Validated Constructive Error Estimations for Biharmonic Problems*

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Abstract

This paper presents some constructive error estimates for two-dimensional biharmonic equations by using verified computational techniques. These estimations are expected to provide valuable information for computer-assisted proofs of nonlinear biharmonic problems. Several numerical examples that confirm the effectiveness are reported.

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. This paper provides a guaranteed error bound for finite-dimensional approximate solutions for the biharmonic problem

$$\begin{cases} \Delta^2 u &= f \quad \text{in} \quad \Omega, \\ u = \dfrac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

for $f \in L^2(\Omega)$. Here, $\partial u/\partial n$ stands for the outer normal derivative of u. The biharmonic problem (1) arises in areas of continuum mechanics, including linear elasticity

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theory and the solution of Stokes flows by using a stream function-vorticity formulation [1, Chapter 7].

For some integer m, let $H^m(\Omega)$ denote the real L^2-Sobolev space of order m on Ω. We define the Hilbert space

$$H_0^2(\Omega) := \left\{ u \in H^2(\Omega) \;\middle|\; u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\} \tag{2}$$

with the inner product $(\Delta u, \Delta v)_{L^2(\Omega)}$ and the norm $\|u\|_{H_0^2(\Omega)} := \|\Delta u\|_{L^2(\Omega)}$, where $(u, v)_{L^2(\Omega)}$ implies the L^2-inner product on Ω. We also define the Hilbert space

$$H_0^1(\Omega) := \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega\} \tag{3}$$

with the inner product $(\nabla u, \nabla v)_{L^2(\Omega)}$ and the norm $\|u\|_{H_0^1(\Omega)} := \|\nabla u\|_{L^2(\Omega)}$, and a Banach space

$$D(\Delta^2) := \{ u \in H_0^2(\Omega) \mid \Delta^2 u \in L^2(\Omega) \} \tag{4}$$

with respect to the norm $\|u\|_{H_0^2(\Omega)} + \|\Delta^2 u\|_{L^2(\Omega)}$.

We assume that for each $f \in L^2(\Omega)$, there exists a unique solution $u \in D(\Delta^2)$ satisfying (1). For example, when Ω is the unit square, the existence of u is assured [5]. We aim to obtain a computable upper bound $C(h) > 0$ such that

$$\|u - u_h\|_{H_0^2(\Omega)} \le C(h)\|f\|_{L^2(\Omega)} \tag{5}$$

for an approximate solution $u_h \in S_h$ of (1) satisfying

$$(\Delta u_h, \Delta v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)}, \quad \forall v_h \in S_h. \tag{6}$$

Here, $S_h \subset H_0^2(\Omega)$ is a finite-dimensional approximation subspace dependent on the parameter $h > 0$. In the computer-assisted proof for nonlinear biharmonic equations, especially, for the two-dimensional Navier-Stokes equations [6, 11], the constant $C(h) > 0$ plays an essential and important role.

Let $P_2 : H_0^2(\Omega) \to S_h$ be the H_0^2-projection defined by

$$(\Delta(\varphi - P_2\varphi), \Delta v_h)_{L^2(\Omega)} = 0, \quad \forall v_h \in S_h. \tag{7}$$

Because the weak formulation of (1) is

$$(\Delta u, \Delta v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}, \quad \forall v \in H_0^2(\Omega), \tag{8}$$

and the approximate solution u_h of (1) satisfies (6), it holds that $u_h = P_2 u$ for the solution $u \in D(\Delta^2)$ of (1). Therefore, the error estimation (5) for the biharmonic problem is equivalent to finding $C(h) > 0$ such that

$$\|u - P_2 u\|_{H_0^2(\Omega)} \le C(h)\|\Delta^2 u\|_{L^2(\Omega)}, \quad \forall u \in D(\Delta^2). \tag{9}$$

In the one-dimensional case in which the domain is $J := (a, b)$, several a priori error estimates satisfying

$$\|u'' - u_h''\|_{L^2(J)} \le \widehat{C}(h)\|u''''\|_{L^2(J)} \tag{10}$$

have been presented [2, 10] with numerically determined values for $\widehat{C}(h) > 0$. Then, for a rectangular domain such that $\Omega = J \times J$, by using the estimation (10), the inequality

$$\|u - u_h\|_{H_0^2(\Omega)} \le \widehat{C}(h)|u|_{H^4(\Omega)} \tag{11}$$

can be derived with the H^4 semi-norm:

$$|u|_{H^4(\Omega)} := \Big(\|u_{xxxx}\|^2_{L^2(\Omega)} + 4\|u_{xxxy}\|^2_{L^2(\Omega)} + 6\|u_{xxyy}\|^2_{L^2(\Omega)} + 4\|u_{xyyy}\|^2_{L^2(\Omega)} + \|u_{yyyy}\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}}.$$

However, it is not so easy to obtain a numerically determined upper bound $\mathcal{C} > 0$ such that

$$|u|_{H^4(\Omega)} \leq \mathcal{C}\|\Delta^2 u\|_{L^2(\Omega)}, \tag{12}$$

even if the domain Ω is a rectangle.

Remark 1 *For example, when Ω is a unit square, by using the Fourier expansion in which $u = \sum_{m,n=1}^{\infty} a_{mn}\psi_{mn}$ with $\psi_{mn} := sin(m\pi x)\sin(n\pi y)/2$, it may appear that (12) has been achieved with $\mathcal{C} = 1$. It is true* if $\widehat{a}_{mn} = ((m\pi)^2 + (n\pi)^2)^2 a_{mn}$ *for the expansion of $\Delta^2 u = \sum_{m,n=1}^{\infty} \widehat{a}_{mn}\psi_{mn} \in L^2(\Omega)$. However, this equality does not hold in general, because the coefficient of the Fourier expansion, $\widehat{a}_{mn} = (\Delta^2 u, \psi_{mn})_{L^2(\Omega)}$, cannot be restored with $a_{mn} = (u, \psi_{mn})_{L^2(\Omega)}$ by partial integration and with the boundary condition $u = \partial u/\partial n = 0$. It has been reported that if $u \in H^4(\Omega)$ satisfies $u = \Delta u = 0$ on $\partial\Omega$,* (12) *holds when $\mathcal{C} = 1$ [3].*

To avoid the need to estimate (12), Nakao et al. [7] proposed a technique that directly determines the constant in the constructive a priori and a posteriori error estimates of (5); they do this by using the finite element approximation. Their procedure is based on verified computational techniques that use the Hermite spline functions for a two-dimensional rectangular domain; several numerical examples have confirmed the effectiveness of this approach.

In this paper, we take another computer-assisted approach that is expected to be applicable to a wide variety of approximation subspaces $S_h \subset H_0^2(\Omega)$.

This paper is organized as follows. Section 2 introduces the notation and several projections with related constants. Section 3 is devoted to some constructive error estimations of biharmonic problems. Several numerical examples are reported in Section 4.

2 Assumptions and Related Notation

We define the H_0^1-projection $P_1 : H_0^1(\Omega) \to S_h$ and the L^2-projection $P_0 : L^2(\Omega) \to S_h$ by

$$(\nabla(\varphi - P_1\varphi), \nabla v_h)_{L^2(\Omega)} = 0, \quad \forall v_h \in S_h, \tag{13}$$

$$(\varphi - P_0\varphi, v_h)_{L^2(\Omega)} = 0, \quad \forall v_h \in S_h, \tag{14}$$

and we assume that the H_0^1-projection P_1 has the following approximation property:

$$\|v - P_1 v\|_{L^2(\Omega)} \leq C_0(h)\|\Delta v\|_{L^2(\Omega)}, \quad \forall v \in D(\Delta^2). \tag{15}$$

Here, $C_0(h) > 0$ is a positive constant that is numerically determined such that $C_0(h) \to 0$ as $h \to 0$. Using $C_0(h)$ of (15), we aim to construct $C(h)$ satisfying (9), namely (5).

We assume that the finite-dimensional approximation subspace S_h belongs to $D(\Delta^2)$, and we define the basis function of S_h by $\{\varphi_i\}_{i=1}^K$ for $K := \dim S_h$ and $K \times K$ matrices A_0, A_1, A_2, A_3, and A_4:

$$[A_0]_{ij} = (\,\varphi_j, \varphi_i\,)_{L^2(\Omega)}, \tag{16}$$

$$[A_1]_{ij} = (\,\Delta\varphi_j, \varphi_i\,)_{L^2(\Omega)} = -(\nabla\varphi_j, \nabla\varphi_i)_{L^2(\Omega)}, \tag{17}$$

$$[A_2]_{ij} = (\Delta\varphi_j, \Delta\varphi_i)_{L^2(\Omega)}, \tag{18}$$

$$[A_3]_{ij} = (\,\Delta^2\varphi_j, \Delta\varphi_i\,)_{L^2(\Omega)}, \tag{19}$$

$$[A_4]_{ij} = (\,\Delta^2\varphi_j, \Delta^2\varphi_i\,)_{L^2(\Omega)}. \tag{20}$$

The matrices A_0, A_1, A_2, and A_4 are symmetric and nonsingular. Because A_0 is positive definite, it can be decomposed as $A_0 = A_0^{1/2} A_0^{T/2}$, where T indicates the transposition, and $A_0^{T/2}$ means $(A_0^{1/2})^T$. Usually, $A_0^{1/2}$ is a lower triangular matrix.

For each $u \in D(\Delta^2)$, by representing the L^2-projection $P_0\Delta^2 u \in S_h$ by (14) and the H_0^2-projection $P_2 u \in S_h$ by (7) as

$$P_0\Delta^2 u = \sum_{i=1}^{K} v_i\varphi_i, \quad \boldsymbol{v} = [v_i] \in \mathbb{R}^K, \tag{21}$$

$$P_2 u = \sum_{i=1}^{K} u_i\varphi_i, \quad \boldsymbol{u} = [u_i] \in \mathbb{R}^K, \tag{22}$$

the definition of projections P_0 and P_2 state that

$$\begin{aligned}(\,P_0\Delta^2 u, \varphi_i\,)_{L^2(\Omega)} &= (\,\Delta^2 u, \varphi_i\,)_{L^2(\Omega)} \\ &= (\Delta u, \Delta\varphi_i)_{L^2(\Omega)} \\ &= (\Delta P_2 u, \Delta\varphi_i)_{L^2(\Omega)} \\ &= (\,P_0\Delta^2 P_2 u, \varphi_i\,)_{L^2(\Omega)}\end{aligned}$$

for all $1 \leq i \leq K$; then, it holds that

$$\boldsymbol{u} = A_2^{-1} A_0 \boldsymbol{v}. \tag{23}$$

We also assume that an element

$$\chi_h = \sum_{i=1}^{K} w_i\varphi_i \in S_h, \quad \boldsymbol{w} = [w_i] \in \mathbb{R}^K \tag{24}$$

can be expressed as

$$\boldsymbol{w} = F\boldsymbol{v}, \tag{25}$$

where $\boldsymbol{v}$ is defined in (21) and $F \in \mathbb{R}^{K\times K}$. The element $\chi_h \in S_h$ is introduced by Lemma 3.1 in the next section, and the relation (25) between $\boldsymbol{w}$ for χ_h and $\boldsymbol{v}$ for $P_0\Delta^2 u$ will be presented in connection with Lemmas 3.2 and 3.3 in the next section.

Finally, we define matrices

$$Q_1 := A_0^{-1/2} A_1 F A_0^{-T/2}, \tag{26}$$

$$Q_2 := -A_0^{T/2} A_2^{-1} A_3^T F A_0^{-T/2}, \tag{27}$$

$$Q_3 := A_0^{T/2} A_2^{-1} A_4 A_2^{-1} A_0^{1/2}, \tag{28}$$

$$Q_4 := A_0^{-1/2} F^T A_2 F A_0^{-T/2}, \tag{29}$$

$$B_1 := Q_2 + Q_2^T + Q_3 + Q_4, \tag{30}$$

$$B_2 := Q_1 + Q_1^T + Q_2 + Q_2^T + Q_3 + Q_4 - I, \tag{31}$$

where I stands for the identity matrix.

3 Constructive Error Estimations of Biharmonic Problems

For the error estimation of the P_2-projection (9) with $C_0(h)$, we begin by showing the following lemma.

Lemma 3.1 *For each $u \in D(\Delta^2)$ and $\chi_h \in S_h$, it is true that*

$$\|u - P_2 u\|_{H_0^2(\Omega)} \le C_0(h) \|\Delta^2(u - P_2 u) + \Delta \chi_h\|_{L^2(\Omega)}. \tag{32}$$

Proof: Set $u_\perp = u - P_2 u \in D(\Delta^2)$. Using (7), two partial integrations, (13), the Cauchy-Schwarz inequality, and (15), we have

$$\begin{aligned}
\|\Delta u_\perp\|_{L^2(\Omega)}^2 &= (\Delta u_\perp, \Delta u_\perp)_{L^2(\Omega)} \\
&= (\Delta u_\perp, \Delta(u_\perp - P_1 u_\perp))_{L^2(\Omega)} \\
&= -(\nabla \Delta u_\perp, \nabla(u_\perp - P_1 u_\perp))_{L^2(\Omega)} \\
&= -(\nabla(\Delta u_\perp + \chi_h), \nabla(u_\perp - P_1 u_\perp))_{L^2(\Omega)} \\
&= (\ \Delta^2 u_\perp + \Delta \chi_h, u_\perp - P_1 u_\perp\ )_{L^2(\Omega)} \\
&\le \|\Delta^2 u_\perp + \Delta \chi_h\|_{L^2(\Omega)}\ \|u_\perp - P_1 u_\perp\|_{L^2(\Omega)} \\
&\le \|\Delta^2 u_\perp + \Delta \chi_h\|_{L^2(\Omega)}\ C_0(h) \|\Delta u_\perp\|_{L^2(\Omega)},
\end{aligned}$$

which implies (32). □

Note that (32) holds for any $\chi_h \in S_h$ and there are some choice of χ_h depending on the finite-dimensional subspace S_h. We show several concrete examples of χ_h in the last section.

Now, we consider the estimation of $C_1(h) > 0$ satisfying

$$\|\Delta^2(u - P_2 u) + \Delta \chi_h\|_{L^2(\Omega)} \le C_1(h) \|\Delta^2 u\|_{L^2(\Omega)}. \tag{33}$$

We show two approaches for $C_1(h)$ satisfying (33). The choice will be depend on S_h and the computational cost. The following lemma is one of the approaches.

Lemma 3.2 *The constant $C_1(h) > 0$ of (33) can be taken as*

$$C_1(h) = 1 + \sqrt{\|B_1\|_2}. \tag{34}$$

Proof: Because

$$\|\Delta^2(u - P_2u) + \Delta\chi_h\|_{L^2(\Omega)} \le \|\Delta^2 u\|_{L^2(\Omega)} + \|\Delta^2 P_2 u - \Delta\chi_h\|_{L^2(\Omega)},$$

using (20), (19), (18), (22), (24), (25), (23), (28), (27), (29), and (30) we obtain

$$\begin{aligned}
&\|\Delta^2 P_2 u - \Delta\chi_h\|^2_{L^2(\Omega)} \\
&\quad = (\,\Delta^2 P_2 u - \Delta\chi_h, \Delta^2 P_2 u - \Delta\chi_h\,)_{L^2(\Omega)} \\
&\quad = (\,\Delta^2 P_2 u, \Delta^2 P_2 u\,)_{L^2(\Omega)} - (\,\Delta^2 P_2 u, \Delta\chi_h\,)_{L^2(\Omega)} \\
&\qquad - (\,\Delta\chi_h, \Delta^2 P_2 u\,)_{L^2(\Omega)} + (\,\Delta\chi_h, \Delta\chi_h\,)_{L^2(\Omega)} \\
&\quad = \boldsymbol{u}^T A_4 \boldsymbol{u} - \boldsymbol{w}^T A_3 \boldsymbol{u} - \boldsymbol{u}^T A_3^T \boldsymbol{w} + \boldsymbol{w}^T A_2 \boldsymbol{w} \\
&\quad = \boldsymbol{v}^T A_0 A_2^{-1} A_4 A_2^{-1} A_0 \boldsymbol{v} - \boldsymbol{v}^T F^T A_3 A_2^{-1} A_0 \boldsymbol{v} - \boldsymbol{v}^T A_0 A_2^{-1} A_3^T F \boldsymbol{v} + \boldsymbol{v}^T F^T A_2 F \boldsymbol{v} \\
&\quad = (A_0^{T/2}\boldsymbol{v})^T \Big(A_0^{T/2} A_2^{-1} A_4 A_2^{-1} A_0^{1/2} - A_0^{-1/2} F^T A_3 A_2^{-1} A_0^{1/2} \\
&\qquad - A_0^{T/2} A_2^{-1} A_3^T F A_0^{-T/2} + A_0^{-1/2} F^T A_2 F A_0^{-T/2} \Big) A_0^{T/2}\boldsymbol{v} \\
&\quad = (A_0^{T/2}\boldsymbol{v})^T \left(Q_2 + Q_2^T + Q_3 + Q_4 \right) A_0^{T/2}\boldsymbol{v} \\
&\quad = (A_0^{T/2}\boldsymbol{v})^T B_1 A_0^{T/2}\boldsymbol{v} \\
&\quad \le \|B_1\|_2 (A_0^{T/2}\boldsymbol{v})^T (A_0^{T/2}\boldsymbol{v}) \\
&\quad = \|B_1\|_2 \; \boldsymbol{v}^T A_0 \boldsymbol{v} \\
&\quad = \|B_1\|_2 \; \|P_0 \Delta^2 u\|^2_{L^2(\Omega)} \\
&\quad \le \|B_1\|_2 \; \|\Delta^2 u\|^2_{L^2(\Omega)},
\end{aligned}$$

then the conclusion. □

Remark 2 *In the case of $\chi_h = 0$, we can take*

$$C_1(h) = 1 + \sqrt{\left\| A_0^{T/2} A_2^{-1} A_4 A_2^{-1} A_0^{1/2} \right\|_2},$$

based on Lemma 3.2, and then $\left\| A_0^{T/2} A_2^{-1} A_4 A_2^{-1} A_0^{1/2} \right\|_2$ coincides with the maximum eigenvalue of the matrix $A_2^{-1} A_4 A_2^{-1} A_0$. For the verified bounds for the 2-norm (spectral norm) of a matrix, see [8].

Now we show an alternative to Lemma 3.2.

Lemma 3.3 *The constant $C_2(h) > 0$ of (33) can be taken as*

$$C_1(h) = \sqrt{1 + \|B_2\|_2}. \tag{35}$$

Proof: When there exists $K_h > 0$ satisfying

$$\|P_0 \Delta^2 u - \Delta^2 P_2 u + \Delta\chi_h\|_{L^2(\Omega)} \le K_h \|P_0 \Delta^2 u\|_{L^2(\Omega)}, \tag{36}$$

using (36) and Hölder's inequality, we obtain

$$\begin{aligned}
\|\Delta^2(u - P_2 u) + \Delta\chi_h\|_{L^2(\Omega)} &= \|(I - P_0)\Delta^2 u + P_0 \Delta^2 u - \Delta^2 P_2 u + \Delta\chi_h\|_{L^2(\Omega)} \\
&\le \|(I - P_0)\Delta^2 u\|_{L^2(\Omega)} + K_h \|P_0 \Delta^2 u\|_{L^2(\Omega)} \qquad (37) \\
&\le \sqrt{1 + K_h^2}\; \sqrt{\|(I - P_0)\Delta^2 u\|^2_{L^2(\Omega)} + \|P_0 \Delta^2 u\|^2_{L^2(\Omega)}} \\
&= \sqrt{1 + K_h^2}\; \|\Delta^2 u\|_{L^2(\Omega)}. \qquad (38)
\end{aligned}$$

For K_h satisfying (36), using partial integration and (16), (18), (19), and (20), we have

$$\begin{aligned}
&\|P_0\Delta^2 u - \Delta^2 P_2 u + \Delta\chi_h\|^2_{L^2(\Omega)} \\
&\quad = (\, P_0\Delta^2 u - \Delta^2 P_2 u + \Delta\chi_h, P_0\Delta^2 u - \Delta^2 P_2 u + \Delta\chi_h\,)_{L^2(\Omega)} \\
&\quad = (\, P_0\Delta^2 u, P_0\Delta^2 u\,)_{L^2(\Omega)} - (\, P_0\Delta^2 u, \Delta^2 P_2 u\,)_{L^2(\Omega)} + (\, P_0\Delta^2 u, \Delta\chi_h\,)_{L^2(\Omega)} \\
&\qquad - (\, \Delta^2 P_2 u, P_0\Delta^2 u\,)_{L^2(\Omega)} + (\, \Delta^2 P_2 u, \Delta^2 P_2 u\,)_{L^2(\Omega)} - (\, \Delta^2 P_2 u, \Delta\chi_h\,)_{L^2(\Omega)} \\
&\qquad + (\, \Delta\chi_h, P_0\Delta^2 u\,)_{L^2(\Omega)} - (\, \Delta\chi_h, \Delta^2 P_2 u\,)_{L^2(\Omega)} + (\, \Delta\chi_h, \Delta\chi_h\,)_{L^2(\Omega)} \\
&\quad = \boldsymbol{v}^T A_0 \boldsymbol{v} - (\, \Delta P_0\Delta^2 u, \Delta P_2 u\,)_{L^2(\Omega)} + \boldsymbol{w}^T A_1 \boldsymbol{v} - (\, \Delta P_2 u, \Delta P_0 \Delta^2 u\,)_{L^2(\Omega)} \\
&\qquad + \boldsymbol{u}^T A_4 \boldsymbol{u} - \boldsymbol{w}^T A_3 \boldsymbol{u} + \boldsymbol{v}^T A_1 \boldsymbol{w} - \boldsymbol{u}^T A_3^T \boldsymbol{w} + \boldsymbol{w}^T A_2 \boldsymbol{w} \\
&\quad = \boldsymbol{v}^T A_0 \boldsymbol{v} - \boldsymbol{u}^T A_2 \boldsymbol{v} + \boldsymbol{w}^T A_1 \boldsymbol{v} - \boldsymbol{v}^T A_2 \boldsymbol{u} + \boldsymbol{u}^T A_4 \boldsymbol{u} - \boldsymbol{w}^T A_3 \boldsymbol{u} \\
&\qquad + \boldsymbol{v}^T A_1 \boldsymbol{w} - \boldsymbol{u}^T A_3^T \boldsymbol{w} + \boldsymbol{w}^T A_2 \boldsymbol{w}.
\end{aligned}$$

Then, noting that $A_0 = A_0^{1/2} A_0^{T/2}$, (22) and (25) can be used to derive

$$\begin{aligned}
&\|P_0\Delta^2 u - \Delta^2 P_2 u + \Delta\chi_h\|^2_{L^2(\Omega)} \\
&\quad = \boldsymbol{v}^T A_0 \boldsymbol{v} - \boldsymbol{v}^T A_0 A_2^{-1} A_2 \boldsymbol{v} + \boldsymbol{v}^T F^T A_1 \boldsymbol{v} \\
&\qquad - \boldsymbol{v}^T A_2 A_2^{-1} A_0 \boldsymbol{v} + \boldsymbol{v}^T A_0 A_2^{-1} A_4 A_2^{-1} A_0 \boldsymbol{v} - \boldsymbol{v}^T F^T A_3 A_2^{-1} A_0 \boldsymbol{v} \\
&\qquad + \boldsymbol{v}^T A_1 F \boldsymbol{v} - \boldsymbol{v}^T A_0 A_2^{-1} A_3^T F \boldsymbol{v} + \boldsymbol{v}^T F^T A_2 F \boldsymbol{v} \\
&\quad = -\boldsymbol{v}^T A_0 \boldsymbol{v} + \boldsymbol{v}^T F^T A_1 \boldsymbol{v} + \boldsymbol{v}^T A_1 F \boldsymbol{v} + \boldsymbol{v}^T A_0 A_2^{-1} A_4 A_2^{-1} A_0 \boldsymbol{v} \\
&\qquad - \boldsymbol{v}^T F^T A_3 A_2^{-1} A_0 \boldsymbol{v} - \boldsymbol{v}^T A_0 A_2^{-1} A_3^T F \boldsymbol{v} + \boldsymbol{v}^T F^T A_2 F \boldsymbol{v} \\
&\quad = (A_0^{T/2}\boldsymbol{v})^T \Big(-I + A_0^{-1/2} F^T A_1 A_0^{-T/2} + A_0^{-1/2} A_1 F A_0^{-T/2} + A_0^{T/2} A_2^{-1} A_4 A_2^{-1} A_0^{1/2} \\
&\qquad - A_0^{-1/2} F^T A_3 A_2^{-1} A_0^{1/2} - A_0^{T/2} A_2^{-1} A_3^T F A_0^{-T/2} + A_0^{-1/2} F^T A_2 F A_0^{-T/2} \Big) A_0^{T/2} \boldsymbol{v} \\
&\quad = (A_0^{T/2}\boldsymbol{v})^T \; B_2 \; A_0^{T/2} \boldsymbol{v} \\
&\quad \le \|B_2\|_2 \; (A_0^{T/2}\boldsymbol{v})^T A_0^{T/2} \boldsymbol{v} \\
&\quad = \|B_2\|_2 \; \boldsymbol{v}^T A_0 \boldsymbol{v} \\
&\quad = \|B_2\|_2 \|P_0 \Delta^2 u\|^2_{L^2(\Omega)}.
\end{aligned}$$

Therefore, we can take $K_h^2 = \|B_2\|_2$. □

Remark 3 *In the case of $\chi_h = 0$ in Lemma 3.3, we can take*

$$C_1(h) = \sqrt{1 + \|A_0^{T/2} A_2^{-1} A_4 A_2^{-1} A_0^{1/2} - I\|_2}.$$

Lemma 3.1, Lemma 3.2, and Lemma 3.3 imply our main result.

Theorem 3.1 *For the solution $u \in D(\Delta^2)$ of the biharmonic equation* (1) *and the approximate solution $u_h \in S_h$ satisfying* (6)*, it is true that*

$$\|u - u_h\|_{H_0^2(\Omega)} \le C(h) \|f\|_{L^2(\Omega)}, \tag{39}$$

with

$$C(h) := C_0(h) \; C_1(h), \tag{40}$$

where $C_1(h)$ is given constructively by (34) *or* (35).

4 Numerical Examples

In this section, we report several numerical examples of a finite-dimensional approximation of $H_0^2(\Omega)$ by Legendre polynomials [2] on the unit square domain $\Omega = (0,1)\times(0,1)$. For $N > 0$, define

$$\psi_n(x) := \frac{(-1)^{n+1}\sqrt{2n+3}}{(n+1)!}\left(\frac{d}{dx}\right)^{n-1}(x-x^2)^{n+1}, \quad 1 \le n \le N, \tag{41}$$

and

$$\varphi_k(x,y) := \psi_m(x) \times \psi_n(y), \tag{42}$$

with some change of indices $(m,n) \to k$. Then, we can assure that $K = N^2$, $h = 1/N$, and $S_h = \mathrm{span}\{\varphi_k\}_{k=1}^K$ is a finite-dimensional subspace of $H_0^2(\Omega)$ satisfying $S_h \subset D(\Delta^2)$. Moreover, $C_0(h) > 0$ of (15) can be taken as

$$C_0(h) = \begin{cases} \sqrt{c_2(N+3)}/4 & \text{if} \quad 1 \le N \le 16, \\ \sqrt{c_3(N+3)}/4 & \text{if} \quad N \ge 17, \end{cases} \tag{43}$$

where

$$\begin{aligned} c_2(L) := {} & \frac{2}{\sqrt{2L-5}(2L-3)^2\sqrt{2L-1}(2L+1)} \\ & + \frac{4}{(2L-3)\sqrt{2L-1}(2L+1)\sqrt{2L+3}(2L+5)} \\ & + \frac{1}{\sqrt{2L-1}(2L+1)(2L+3)(2L+5)\sqrt{2L+7}} \\ & + \frac{10L-3}{(2L-3)^2(2L-1)(2L+1)(2L+3)}, \end{aligned} \tag{44}$$

and

$$\begin{aligned} c_3(L) := {} & \frac{1}{\sqrt{2L-5}(2L-3)(2L-1)(2L+1)\sqrt{2L+3}} \\ & + \frac{4}{(2L-3)\sqrt{2L-1}(2L+1)\sqrt{2L+3}(2L+5)} \\ & + \frac{6}{(2L-1)(2L+1)(2L+5)(2L+7)} \\ & + \frac{4}{(2L+1)\sqrt{2L+3}(2L+5)\sqrt{2L+7}(2L+9)} \\ & + \frac{1}{\sqrt{2L+3}(2L+5)(2L+7)(2L+9)\sqrt{2L+11}}. \end{aligned} \tag{45}$$

Note that by using Theorem 3.7 in [2], it would be possible to further improve $C_0(h)$.

Table 1 shows the bounds of $C_1(h)$ obtained by Wolfram Mathematica 10.0.2.0 with 100-digit multiple precision. To avoid rounding-error effects, this should be confirmed analytically, which can be accomplished by interval arithmetic software (e.g., [4, 9]). In Table 1, we consider three types of the matrix F. The notation "0" indicates $\chi_h = 0$, "$A_2^{-1}A_3A_2^{-1}A_0$" indicates that $\boldsymbol{w}$ in (24) satisfies

$$(\,\Delta\chi_h - \Delta^2 P_2 u, \Delta\varphi_i\,)_{L^2(\Omega)} = 0, \quad 1 \le i \le K,$$

which ensures that $Q_2 + Q_4 = 0$, and "$A_2^{-1}(A_3 A_2^{-1} A_0 - A_1)$" indicates that $\boldsymbol{w}$ is taken such that

$$(\Delta\chi_h - \Delta^2 P_2 u + P_0 \Delta^2 u, \Delta\varphi_i)_{L^2(\Omega)} = 0, \quad 1 \le i \le K.$$

The simplest case, $F = 0$, is very unstable; in other cases, there is some improvement in $C_1(h)$.

Table 1: Constructive constants of $C_1(h)$ in Lemma 3.2 and Lemma 3.3.

F	0		$A_2^{-1}A_3A_2^{-1}A_0$		$A_2^{-1}(A_3A_2^{-1}A_0 - A_1)$	
N	Lemma 2	Lemma 3	Lemma 2	Lemma 3	Lemma 2	Lemma 3
5	3.3305	2.3305	2.8906	1.9895	3.0421	1.7912
10	5.7256	4.7256	3.9293	2.9970	4.0323	2.8774
15	8.6612	7.6612	5.0966	4.1518	5.1680	4.0723
20	12.0622	11.0622	6.3069	5.3539	6.3601	5.2962

Table 2 shows the bounds of each constant by using Lemma 3 with

$$F = A_2^{-1}(A_3 A_2^{-1} A_0 - A_1).$$

$C(h)$ seems to be approximately $O(h)$, which means it should provide a "good" verification of nonlinear biharmonic problems.

Table 2: Constructive error estimates for the biharmonic equation.

N	$C(h)$	$C_0(h)$	$C_1(h)$
10	3.7742×10^{-3}	1.3117×10^{-3}	2.8774
20	2.2329×10^{-3}	4.2161×10^{-4}	5.2962
30	1.6453×10^{-3}	2.1133×10^{-4}	7.7851
40	1.3051×10^{-3}	1.2672×10^{-4}	10.2997
50	1.0823×10^{-3}	8.4375×10^{-5}	12.8265

It is not clear why $C_1(h)$ shows a tendency to become large as $h \to 0$. As an area of future work, we intend to investigate much finer spacing of F for $C(h)$ and to use another finite-dimensional basis, e.g., finite element functions; we also will try to verify these solutions of nonlinear biharmonic equations, especially the two-dimensional Navier-Stokes equations.

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