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TRACING CHARACTERISTICS OF SMOOTH NONLINEAR RESISTIVE CIRCUITS BY INTERVAL ANALYSIS

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ABSTRACT

An algorithm for tracing solution curves of one-port nonlinear resistive networks, based on the covering of the solution set by a collection of n-dimensional boxes, is presented. The covering algorithm, developed from interval techniques, guarantees that every unicursal branch of the characteristic is captured and it is well suited to render the resulting curves on a graphics display. In addition, a new zero inclusion test for circuit equations in the hybrid form is introduced. This strategy, on the one hand has been proved to be effective in improving the quality of the analysis, while on the other hand has been able to considerably increase the performance of the method. As an application of the method the driving-point and transfer characteristics of a tunnel diode circuit are finally presented.

1. INTRODUCTION

To obtain the driving-point or transfer characteristics of a nonlinear resistive circuit one has to solve a system of $n-1$ nonlinear equations in n unknowns:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{x} \in D \subseteq \mathbb{R}^n, \qquad \mathbf{f}(\cdot) \in \mathbb{R}^{n-1}. \tag{1}$$

If $\mathbf{f}(\cdot)$ is regular in D, i.e. it is continuously differentiable and the Jacobian $\mathbf{J}(\mathbf{x})$ has rank $n-1$ in a neighborhood of the zeroes set

$$S = \{\mathbf{x} \in D : \mathbf{f}(\mathbf{x}) = \mathbf{0}\}, \tag{2}$$

then S is a one-dimensional manifold in $\mathbb{R}^n$. For nonlinear circuits, S is the union of several branches, even when all nonlinear resistive elements exhibit a single-branch characteristic [1]. Moreover, when projecting S onto plane x_i-x_j, where i and j select entries of $\mathbf{x}$, the resulting solution curve may be represented by a multi-valued function. In general, the number of branches of this curve is not known *a priori*, except for nonlinear series-parallel one-ports and, moreover, bifurcation points may appear [2]. Existing algorithms for tracing solution curves, based on continuation methods, start from a point located on a branch and follow it. The main difficulty of this approach lies in finding a starting point for each branch and in the determination of the total number of branches. Other problems occur when tracing characteristics by predictor-corrector methods like the backward differentiation formula (BDF) algorithm as proposed by Ushida and Chua [1]. In the presence of sharp turning points, in fact, the result of the predictor step may yield a poor approximation to the solution curve. Thus, a suitable step size control algorithm has to be added to avoid non-convergence of the Newton-Raphson method implemented in the corrector step.

In a recent paper by Yamamura [3], an algorithm employing hyper-spheres instead of hyper-planes for tracing solution curves has been proposed. Control of the sphere's radius is necessary in order to avoid a "reversion" phenomenon, i.e. tracing the curve in the backward direction. The authors therein propose a modified version of the algorithm able to overcome this possibility, even though the chance to take a step in the wrong direction remains.

In [1] the authors said that "it is unlikely that a completely general algorithm can be developed to find all branches". Nevertheless, the algorithm subsequently presented in [4] allows one to find all the branches for the class of piecewise-linear (PWL) circuits. In this work we direct our attention to the class of smooth circuits and present an algorithm, based on interval analysis [5], able to find every branch with mathematical certainty.

2. BOX COVERING ALGORITHM

Let us call *interval number* a closed set of real numbers, e.g. $X = \{x \in \mathbb{R} : a \leq x \leq b\}$. Interval quantities are denoted by capital letters, while non-interval (real) numbers are typeset in lower case. $\mathbb{IR}$ and $\mathbb{IR}^n$ stand respectively for the set of real intervals and the set of interval vectors of dimension n. The *width* of X is defined as $w(X) = b - a$. If $\mathbf{X} \in \mathbb{IR}^n$ then $w(\mathbf{X}) = \max\{w(X_i),\ i = 1, \ldots, n\}$. In the sequel, the expressions interval vector of dimension n and box will be used interchangeably. $f(X)$ and $F(X)$ denote respectively the range and the natural interval extension [5] of $f(\cdot)$ over X.

The underlying idea of the proposed method is based on interval analysis, whose applications to circuit analysis and design are recently drawing the attention of researchers [6]. The tracing method proposed in this work is a modification of Neumaier's algorithm for the enclosure of the solution set of parameter-dependent systems of equations [7]. This algorithm is able to cover the solution set of (1) by a collection of n-dimensional intervals. A driving point or transfer characteristic is approximated by projecting this set onto appropriate coordinate axes, defining a set of rectangles in a plane. Starting from an initial box $\mathbf{B}^{(0)} \in \mathbb{IR}^n$ the method generates—by repeated subdivisions and contractions—a number of boxes whose width is progressively reduced. At each stage of the process this set gives an increasingly accurate information about the location of the solution curve. When the width of each component is lower than a prescribed resolution, i.e. when

$$w(B_i) \leq w(B_i^{(0)})/N_i, \qquad i = 1, \ldots, n, \tag{3}$$

for each generated box $\mathbf{B}$, we may represent the resulting image on a computer screen. When a box, say $\mathbf{X}^{(\mathrm{k})}$, is taken into account in the first step of Neumaier's algorithm, the natural interval extension $\mathbf{F}(\mathbf{X}^{(\mathrm{k})})$ is evaluated by the rules of standard interval arithmetic. From the fundamental theorem of interval analysis [5] we have

$$f(\mathbf{x}^{(k)}) \subseteq F(\mathbf{X}^{(k)}), \qquad \mathbf{x}^{(k)} \in \mathbf{X}^{(k)}. \tag{4}$$

A necessary condition for (1) to have a solution in $\mathbf{X}^{(k)}$ is given by

$$0 \in F_i(\mathbf{X}^{(k)}), \qquad i = 1, \ldots, n. \tag{5}$$

Thus, if (5) does not hold for one or more values of i we can safely discard box $\mathbf{X}^{(k)}$ from further subdivisions. The converse, however, is not true, so it is not guaranteed that every printed box contains a solution point. An obvious cure to this problem is to increase N_i. In the case of a continuous interval extension, in fact, it can be shown that (5) excludes any point which does not belong to the solution set for sufficiently small boxes. This procedure, however, has the disadvantage of increasing the memory requirements and computation times of the algorithm. To address this problem we have developed a zero inclusion test based on a general property of circuit equations, that is presented in the next section. This property makes it possible to compute an enclosure for each component of $\mathbf{f}(\cdot)$ over a box yielding the true value of the range when exact interval arithmetic is used. Even if rounding occurs, as in any actual implementation of the algorithm, the intervals provided by this method are considerably narrower than those produced by the natural interval extension of (4) or other types of interval extension found in literature.

3. ZERO INCLUSION TEST FOR CIRCUIT EQUATIONS

Let us assume that (1) admits the hybrid representation

$$\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) - \mathbf{H}\mathbf{x} - \mathbf{s} = \mathbf{0},$$
$$g_i(\mathbf{x}) = g_i(x_i).$$

Taking into account a single component $f_i(\cdot)$ we can write

$$\begin{aligned} f_i(\mathbf{x}) \quad &= \quad g_i(x_i) - \sum_{j=1}^{n} h_{ij}x_j - s_i = \\ & \qquad g_i(x_i) - h_{ii}x_i - \sum_{j=1,\, j\neq i}^{n} h_{ij}x_j - s_i. \end{aligned} \tag{6}$$

If we let

$$\phi_{ij}(x_i) = \begin{cases} g_i(x_i) - h_{ii}x_i & \text{if } i = j \\ -h_{ij}x_j & \text{if } i \neq j \end{cases}, \tag{7}$$

Equation (6) rewrites to

$$f_i(\mathbf{x}) = \sum_{j=1}^{n} \phi_{ij}(x_j) - s_i. \tag{8}$$

In this way each circuit variable occurs only once in each equation, either as an argument of a nonlinear function or in a linear term. The zero-inclusion test is derived by the following proposition.

Proposition 3.1 *Let* $f\,(x_1, x_2, \ldots, x_n)\,, \quad f\,:\,\mathbb{R}^n \longrightarrow \mathbb{R}$ *be a continuously differentiable function of the form:*

$$f\,(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} f_j\,(x_j)$$

where $f_j : \mathbb{R} \longrightarrow \mathbb{R}$, *and* $x_j \in X_j = [a_j, b_j]\,,\ j = 1, \ldots, n$. *Let* $f_j\,(X_j)$ *be the range of* $f_j(x_j)$ *over* $X_j,\ j = 1, \ldots, n$. *Then:*

$$f\,(X_1, X_2, \ldots, X_n) = \sum_{j=1}^{n} f_j\,(X_j) \tag{9}$$

The proof of this proposition is given in Appendix A.
Remark. If $f_j(\cdot)$ is monotone over X_j, $f(X_j)$ is easily computed as $[f(a_j), f(b_j)]$. For a rational function, instead, the range can be approximated in the way detailed in section 6.4 of [5], independently of monotonicity.

4. RESULTS

The effectiveness of the presented algorithm has been tested with a simple tunnel diode circuit [1], described by the following equations:

$$\begin{aligned} v_{IN} &= R \cdot i + v_1 + v_2, & R = 3.2k\Omega \\ i &= 0.43 \cdot {v_1}^3 - 2.69 \cdot {v_1}^2 + 4.56 \cdot v_1, \\ i &= 2.5 \cdot {v_2}^3 - 10.5 \cdot {v_2}^2 + 11.8 \cdot v_2. \end{aligned}$$

In Fig. 1, reporting the $v_2(v_{IN})$ transfer characteristic, several spurious boxes are manifest. To overcome this problem, we have embedded the zero inclusion test in Neumaier's algorithm as follows. Whenever a box is generated, either by bisection or contraction, a copy of it is pushed into a stack. The box at the top of the stack, say $\mathbf{X}$, is taken into account in the first step of the method. If

$$0 \notin f_i\,(X_1, X_2, \ldots, X_n)$$

holds for some i, where the range is computed by (9), the box is discarded. When the zero inclusion test has been employed in the computation of the same characteristic, only one box surely not including any point of the solution set was found. This box, indicated by an arrow in Fig. 2, has not been eliminated by the zero inclusion test because of rounding errors. The computation of the solution curve took, respectively, 100.28 sec and 63.22 sec on a 90 MHz Pentium PC. We also monitored the total number N_G of generated boxes in both cases. In the first case we observed $N_G = 5944$, while in the latter a value of 3701 boxes was obtained. The resulting $i(v_{IN})$ transfer characteristic is depicted in Fig. 3, where two unicursal branches are evident. All results presented in this section have been produced by a program written in PASCAL-XSC [8], a language developed for programming algorithms based on interval analysis.

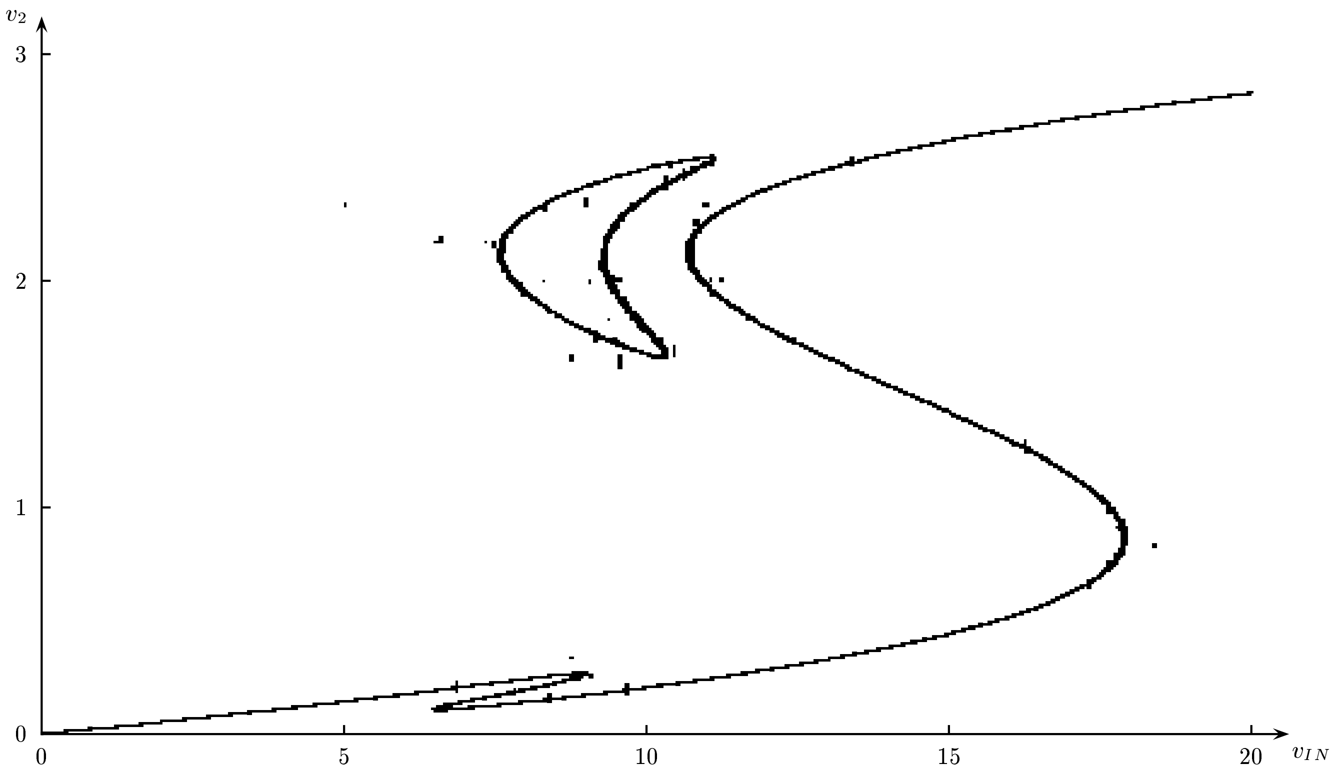

Figure 1: $v_2(v_{IN})$ transfer characteristic without zero inclusion test.

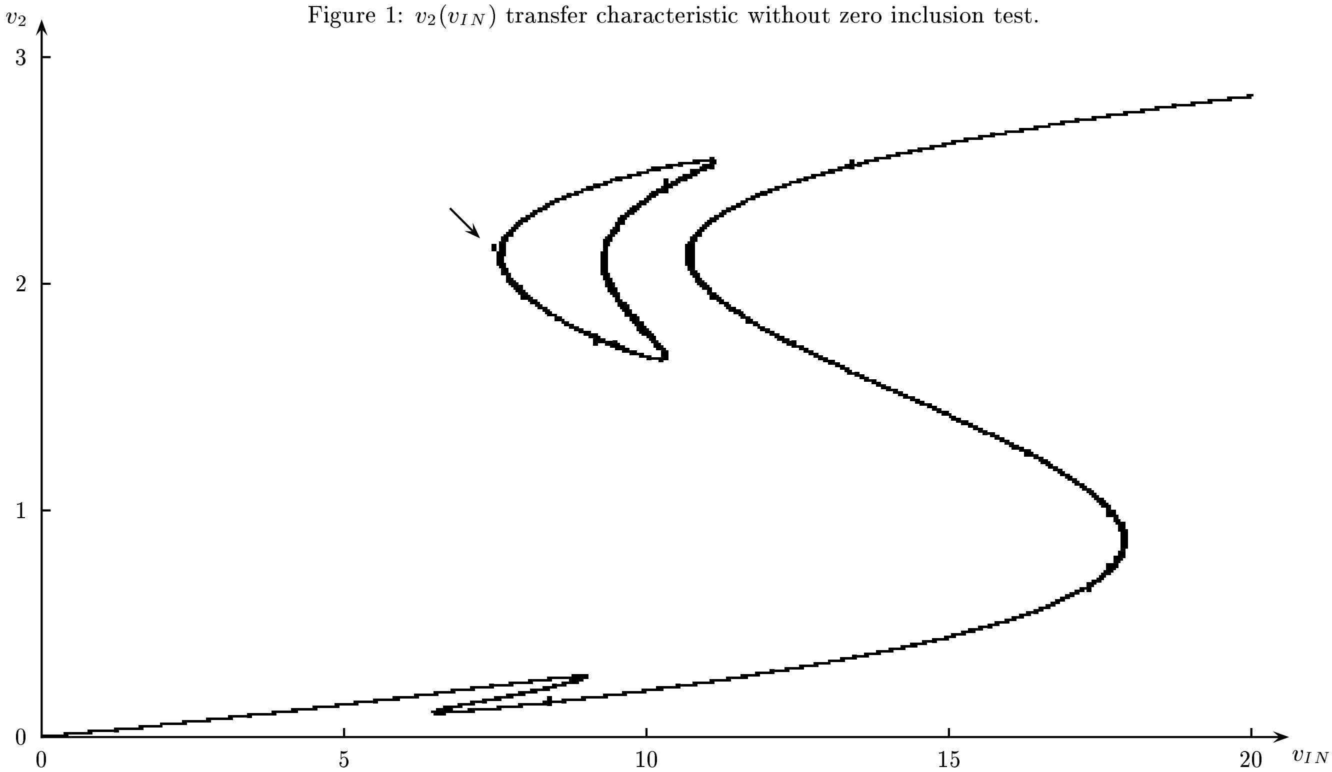

Figure 2: $v_2(v_{IN})$ transfer characteristic with zero inclusion test.

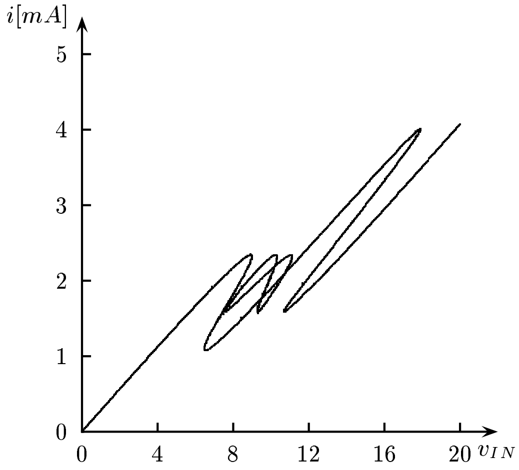

Figure 3: i-v_{IN} driving point characteristic

5. CONCLUSIONS AND FUTURE WORK

In this paper we have presented a method able to trace solution curves of smooth nonlinear circuits. By the global properties of interval analysis every branch of the solution set is captured with mathematical certainty. It has been shown how a new zero inclusion test based on the general structure of circuit equations, written in hybrid form, allows to considerably reduce the number of generated boxes; furthermore it decreases the run time of the algorithm. The application of the inclusion test is not limited to the curve tracing problem and we believe that it might prove to be useful also in other contexts, such as the determination of all equilibrium points of nonlinear resistive networks [6]. The computation of the driving point and transfer characteristics of a simple diode circuit proved the effectiveness of the method for problems defined in low dimension spaces. Application to higher order systems could lead to excessive requirements in terms of time and memory due to the increased number of generated boxes. Hence we suggest it as a subject of further investigation.

6. ACKNOWLEDGMENTS

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A. PROOF OF THE ZERO INCLUSION TEST

Proof. By the hypothesis we have:

$$a_j \leq f_j(x_j) \leq b_j, \quad x_j \in X_j, \quad j = 1, \ldots, n. \tag{10}$$

If we let

$$\Sigma_a = \sum_{j=1}^{n} a_j, \qquad \Sigma_b = \sum_{j=1}^{n} b_j,$$

from (10) follows:

$$\Sigma_a \leq \sum_{j=1}^{n} f_j(x_j) \leq \Sigma_b, \qquad x_j \in X_j, \quad j = 1, \ldots, n.$$

Hence, on the one hand we have:

$$f(X_1, X_2, \ldots, X_n) = [\Sigma_a, \Sigma_b] \subseteq \sum_{j=1}^{n} f_j(X_j). \tag{11}$$

On the other hand, if we suppose

$$\alpha \in [\Sigma_a, \Sigma_b],$$

by the definition of the intervals addition we have that there is at least a $n-$ple

$$\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \mid \alpha_1 \in X_1, \alpha_2 \in X_2, \ldots, \alpha_n \in X_n,$$

such that $\alpha = \sum_{j=1}^{n} \alpha_j$. Since $[a_j, b_j]$, $j = 1, \ldots, n$ is the range of $f_j(\cdot)$ over X_j, there exists at least a $n-$ple $\{\xi_j\}_{j=1}^{n}$ with $\xi_j \in X_j$, such that $f_j(\xi_j) = \alpha_j$, $j = 1, \ldots, n$. Therefore we obtain:

$$\begin{aligned} \alpha &= \sum_{j=1}^{n} f_j(\xi_j) = \\ f(\xi_1, \xi_2, \ldots, \xi_n) &\in f(X_1, X_2, \ldots, X_n) \implies \\ \sum_{j=1}^{n} f_j(X_j) &\subseteq f(X_1, X_2, \ldots, X_n). \end{aligned} \tag{12}$$

By (11) and (12) we have the thesis. ∎

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