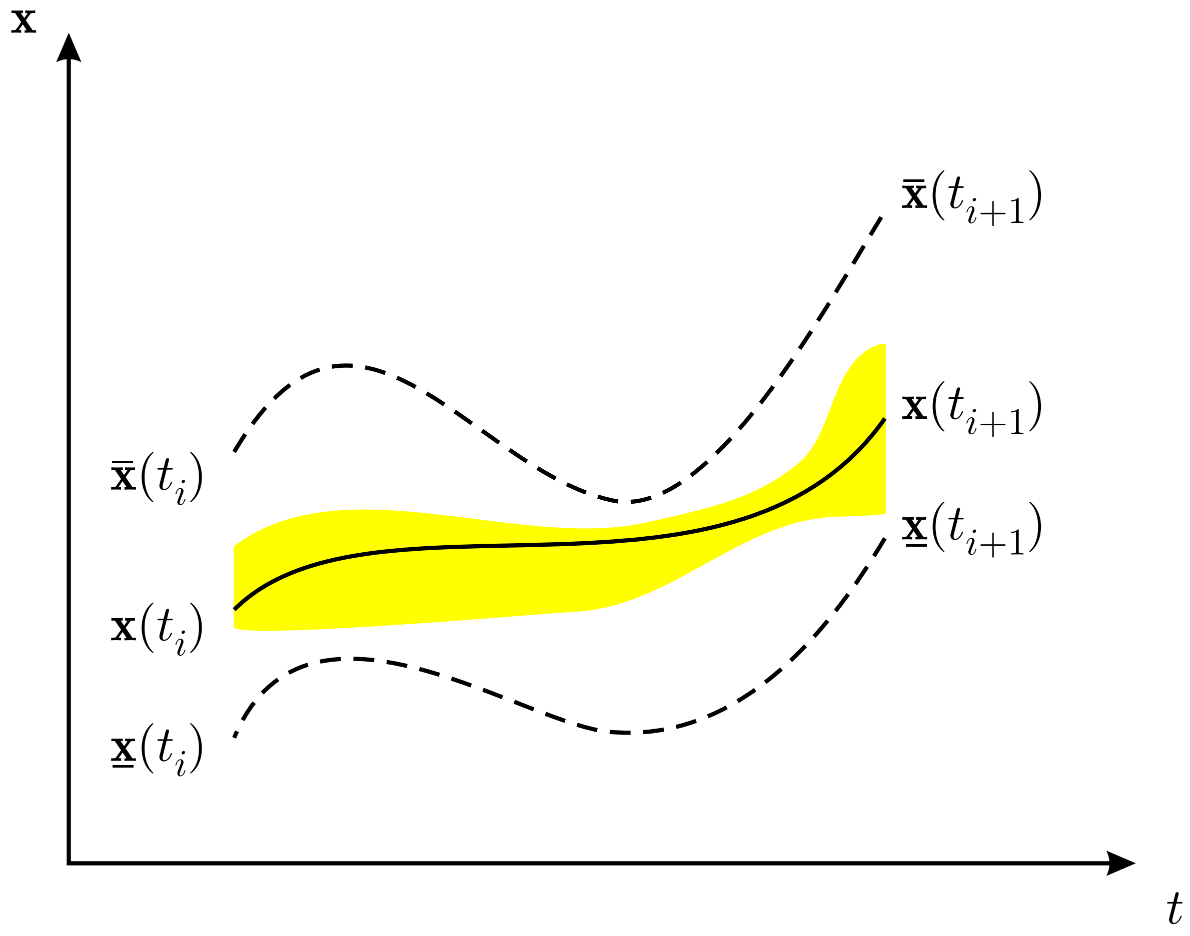
then

$$\underline{\mathbf{x}}\left(t\right) \leqslant \mathbf{x}\left(t\right) \leqslant \overline{\mathbf{x}}\left(t\right), \text{ for any } t \geqslant 0,$$

with

– $\underline{\mathbf{x}}\left(t\right) = \underline{\phi}\left(\underline{\mathbf{x}}_0, t\right)$ *the flow corresponding to* $\left\{\underline{\mathbf{x}}' = \underline{\mathbf{f}}\left(\underline{\mathbf{x}},t\right), \underline{\mathbf{x}}\left(0\right) = \underline{\mathbf{x}}_0\right\}$

– $\overline{\mathbf{x}}\left(t\right) = \overline{\phi}\left(\overline{\mathbf{x}}_0, t\right)$ *the flow corresponding to* $\left\{\overline{\mathbf{x}}' = \overline{\mathbf{f}}\left(\overline{\mathbf{x}},t\right), \overline{\mathbf{x}}\left(0\right) = \overline{\mathbf{x}}_0\right\}$.


x
$\bar{\mathbf{x}}(t_i)$
$\mathbf{x}(t_i)$
$\underline{\mathbf{x}}(t_i)$
$\bar{\mathbf{x}}(t_{i+1})$
$\mathbf{x}(t_{i+1})$
$\underline{\mathbf{x}}(t_{i+1})$
t

Steps to build an inclusion function for $\mathbf{y}_{\mathrm{m}}([\mathbf{p}], t_i)$

1. Find a pair of cooperative systems satisfying

$$\underline{\mathbf{f}}(\mathbf{x},t) \leqslant \mathbf{f}(\mathbf{x},\mathbf{p},\mathbf{w},\mathbf{u}) \leqslant \overline{\mathbf{f}}(\mathbf{x},t),$$

for all $\mathbf{p} \in [\underline{\mathbf{p}}, \overline{\mathbf{p}}]$, $\mathbf{w} \in [\underline{\mathbf{w}}(t), \overline{\mathbf{w}}(t)]$, $t \geqslant 0$ and $\mathbf{x} \in \mathcal{D}$.

2. Integrate

$$\begin{cases} \underline{\mathbf{x}}' = \underline{\mathbf{f}}(\underline{\mathbf{x}},t) \\ \overline{\mathbf{x}}' = \overline{\mathbf{f}}(\overline{\mathbf{x}},t) \end{cases} \text{with} \begin{cases} \underline{\mathbf{x}}(0) = \underline{\mathbf{x}}_0 \\ \overline{\mathbf{x}}(0) = \overline{\mathbf{x}}_0 \end{cases}$$

with a guaranteed ODE solver to get

$$\begin{cases} [\underline{\phi}(\underline{\mathbf{x}}_0, t_i)] = \left[\underline{\underline{\phi}(\underline{\mathbf{x}}_0, t_i)}, \overline{\underline{\phi}(\underline{\mathbf{x}}_0, t_i)}\right] \\ [\overline{\phi}(\overline{\mathbf{x}}_0, t_i)] = \left[\underline{\overline{\phi}(\overline{\mathbf{x}}_0, t_i)}, \overline{\overline{\phi}(\overline{\mathbf{x}}_0, t_i)}\right] \end{cases}, \quad i = 1, \ldots, N$$

3. The box-valued function

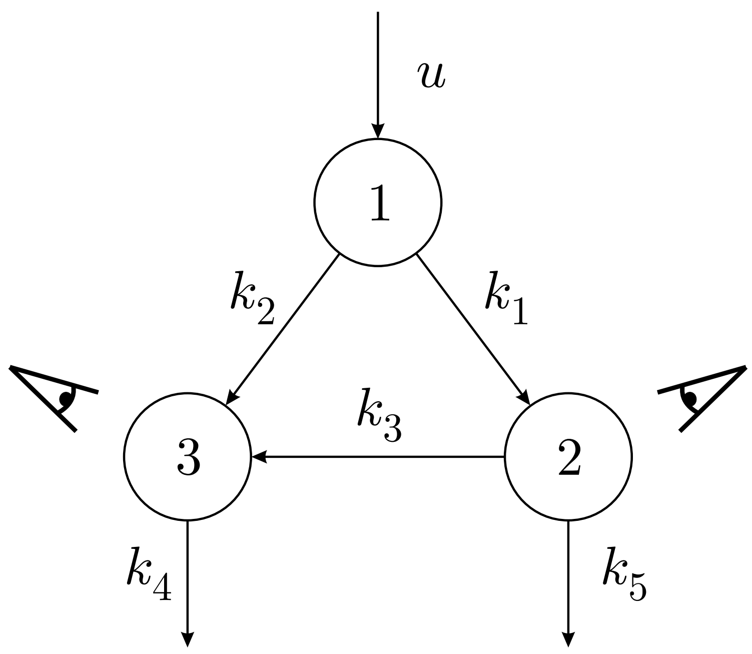
$$[[\phi]]\left([\mathbf{x}], t_i\right) = \left[\underline{\underline{\phi}\left(\underline{\mathbf{x}}, t\right)}, \overline{\overline{\phi}\left(\overline{\mathbf{x}}, t_i\right)}\right]$$

is an inclusion function for $\mathbf{x}(t_i)$ and the box-valued function

$$[\mathbf{h}]\left([[\phi]]\left([\mathbf{x}], t_i\right)\right)$$

is thus an inclusion function for $\mathbf{y}_{\mathrm{m}}\left([\mathbf{p}], t_i\right)$.

1.9 Example



Compartmental model of the behaviour of a drug (Glafenine) administered orally.

$$\begin{cases} x_1' = -(k_1 + k_2)x_1 + u \\ x_2' = k_1 x_1 - (k_3 + k_5)\, x_2 \\ x_3' = k_2 x_1 + k_3 x_2 - k_4 x_3 \end{cases}$$

$$\mathbf{y}_{\mathrm{m}}(\mathbf{p},t) = \left(x_2(\mathbf{p},t),\; x_3(\mathbf{p},t)\right)^{\mathrm{T}}.$$

Unknown parameter vector $\mathbf{p}^* = (k_1, k_2, k_3, k_4, k_5)^{\mathrm{T}}$, with $\mathbf{p}^* > \mathbf{0}$.

Can be bounded between

$$\left\{\begin{aligned} \underline{x}_1' &= -(\overline{k}_1 + \overline{k}_2)\underline{x}_1 + u \\ \underline{x}_2' &= \underline{k}_1\underline{x}_1 - \left(\overline{k}_3 + \overline{k}_5\right)\underline{x}_2 \\ \underline{x}_3' &= \underline{k}_2\underline{x}_1 + \underline{k}_3\underline{x}_2 - \overline{k}_4\underline{x}_3 \end{aligned}\right. \quad \text{and} \quad \left\{\begin{aligned} \overline{x}_1' &= -(\underline{k}_1 + \underline{k}_2)\overline{x}_1 + u \\ \overline{x}_2' &= \overline{k}_1\overline{x}_1 - \left(\underline{k}_3 + \underline{k}_5\right)\overline{x}_2 \\ \overline{x}_3' &= \overline{k}_2\overline{x}_1 + \overline{k}_3\overline{x}_2 - \underline{k}_4\overline{x}_3 \end{aligned}\right.$$

$\Longrightarrow$ two cooperative systems

Guaranteed numerical integration provides
inclusion function for $\mathbf{y}_{\mathrm{m}}(\mathbf{p},t)$, here minimal.

Simulation conditions

$\mathbf{p}^* = (0.6,\ 0.8,\ 1,\ 0.2,\ 0.4)^{\mathrm{T}}$

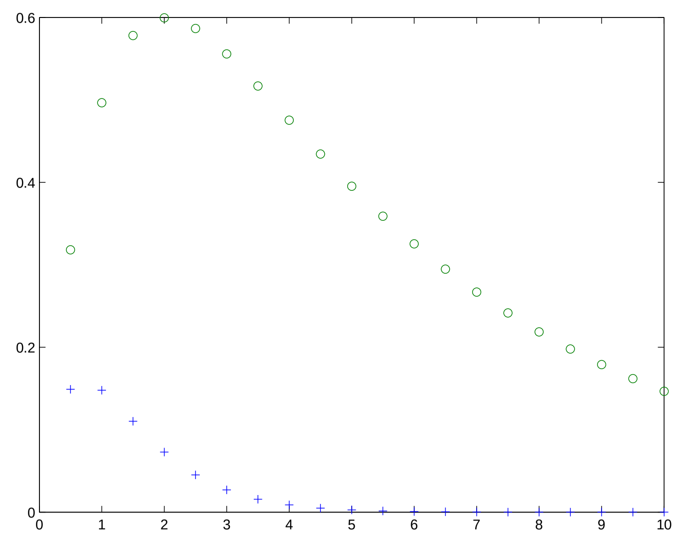
Imput $u(t) = \delta(t)$

Outputs of Compartments 2 and 3 have been measured at $t_i = 0.5i,\ i = 1, \ldots, 20.$

Introduction of bounded relative random noise

$$\widetilde{y}_i \rightarrow y_i (1 + \epsilon_i)$$

with ϵ_i random in $[-0.01,\ 0.01]$.



Compartment 2 (+)
Compartment 3 (o)

Solution

Precision parameter : $\epsilon = 0.01$

Computing time : 15 mn on an Athlon 1800

Bounding box :

$$\mathcal{S} \subset [0.586, 0.625] \times [0.74, 0.85] \\ \times [0.81, 1.25] \times [0.185, 0.215] \times [0.235, 0.56]$$

contains $\mathbf{p}^*$.

Conclusions

– Interval techniques provide guaranteed enclosure of the solution

– ICP or SIVIA + ICP allows more unknown parameters than SIVIA
but require an explicit solution for the model

– Alternative approach needs only state equation
but still time-consuming
← Guaranteed integration of ODE
← Contractors not usable

Further work

Each cooperative system is

- non-linear in parameters
- linear in $\mathbf{x}$

$\Longrightarrow$ Use explicit solution of the form

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau.$$

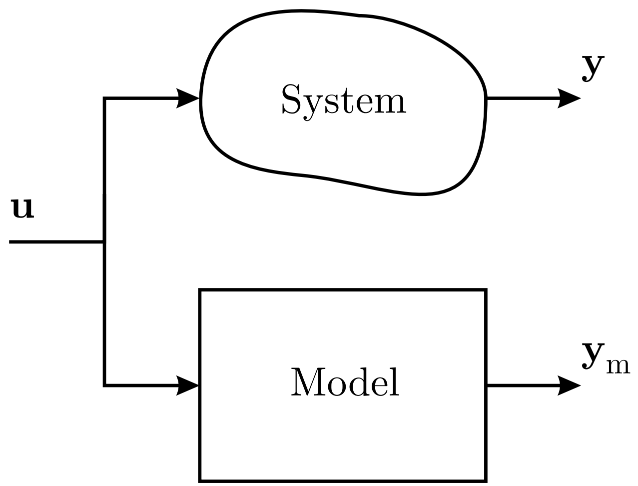
$\Longrightarrow$ Requires guaranteed computation of solution with A and B point matrices.

Questions

- Efficient algorithms for such problem ?
- Performances compared to guaranteed numerical integration ?
- May contractors be used again ?

2 State estimation

2.1 Introduction



Continuous-time state equation :

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \mathbf{f}\left(\mathbf{x}, \mathbf{w}, \mathbf{u}\right).$$

Observation equation :

$$\mathbf{y}\left(t_i\right) = \mathbf{h}\left(\mathbf{x}\left(t_i\right)\right) + \mathbf{v}\left(t_i\right),$$
$$i = 1, \ldots, N.$$

Problem :

Evaluate state $\mathbf{x}$ using all available information.

2.2 Recursive state estimation

Technique depends on

- whether **f** and **h** linear
- type of description of uncertainty.

	Probabilistic	**Bounding**
Linear	Kalman filter (Kalman, 60. . .)	Ellipsoidal, polytopic (Chernousko, 94 ; Kurzhanski, 97. . .)
Nonlinear	EKF (Gelb, 74. . .) No guarantee	Here Guaranteed

2.3 Recursive nonlinear state estimation in a bounded-error context

Hypotheses :

- at t_0, $\mathbf{x}(t_0) \in [\mathbf{x}_0]$,
- $\mathbf{w}(t) \in [\underline{\mathbf{w}}(t), \overline{\mathbf{w}}(t)]$, known for all past t,
- $\mathbf{v}(t_i) \in [\underline{\mathbf{v}}(t_i), \overline{\mathbf{v}}(t_i)]$, known for all past t_i,
- $\mathbf{u}(t)$, known for all past t.

Problem :

Characterize set $\mathcal{X}(t)$ of all $\mathbf{x}(t)$ compatible with hypotheses.

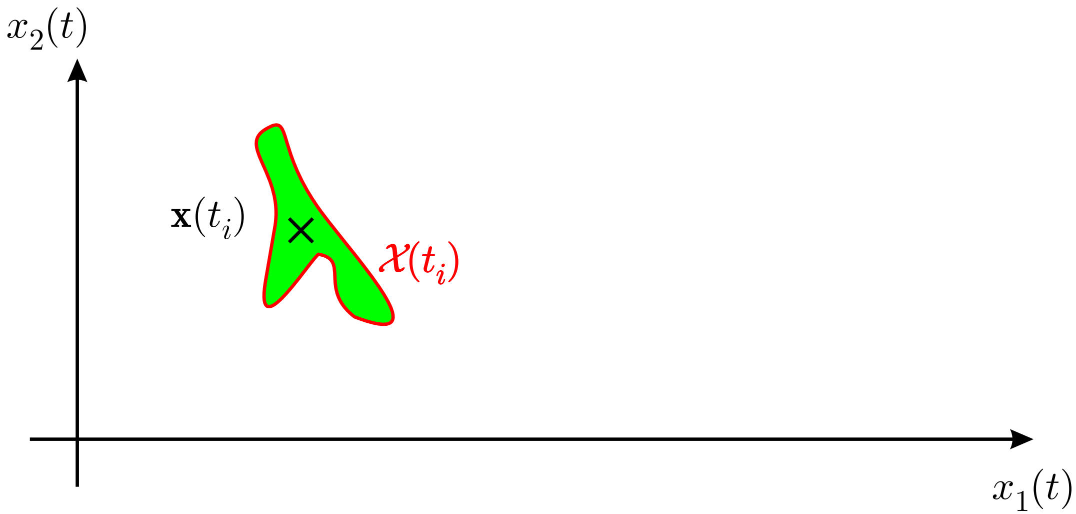
Previous results

- (Kieffer *et al*, CDC, 98) :
  - discrete-time,
  - set description with subpavings.
- (Gouzé *et al*, J. Ecol. Mod., 00) :
  - continuous-time,
  - uncertain state equation,
  - state equation bounded between cooperative systems,
  - set description with boxes.
- (Jaulin, Automatica, 02) :
  - continuous-time,
  - no state perturbations,
  - guaranteed numerical integration of non-punctual boxes,
  - set description with subpavings.

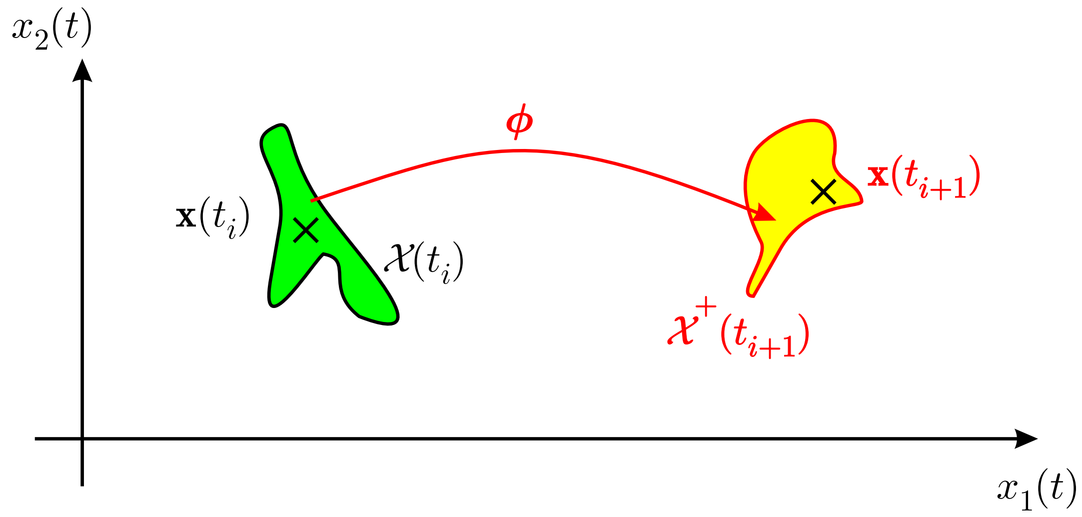
Present context

- continuous-time,
- state perturbations,
- set description with subpavings,
- cooperative systems.

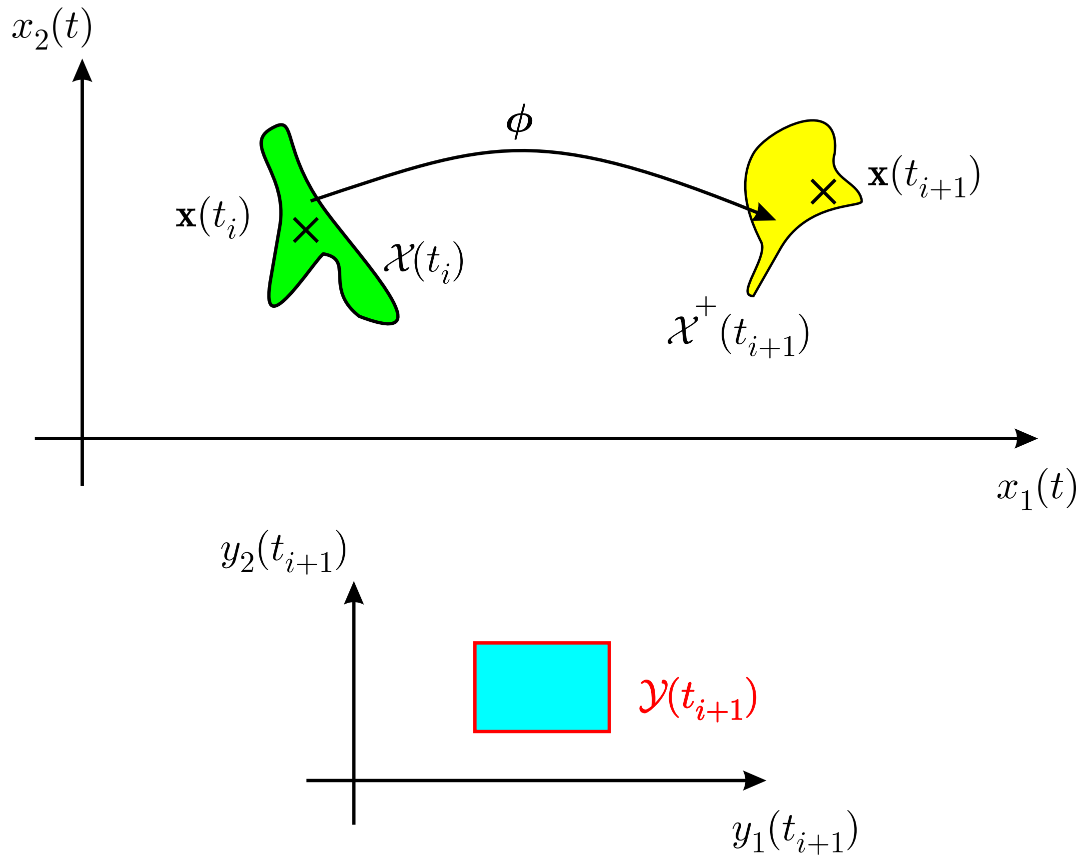
2.4 Idealized recursive state estimation



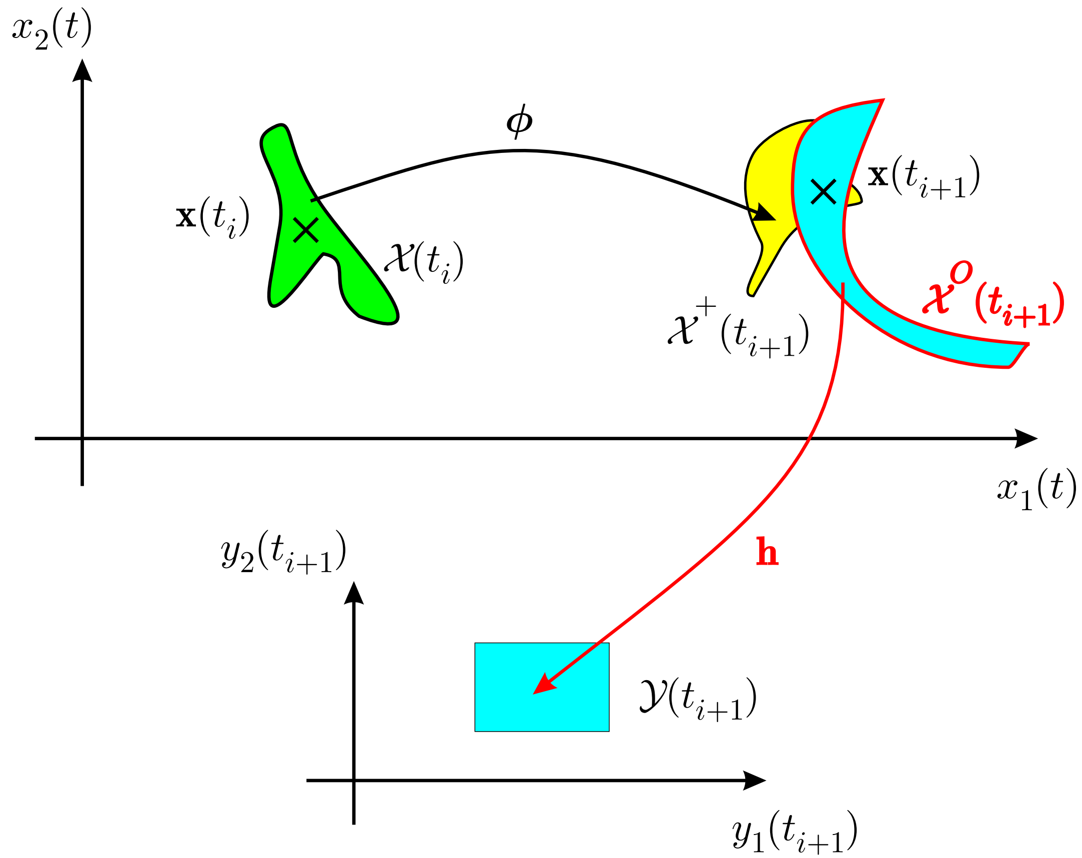
Idealized recursive state estimation



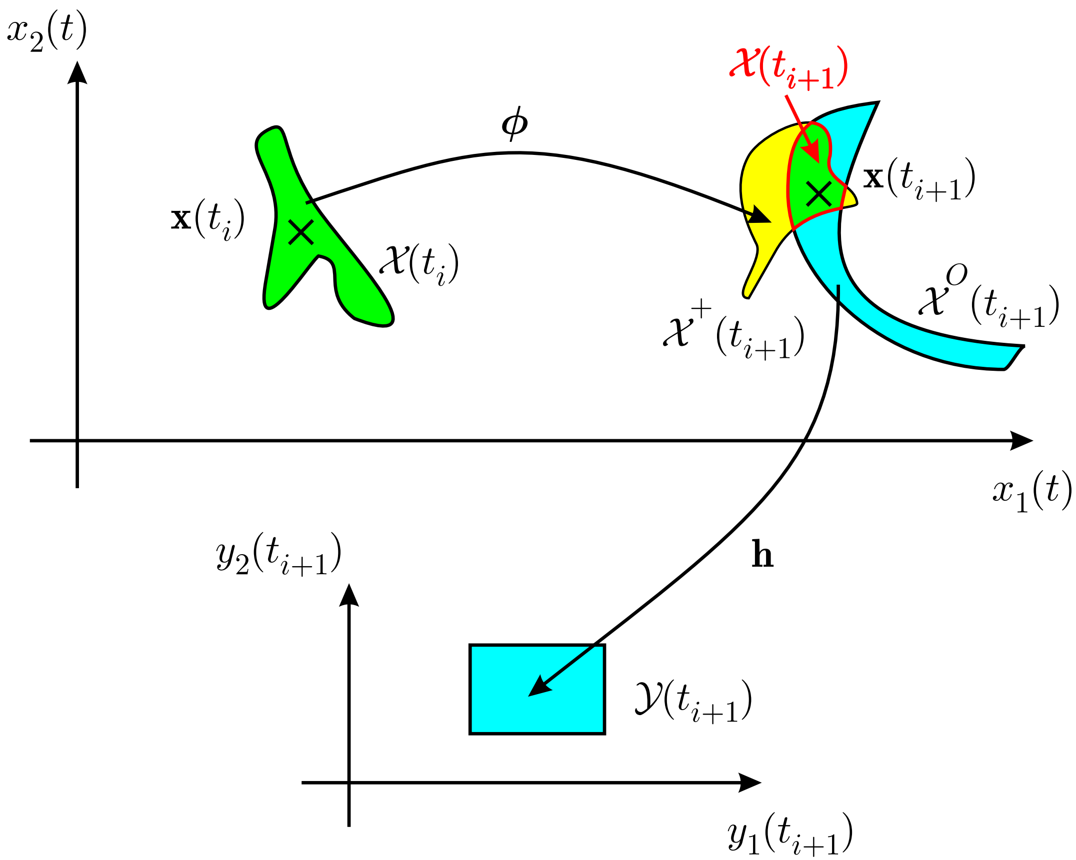
Idealized recursive state estimation



Idealized recursive state estimation

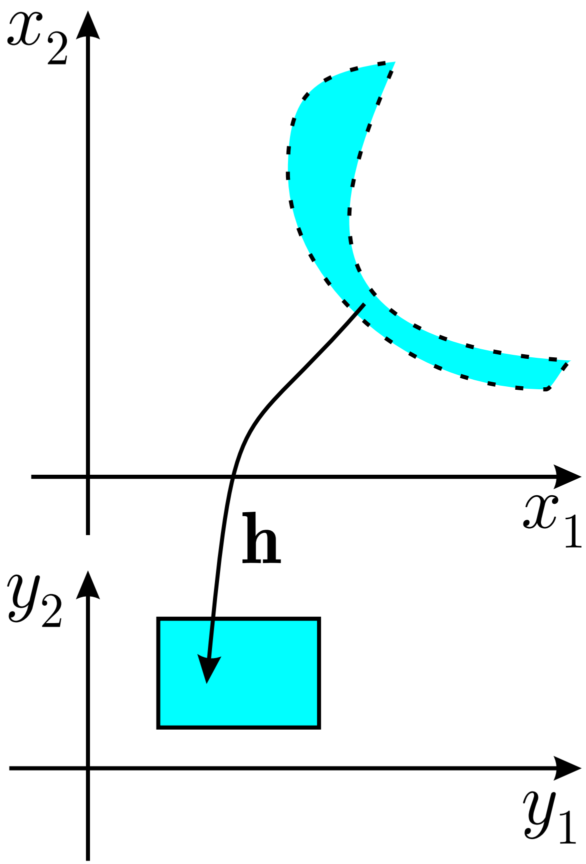


Idealized recursive state estimation



Prediction and correction steps alternate

2.5 Correction step



Set-inversion problem :

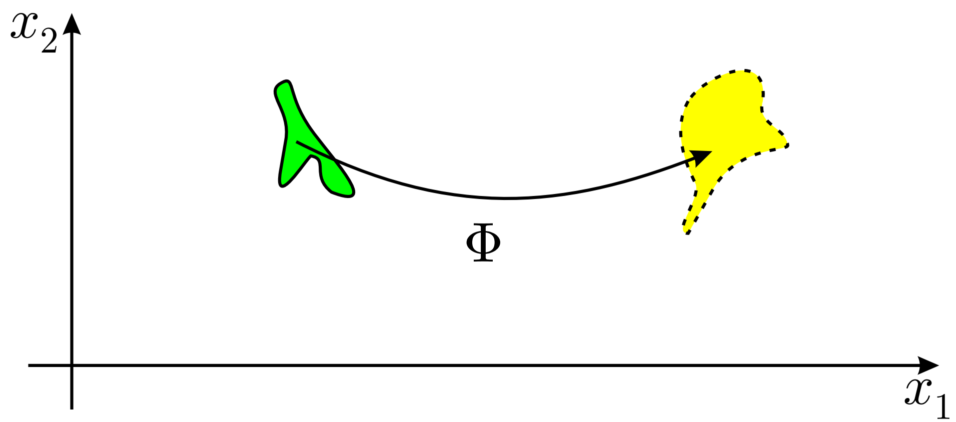
Find

$$\mathcal{X}^O(t_i) = \mathbf{h}^{-1}(\mathbf{y}(t_i) - [\mathbf{v}(t_i)]).$$

Solution provided by Sivia.

(similar to parameter estimation)

2.6 Prediction step

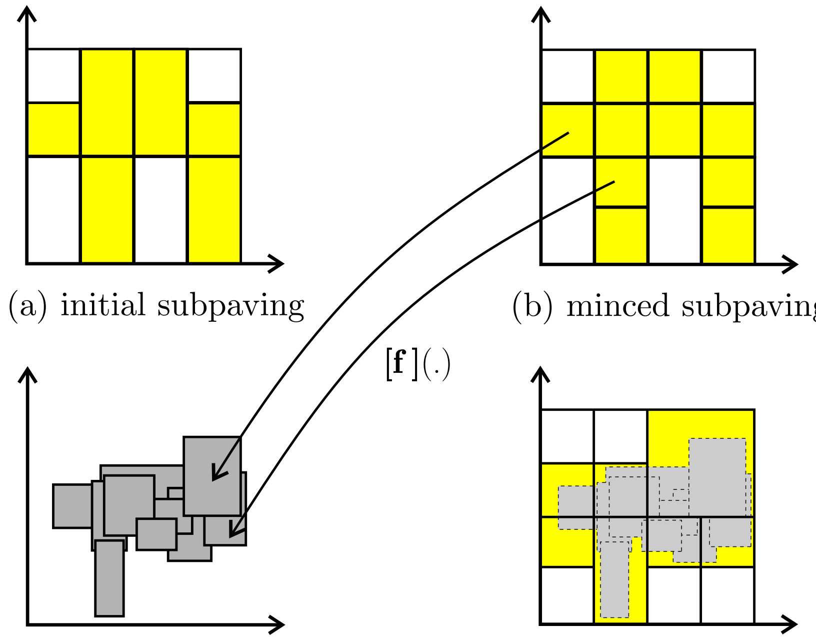


Flow ϕ difficult to obtain in general.

Situation much simpler with discrete-time state equation

$$\mathbf{x}_{k+1} = \mathbf{f}_k\left(\mathbf{x}_k, \mathbf{w}, \mathbf{u}_k\right).$$

$\Longrightarrow$ IMAGESP



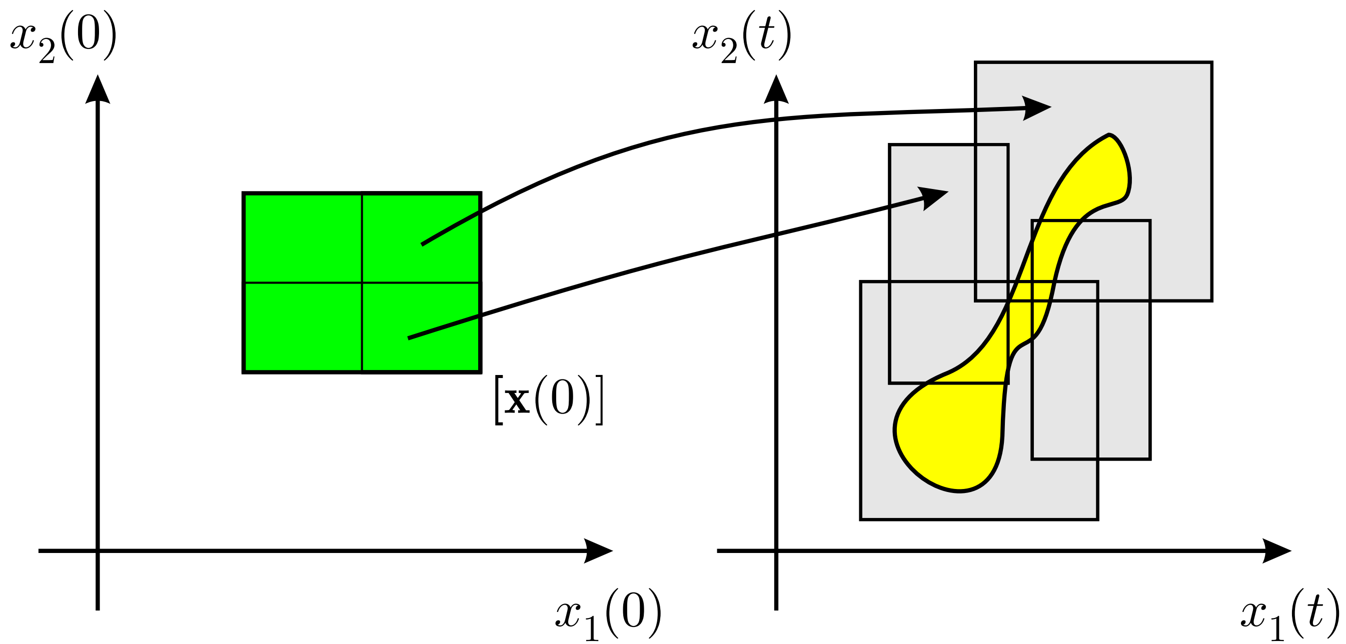
(a) initial subpaving (b) minced subpaving

(c) image boxes (d) image subpaving

Guaranteed numerical integration of continuous-time state equation

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \mathbf{f}\left(\mathbf{x}, \mathbf{w}, \mathbf{u}\right)$$

combined with IMAGESP.



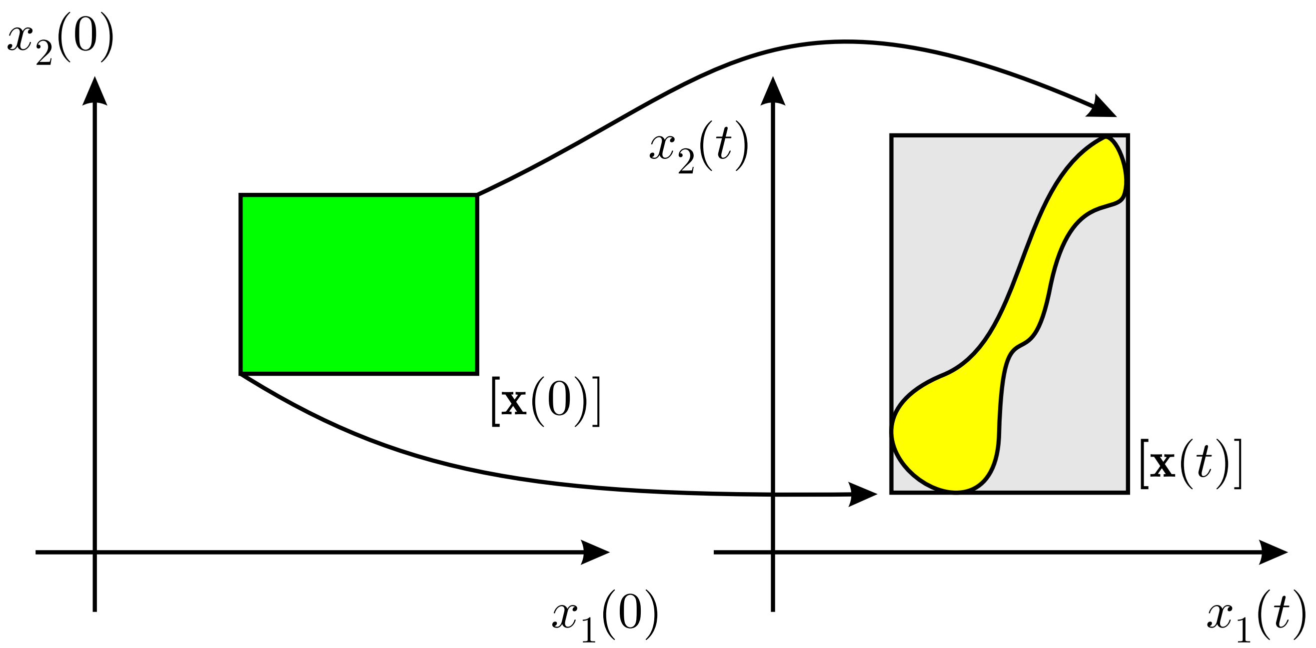
State equation/initial conditions not well known

$\Longrightarrow$ no accurate box enclosures

Enclosure of state equation between two cooperative systems

+

Guaranteed numerical integration of the cooperative systems



Guaranteed version of Gouzé *et al.*'s interval observer

Accurate box enclosure

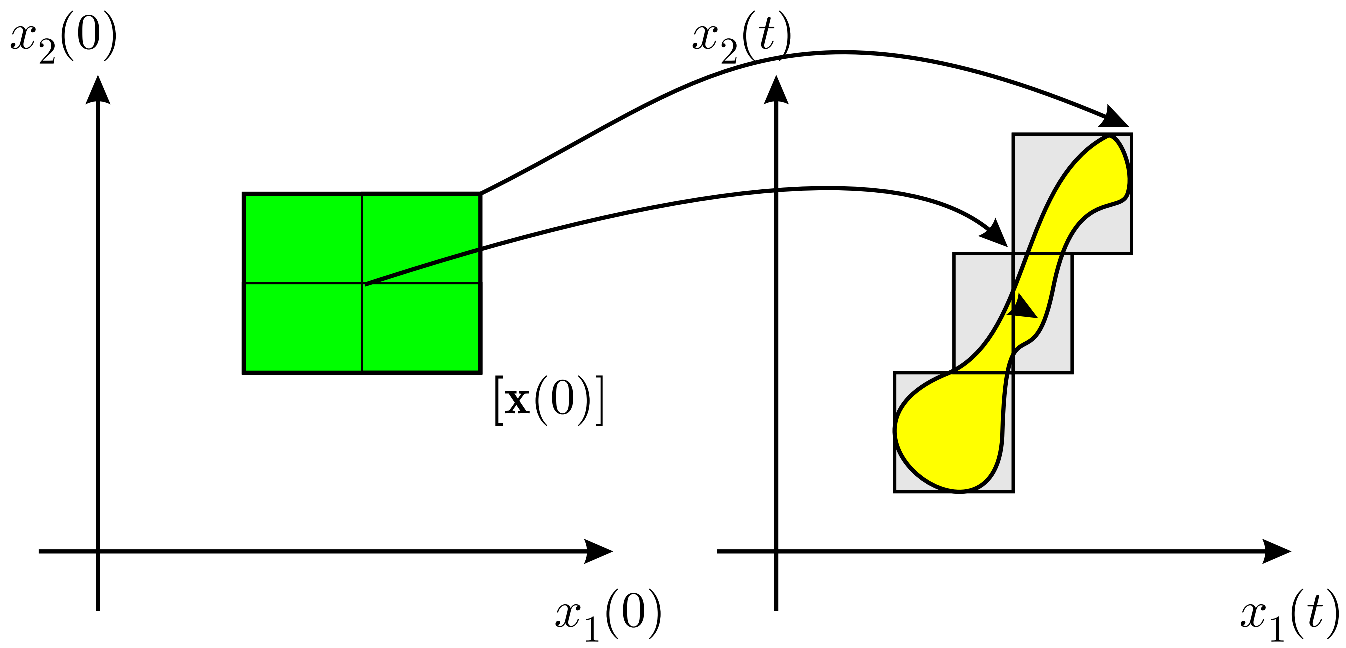
Enclosure of state equation between two cooperative systems

+

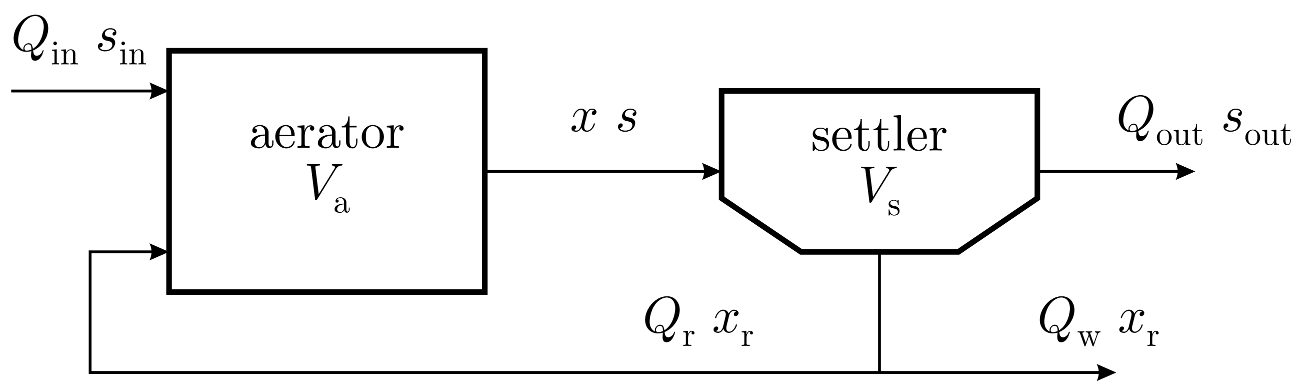
Guaranteed numerical integration of these cooperative systems

+

IMAGESP



2.7 Example (Gouzé *et al.*)



Model

$$\begin{cases} \dot{x} = \mu_1(.)\,x - (1+r)\,D(t)\,x + r D(t)\,x_{\mathrm{r}} \\ \dot{s} = -\dfrac{\mu_1(.)\,x}{Y_{\mathrm{s}}} - (1+r)\,D(t)\,s + D(t)\,s_{\mathrm{in}} \\ \dot{x}_{\mathrm{r}} = v\,(1+r)\,D(t)\,x - v\,(w+r)\,D(t)\,x_{\mathrm{r}} \end{cases}$$

with

$$D(t) = \frac{Q_{\mathrm{in}}}{V_{\mathrm{a}}},\ r = \frac{Q_{\mathrm{r}}}{Q_{\mathrm{in}}},\ w = \frac{Q_{\mathrm{w}}}{Q_{\mathrm{in}}},\ v = \frac{V_{\mathrm{a}}}{V_{\mathrm{s}}}.$$

With s assumed perfectly measured,
via linear change of variables

$$\Longrightarrow$$

reduced linear system that can be enclosed
between two cooperative systems

2.8 Simulation results

Simulation conditions are as in Z. M. Hadj-Sadok's PhD dissertation

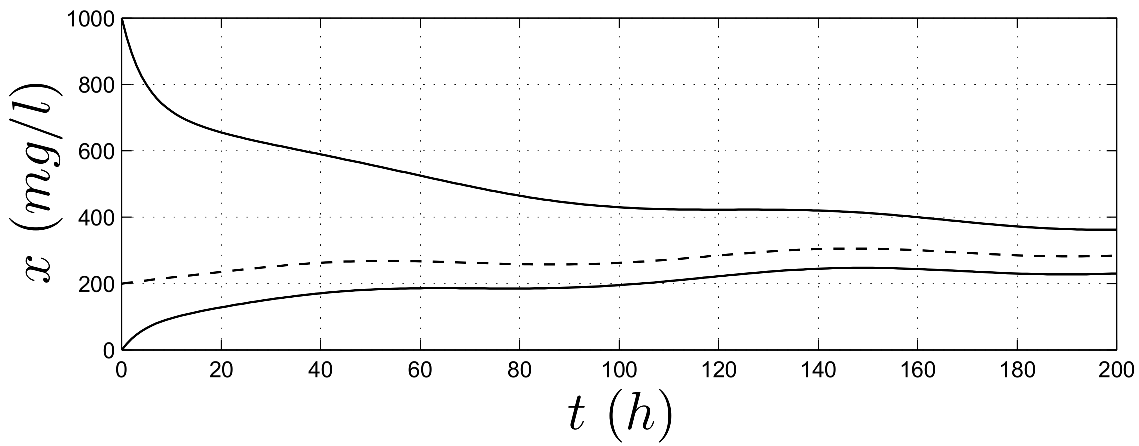
- $s(t)$ perfectly measured.
- No other measurement available (prediction only)

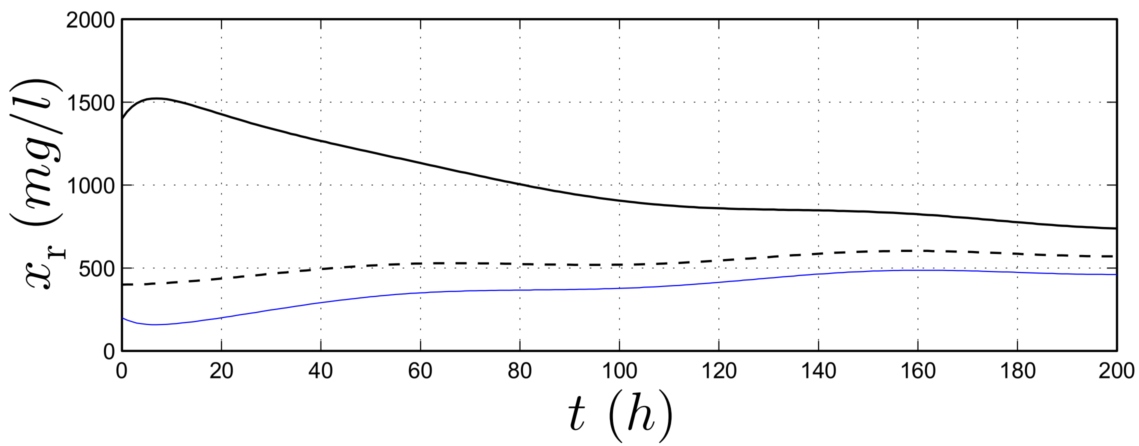
Initial conditions

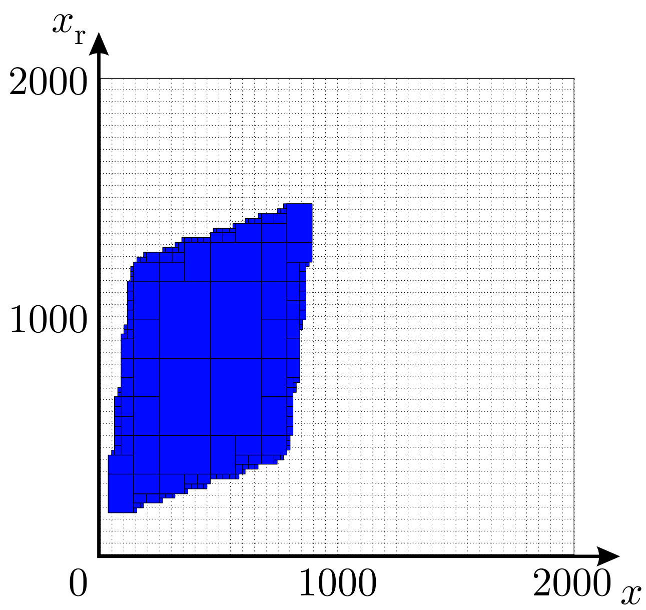
$$s(0) = 50 \text{ mg/l}, \; x(0) = 200 \text{ mg/l}, \; x_{\mathrm{r}}(0) = 400 \text{ mg/l}.$$

$$[x(0)] = [0, 1000] \text{ mg/l}, \; [x_{\mathrm{r}}(0)] = [200, 1400] \text{ mg/l}$$

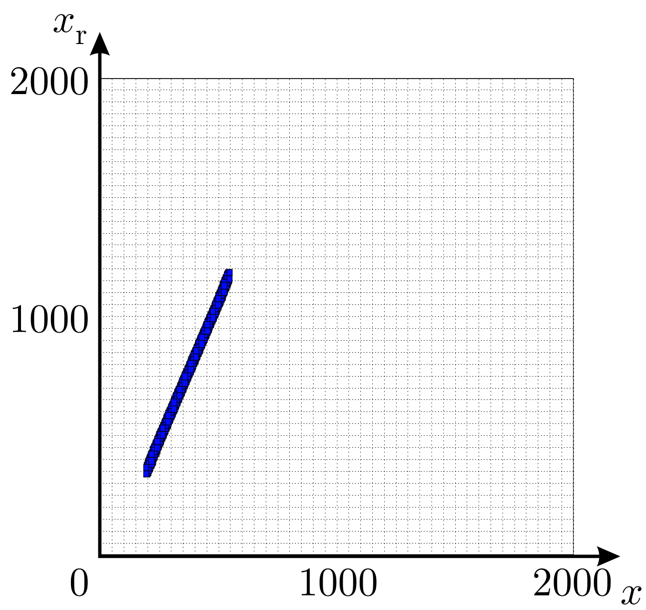
Results (prediction only)



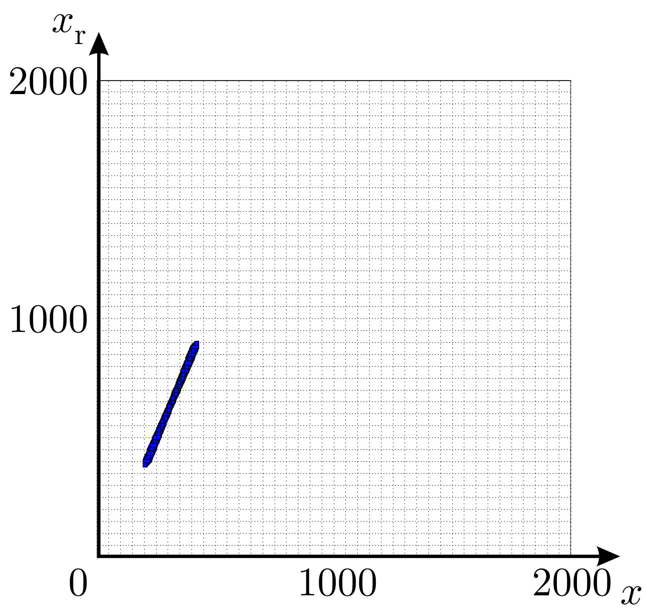




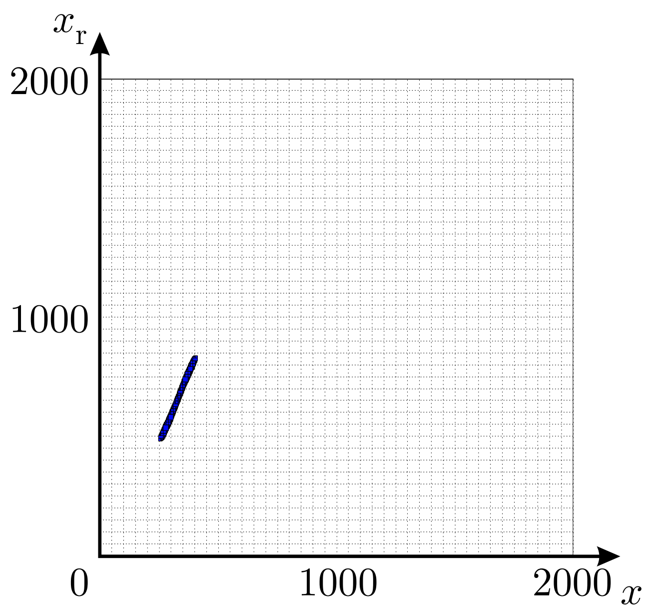
$t = 2$h



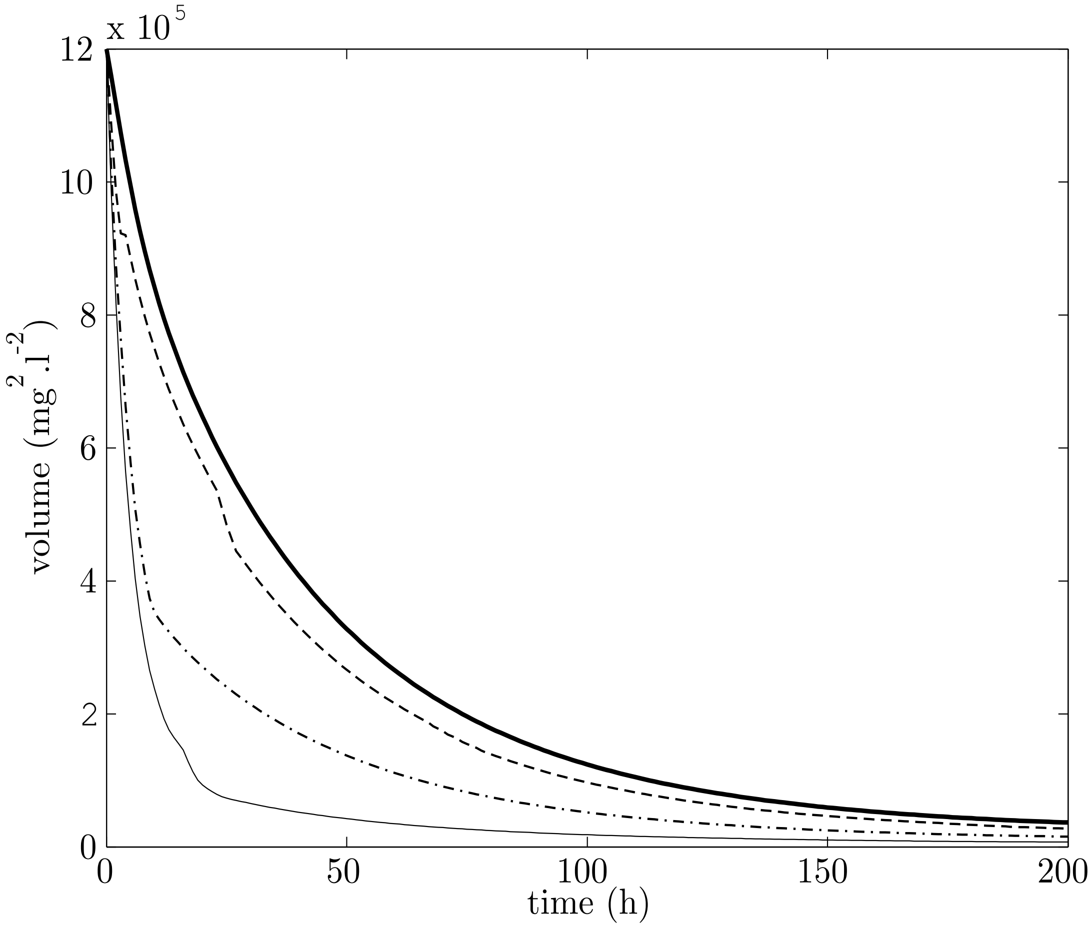
$t = 50$h



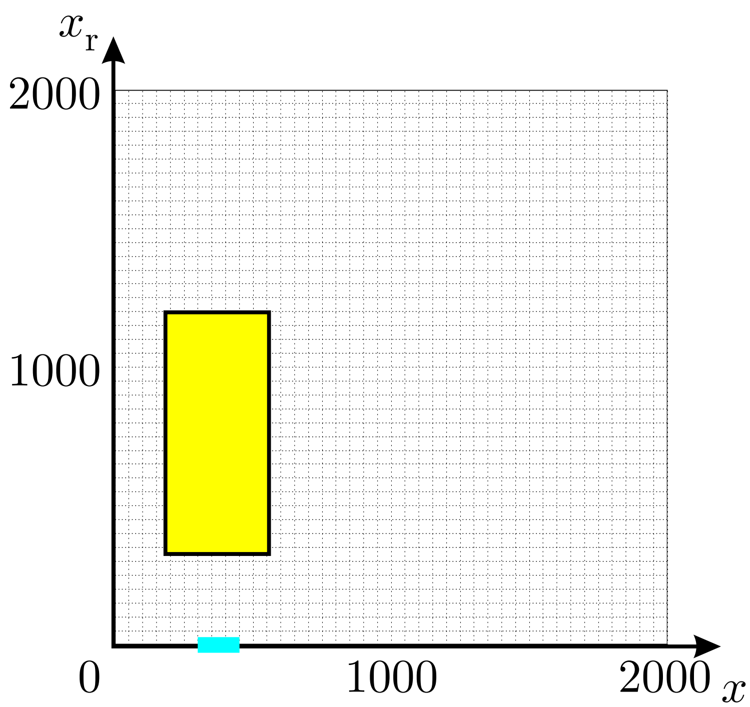
$t = 100\text{h}$



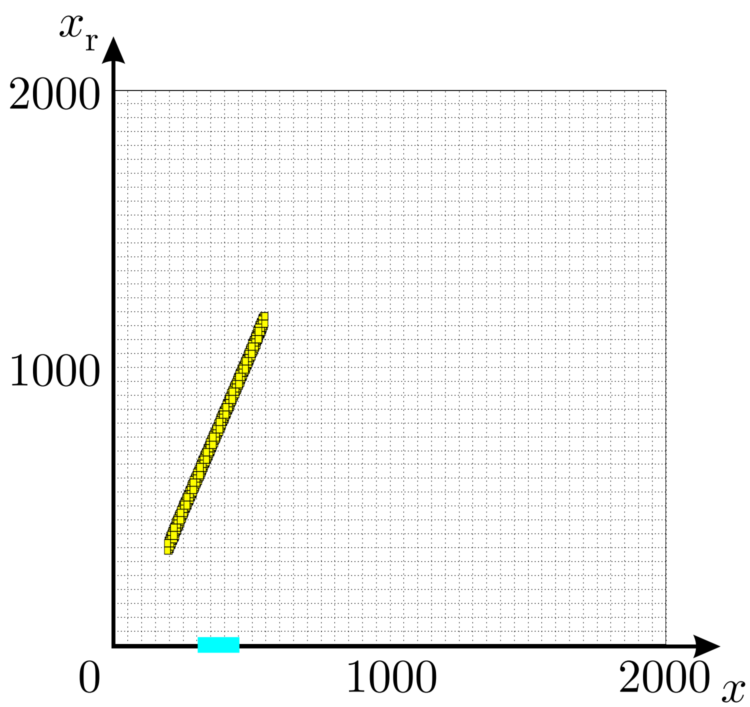
$t = 150\text{h}$


x 10^5
12
10
8
6
4
2
0
volume (mg^2.l^-2)
0
50
100
150
200
time (h)

What if measurements are available ?

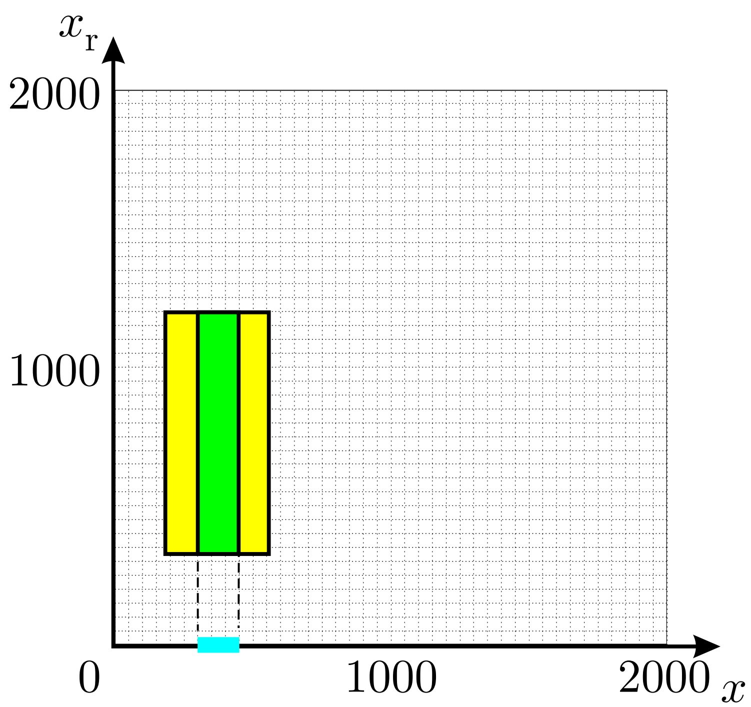


$t = 50$h
Interval observer

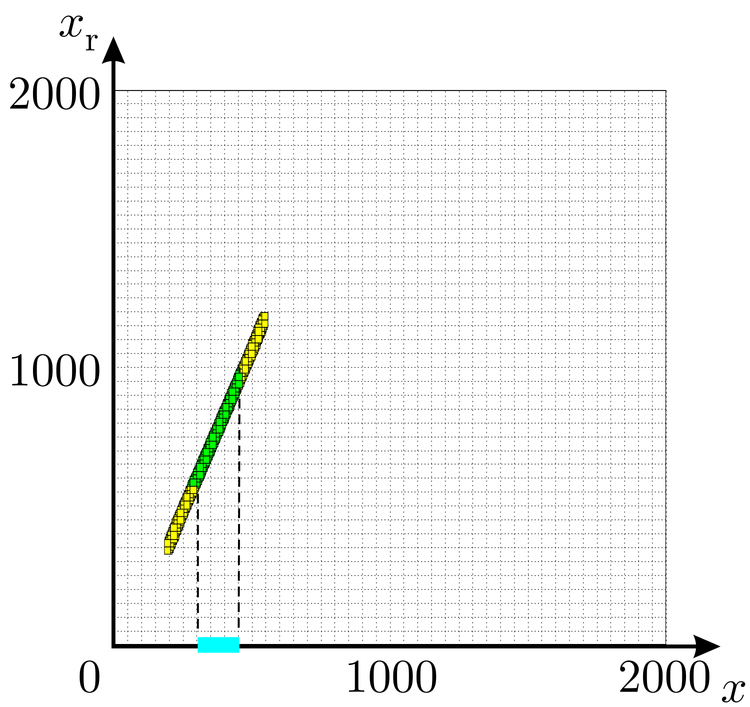


$t = 50$h
Set observer

What if measurements are available ?

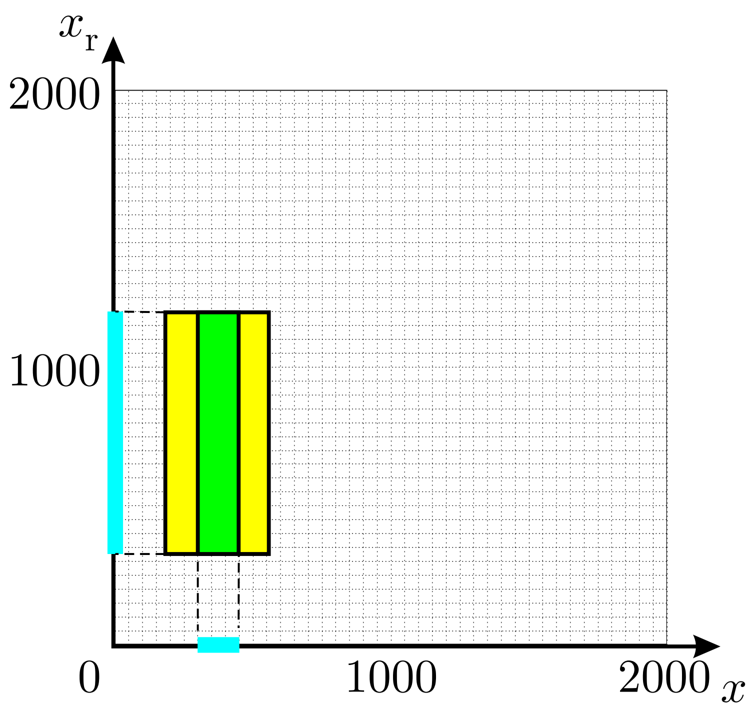


$t = 50$h
Interval observer

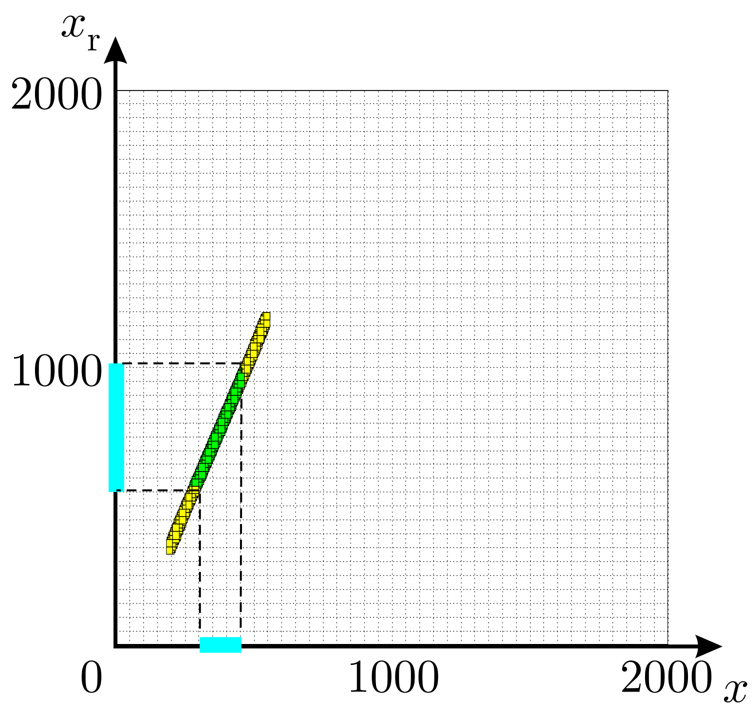


$t = 50$h
Set observer

What if measurements are available ?



$t = 50$h
Interval observer



$t = 50$h
Set observer

2.9 Joint parameter and state estimation

Continuous-time state equation

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \mathbf{f}\left(\mathbf{x}, \mathbf{p}, \mathbf{w}, \mathbf{u}\right).$$

Uncertain parameters $\mathbf{p}$ to be estimated jointly with $\mathbf{x}$.

Hypothesis needed on evolution of $\mathbf{p}$.

Possible choices are :

- $\mathbf{p}' = \mathbf{0}$
- $\mathbf{p}' = \mathbf{w}_{\mathrm{p}}$
- $\mathbf{p}' = \mathbf{f}_{\mathrm{p}}\left(\mathbf{x}, \mathbf{w}_{\mathrm{p}}, \mathbf{u}\right)$.

Extended state vector

$$\mathbf{x}_{\mathrm{e}} = \begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix}$$

Extended state equation

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix}' = \begin{pmatrix} \mathbf{f}\left(\mathbf{x}, \mathbf{w}, \mathbf{u}\right) \\ \mathbf{f}_{\mathrm{p}}\left(\mathbf{x}, \mathbf{w}_{\mathrm{p}}, \mathbf{u}\right) \end{pmatrix}$$

$$\mathbf{x}_{\mathrm{e}}' = \mathbf{f}_{\mathrm{e}}\left(\mathbf{x}_{\mathrm{e}}, \mathbf{w}, \mathbf{w}_{\mathrm{p}}, \mathbf{u}\right).$$

Conclusions

1. Recursive state estimation algorithm for continuous-time systems (for models that can be bounded by cooperative systems).
2. Guaranteed numerical integration still needed but with infinitesimal boxes
   $\hookrightarrow$ more suitable conditions for using guaranteed solvers than in (Jaulin, 02)
3. Uncertain parameters or state equations can be considered.
4. Real-time state estimation possible for systems with large time constants such as those encountered in bio-chemical processes...