Fast Determination of the Tensorial and Simplicial Bernstein Forms of Multivariate Polynomials and Rational Functions*

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Abstract

Tests for speeding up the determination of the Bernstein enclosure of the range of a multivariate polynomial and a rational function over a box and a simplex are presented. In the polynomial case, this enclosure is the interval spanned by the minimum and the maximum of the Bernstein coefficients which are the coefficients of the polynomial with respect to the tensorial or simplicial Bernstein basis. The methods exploit monotonicity properties of the Bernstein coefficients of monomials as well as a recently developed matrix method for the computation of the Bernstein coefficients of a polynomial over a box.

Keywords: Multivariate polynomial, multivariate rational function, Bernstein coefficient, tensorial Bernstein form, simplicial Bernstein form, range enclosure

1 Introduction

Solving global optimization problems is of paramount importance in many real-life and scientific problems; polynomial global optimization problems form a significant part of them. A commonly used approach for solving global optimization problems is the branch and bound method. This is summarized as splitting the search region into smaller parts and using suitable means to discard subregions that cannot contain any global optimizer. The latter ones require the ability to compute tight bounds for the range of the objective function and constraint functions over the considered search region. In the case of polynomial optimization problems one can make use of

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the expansion of a polynomial into Bernstein polynomials, see [12], [13], [15]. Then the minimum and maximum of the coefficients of this expansion, the so-called Bernstein coefficients, provide bounds for the range of the polynomial over the search region. Other fields, where this range enclosing property has been employed, include stability analysis, e.g., [14], static analysis of computer programs [1], and the verified solution of finite element models with uncertain parameters, e.g., [5]. As a particularly promising field appears automatic theorem proving. This application includes the proof of nonlinear inequalities in the *flyspeck project* which aimed at a formal proof of Keplers Conjecture on the density of spheres [6] and the formalization of the representation of Bernstein polynomials in the higher-order logic of the mechanical theorem prover *Prototype Verification System* (PVS) [9] with application to the development of formally verifiable conflict detection algorithms for aircraft flying arbitrary, nonlinear trajectories [11].

The traditional approach, e.g., [3], [7], [12], requires that all Bernstein coefficients have to be computed. This is not recommended since their number is exponentially growing in the number of the variables. In [15], a method was proposed by which the Bernstein coefficients over a box are represented implicitly and which employs three tests to reduce the search space for the minimum and maximum coefficient. This approach is advantageous for many types of sparse polynomials typically encountered in global optimization problems because the computational complexity becomes nearly linear with respect to the number of the terms of the polynomial. We combine these tests with a recently developed method [17] for the computation of the Bernstein coefficients over a box. Also, we formulate the tests to localize the minimum and maximum coefficients of the Bernstein expansion of a polynomial over a simplex. In [10] the Bernstein enclosure for polynomials was extended to rational functions. Here the enclosure is provided by the minimum and maximum of the ratio of the Bernstein coefficients of the numerator and denominator polynomials. This allows us to extend our results derived in the polynomial case to the rational case.

The organization of our paper is as follows: In the next section, we give the notation that is used throughout the paper. In Section 3, we briefly recall the Bernstein expansion of a polynomial over a box and a simplex. In Sections 4 and 5, we present the determination of the Bernstein enclosure for polynomials over a box and a simplex, respectively. Finally in Section 6, we extend the results to rational functions.

2 Notation

In this section we introduce the notation that we are using throughout this paper. Let $n \in \mathbb{N}$ (set of the nonnegative integers) be the number of variables. A multi-index $(i_1, \ldots, i_n) \in \mathbb{N}^n$ is abbreviated by i. In particular, we write 0 for $(0, \ldots, 0)$. Comparison between multi-indices and arithmetic operations using them are understood entry-wise. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, its *monomials* are defined as $x^i := \prod_{s=1}^{n} x_s^{i_s}$. For $d = (d_1, \ldots, d_n) \in \mathbb{N}^n$ such that $i \leq d$, we use the compact notations $\sum_{i=0}^{d} := \sum_{i_1=0}^{d_1} \cdots \sum_{i_n=0}^{d_n}$ and $\binom{d}{i} := \prod_{s=1}^{n} \binom{d_s}{i_s}$. Finally, we define $|x| := x_1 + \ldots + x_n$.

Let $\mathbb{IR}$ be the set of compact, nonempty real intervals $[x] = [\underline{x}, \overline{x}]$, $\underline{x} \leq \overline{x}$. A *box* $\boldsymbol{x}$

of $\mathbb{R}^n$ is a vector with n components from $\mathbb{IR}$.

3 Bernstein Form over the Unit Box and the Standard Simplex

In this section we present fundamental properties of the Bernstein expansion over a box [2], [3] and over the standard simplex [7], [8] that are employed throughout the paper.

3.1 Tensorial Bernstein form

For simplicity we consider the unit box $\boldsymbol{u} := [0,1]^n$ since any compact nonempty box $\boldsymbol{x}$ of $\mathbb{R}^n$ can be mapped affinely onto $\boldsymbol{u}$. Let $l \in \mathbb{N}^n$ and p be an n-variate polynomial with the power representation

$$p(x) = \sum_{i=0}^{l} a_i x^i. \tag{1}$$

We expand p with respect to the basis of the Bernstein polynomials of degree d, $d \geq l$, over $\boldsymbol{u}$ as

$$p(x) = \sum_{i=0}^{d} b_i^{(d)} B_i^{(d)}(x), \tag{2}$$

where $B_i^{(d)}$ is the *i-th Bernstein polynomial of degree d over* $\boldsymbol{u}$, defined as

$$B_i^{(d)}(x) := \binom{d}{i} x^i (1-x)^{d-i}, \tag{3}$$

and $b_i^{(d)}$ is the *i-th Bernstein coefficient of p of degree d over* $\boldsymbol{u}$ which is given by

$$b_i^{(d)} = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{d}{j}} a_j, \quad 0 \leq i \leq d, \tag{4}$$

with the convention that $a_j := 0$ for $j \geq l$, $j \neq l$. We arrange the Bernstein coefficients in a multidimensional array $B(\boldsymbol{u}) = (b_i^{(d)})_{0 \leq i \leq d}$, the so-called *Bernstein patch*. The Bernstein coefficients provide lower and upper bounds for the range of p over $\boldsymbol{u}$,

$$\min_{i=0}^{d} b_i^{(d)} \leq p(x) \leq \max_{i=0}^{d} b_i^{(d)}, \quad \text{for all } x \in \boldsymbol{u}. \tag{5}$$

This property is called the *range enclosure property* and the enclosure (5) itself the *tensorial Bernstein form of* p. Equality holds in the left or right inequality of (5) if and only if the minimum or the maximum, respectively, is attained at a vertex of $B(\boldsymbol{u})$, i.e., if $i_s \in \{0, d_s\}$, $s = 1, \ldots, n$. This condition is known as the *vertex condition*. Another property of the Bernstein coefficients is their *linearity*: Let $p = \alpha p_1 + \beta p_2,\ \alpha, \beta \in \mathbb{R}$, where the degrees of p_1 and p_2 is less than or equal to d. Then

$$b_i^{(d)}(p) \ = \ \alpha b_i^{(d)}(p_1) + \beta b_i^{(d)}(p_2), \text{ for all } \ 0 \leq i \leq d, \tag{6}$$

where $b_i^{(d)}(p_1)$ and $b_i^{(d)}(p_2)$ are the i-th coefficients of the Bernstein expansions of degree d of p_1 and p_2, respectively. In the following text, we choose $d = l$ and suppress the upper index d.

3.2 Simplicial Bernstein form

Let $v_0, \ldots, v_n$ be $n+1$ points of $\mathbb{R}^n$. The ordered list $V = [v_0, \ldots, v_n]$ is called *simplex of the vertices* $v_0, \ldots, v_n$. The *realization* $|V|$ of the simplex V is the set of $\mathbb{R}^n$ defined as the convex hull of the points $v_0, \ldots, v_n$. We will consider here only the *standard simplex* $\Delta := [e_0, e_1, \ldots, e_n]$, where e_0 is the zero vector in $\mathbb{R}^n$ and e_s is the s-th vector of the canonical basis of $\mathbb{R}^n$, $s = 1, \ldots, n$. This is not a limitation since any non-degenerate simplex V in $\mathbb{R}^n$ can be mapped affinely upon Δ, see, e.g., [7], [8]. Let $k \in \mathbb{N}$. If $|i| \leq k$, we further use the notation $\binom{k}{i} := \frac{k!}{i_1! \ldots i_n!(k-|i|)!}$.

The *Bernstein polynomials of degree* k *over* Δ are the polynomials $(B_i^{(k)})_{|i| \leq k}$, defined as

$$B_i^{(k)}(x) := \binom{k}{i} x^i (1 - |x|)^{k-i}. \tag{7}$$

Let the polynomial p be given in its power representation (1). We define

$$l' := \max \{|i| \mid i = 0, \ldots, l \text{ with } a_i \neq 0\}.$$

We expand p with respect to the basis of the Bernstein polynomials of degree k, $l' \leq k$, over Δ as

$$p(x) = \sum_{|i| \leq k} b_i^{(k)} B_i^{(k)}(x). \tag{8}$$

Herein the $b_i^{(k)}$ are the *Bernstein coefficients of* p *of degree* k *over* Δ which are given by

$$b_i^{(k)} = \sum_{m \leq i} \frac{\binom{i}{m}}{\binom{k}{m}} a_m \tag{9}$$

with the convention that

$$a_m := 0 \ \text{ for } \ m_s > l_s \text{ for at least one } s \in \{1, \ldots, n\}.$$

We arrange again the Bernstein coefficients in the Bernstein patch $B(\Delta) = (b_i^{(k)})_{|i| \leq k}$.

As in the tensorial case, the Bernstein coefficients are linear, see (6), and provide the range enclosure property (with the sharpness of the bounds if the respective vertex condition is fulfilled), see [19, Proposition 2]. The interval $[\min_{|i| \leq k} b_i^{(k)}, \max_{|i| \leq k} b_i^{(k)}]$ is called the *simplicial Bernstein form of* p. If in the sequel the degree of the Bernstein expansion will be clear from the context, we suppress the upper index k.

4 Determination of the Tensorial Bernstein Form for Polynomials

In this section we present an efficient method for the determination of the Bernstein form for a multivariate polynomial over the unit box which is spanned by the minimum and maximum Bernstein coefficients. The proposed method relies on the matrix method presented in [17] which we combine with a method [15] for speeding up the determination of the Bernstein form.

4.1 Matrix method for the computation of the tensorial Bernstein coefficients

In [17] we propose a matrix method for the computation of the Bernstein coefficients over the unit and a general box. This method has complexity $O(n\kappa^{n+1})$, where κ is the maximum degree of the variables. Let p be an n-variate polynomial given by (1). The coefficients of p are arranged in an $(l_1+1)\times l^*$ matrix A, where $l^* := \prod_{s=2}^{n}(l_s+1)$. The correspondence between the coefficients $a_{i_1,\ldots,i_n}$ of p and the entry of A in row i and column j is as follows:

$$\begin{aligned} i &= i_1+1, \\ j &= i_2+1+\sum_{s=3}^{n} i_s(l_2+1)\cdot\ldots\cdot(l_{s-1}+1). \end{aligned}$$

We introduce the following matrices of $\mathbb{R}^{l_s+1,l_s+1}$, $s=1,\ldots,n$. The *lower triangular Pascal matrix* P_s is defined as

$$(P_s)_{ij} := \begin{cases} \binom{i-1}{j-1}, & \text{if } j \le i, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

The matrices K^s_μ, $\mu = 1,\ldots,l_s$, are given by

$$(K^s_\mu)_{ij} := \begin{cases} 1, & \text{if } i=j, \\ 1, & \text{if } i=j+1,\ \ l_s-\mu+1\le j\le l_s, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

We will make use of the following factorization, e.g., [16, Lemma 2.4],

$$P_s = \prod_{\mu=1}^{l_s} K^s_\mu. \tag{12}$$

For the computation of the Bernstein patch of p over $\boldsymbol{u}$ we first multiply the entries $a_{i_1,\ldots,i_n}$ of A by $\binom{l_1}{i_1}^{-1}\ldots\binom{l_n}{i_n}^{-1}$. We name the resulting matrix $\Lambda(\boldsymbol{u})$, put $\Lambda_0 := \Lambda(\boldsymbol{u})$, and define for $s=1,\ldots,n$

$$\Lambda_s := (P_s\Lambda_{s-1})^c. \tag{13}$$

The superscript c denotes the *cyclic ordering* of the sequence of the indices, i.e., the order of the indices of the entries of the array under consideration is changed cyclically. This means that the index in the first position is replaced by the index in the second one, the index in the second position by the one in the third, ... , the index in the n-th position by the one in the first position, so that after n such steps the sequence of the indices is again in its initial order, see Figure 1 in [17] for an illustration in the trivariate case. Note that in the bivariate case the cyclic ordering is just the usual matrix transposition. The Bernstein patch $B(\boldsymbol{u})$ arranged accordingly in an $(l_1+1)\times l^*$ matrix is given by Λ_n. For simplicity we assume that $l_s=\kappa, s=1,\ldots,n$. Therefore, we suppress the subscript of P_s and the superscript of K^s_μ, $s=1,\ldots,n$. In [17], a method, named Method 1, is proposed for the computation of $B(\boldsymbol{u})$ according to (13) which relies on the factorization (12). In this method we first multiply K_κ and $\Lambda(\boldsymbol{u})$ and multiply the resulting matrix by $K_{\kappa-1}$ and so on. The main advantage of using factorization (12) of the Pascal matrix is that it allows us to get rid of the multiplication operations which are required when we multiply by the Pascal matrix. This method requires $n\frac{\kappa(\kappa+1)^n}{2}$ additions and $n(\kappa+1)^n$ multiplications.

In passing we note that an alternative factorization of the Pascal matrix into a Toeplitz matrix and two diagonal matrices allows the use of the Fast Fourier Transform hereby reducing the amount of the arithmetic operations to $O(n\kappa^n\log_2\kappa)$; for details see Method 2 in [17].

4.2 Determination of the Bernstein form over the unit box

In this subsection we apply the method that is briefly presented in the previous subsection for the determination of the Bernstein form over $\boldsymbol{u}$.

In [15], a method called *implicit Bernstein form* for the representation and computation of the Bernstein coefficients of a multivariate polynomial is introduced. In this method one only needs to compute the univariate Bernstein coefficients of each univariate component monomial of each term. Therefore, the computation of all the Bernstein coefficients is not required. The calculation of a single Bernstein coefficient requires then $(n+1)t-1$ arithmetic operations, where t is the number of terms in the n-variate polynomial [15]. The minimum and maximum Bernstein coefficients are referenced by multiindices which we label $i_{\min}$ and $i_{\max}$. We want to determine the value of $i_{\min}$ in each coordinate direction (for $i_{\max}$ we proceed similarly). This task is facilitated by three tests introduced in [15], see Subsection 5.2. We explain the application of the method from Subsection 4.1 by an expository example. Let us assume that we have already determined the first two components of the multiindex sought, $i_{\min}=(0,\kappa,i_3,\ldots,i_n)$ say.

The i-th Bernstein coefficient given by (4) can be represented as

$$b_{i_1,\ldots,i_n}=\sum_{j_n=0}^{i_n}\frac{\binom{i_n}{j_n}}{\binom{\kappa}{j_n}}\cdots\sum_{j_2=0}^{i_2}\frac{\binom{i_2}{j_2}}{\binom{\kappa}{j_2}}\sum_{j_1=0}^{i_1}\frac{\binom{i_1}{j_1}}{\binom{\kappa}{j_1}}a_{j_1,j_2,\ldots,j_n}. \tag{14}$$

We start with our knowledge of the first two components. Since $i_{\min_1}=0$ we need to compute all the Bernstein coefficients $b_{0,i_2,\ldots,i_n}, 0\le i_s\le\kappa, s=2,\ldots,n$. Therefore, the first inner-most sum in (14) does not contribute to the calculation of $b_{i_{\min}}$ and we fix $i_1=0$, which corresponds to the first row vector in A of length $(\kappa+1)^{n-1}$. So we only multiply the first row in P, which is $[1\ 0\ \ldots\ 0]$, by A which requires no arithmetic operations. After that we apply the cyclic ordering and we get an $(\kappa+1)\times(\kappa+1)^{n-2}$

matrix named A'.

Now by $i_{\min_2} = \kappa$, the computation of all the Bernstein coefficients $b_{0,\kappa,i_3,\ldots,i_n}$ is required. Therefore, in the second inner-most sum we fix $i_2 = \kappa$, which corresponds to the last row vector in A' of length $(\kappa+1)^{n-2}$. So, we need to multiply the matrix A' by the row vector which is obtained by division of each entry in last row of P by the corresponding binomial coefficient $\binom{\kappa}{j_2}$, $j_2 = 0, \ldots, \kappa$, i.e., by $[1\ 1\ \ldots\ 1]$. Then we need κ additions for each column of A' which makes $\kappa(\kappa+1)^{n-2}$ additions. After the application of the cyclic ordering we obtain a $(\kappa+1)\times(\kappa+1)^{n-3}$ matrix, that we will denote by A''.

The remaining indices are not determined. We need to compute all the Bernstein coefficients $b_{0,\kappa,i_3,\ldots,i_n}$. We define $\Lambda''(\boldsymbol{u})$ from A'' by dividing each of its entries by $\binom{i_3}{j_3}\ldots\binom{i_n}{j_n}$. Then we apply the method from the Subsection 4.1 and use the factorization (12) of the Pascal matrix $n-2$ times. The number of arithmetic operations that are required to calculate these Bernstein coefficients is $(n-2)\frac{\kappa(\kappa+1)^{n-2}}{2}$ additions and $(n-2)(\kappa+1)^{n-2}$ multiplications /divisions.

In the general situation, we first rearrange the matrix A in such a way that zero indices appear first followed by the ones which are equal to κ, and that the remaining ones are undetermined. Let σ and ζ be the number of the variables x_s with $i_{\min_s} = 0$ and $i_{\min_s} = \kappa$, respectively. Then the complexity for the proposed method is for $\kappa \geq 2$

$$\begin{cases} O((n-\sigma)\kappa^{n-\sigma+1}), & \text{if } \zeta = 0, \\ O(\frac{n-\sigma+1}{2}\kappa^{n-\sigma}), & \text{if } \zeta = 1, \\ O(\max\left\{\kappa^{n-\sigma}, \frac{n-\sigma-\zeta}{2}\kappa^{n-\sigma-\zeta+1}\right\}), & \text{if } \zeta > 1. \end{cases}$$

The case $\zeta = 0$ follows immediately from the case of n undetermined variables by replacing n by $n-\sigma$. If $\zeta \geq 1$ we need $\kappa(\kappa+1)^{n-\sigma-1}$ additions for the first variable x_s with $i_{\min_s} = \kappa$ and $O(\kappa^{n-\sigma-\zeta+1})$ for the ζ-th of such variables (if there is any). The amount of operations for the $n-\sigma-\zeta$ undetermined variables is $\frac{n-\sigma-\zeta}{2}\kappa(\kappa+1)^{n-\sigma-\zeta}$ additions and $(n-\sigma-\zeta)(\kappa+1)^{n-\sigma-\zeta}$ multiplications.

5 Determination of the Simplicial Bernstein Form for Polynomials

In this section, we consider the determination of the Bernstein form of a multivariate polynomial p over the standard simplex.

5.1 Monotonicity of the simplicial Bernstein coefficients of multivariate monomials

Recall that the Bernstein coefficients are linear with respect to the polynomial to which they are associated. Therefore, we may consider the case of a polynomial consisting of a single term and may assume without loss of generality that the coefficient of the monomial is 1.

Proposition 5.1 *Let $p(x) = x^r$, $x = (x_1, \ldots, x_n)$, and $|r| = r' \leq k$. Then its Bernstein coefficients b_i of degree k are increasing with respect to i.*

Proof: From (9) it is easy to see that the first nonzero Bernstein coefficient appears for $m = r$.

We have for $1 \leq s \leq n$ and $0 \leq t \leq k - |r|$

$$b_{r+te_s} = \frac{\binom{r_s+t}{r_s}}{\binom{k}{r}} \tag{15}$$

and therefore

$$b_{r+te_s} \leq b_{r+(t+1)e_s}, \tag{16}$$

provided that $t + 1 \leq k - |r|$. □

Remark 5.1 *From Proposition 5.1 we conclude that the maximum Bernstein coefficient is attained at one of the following Bernstein coefficients*

$$b_{r+y} = \frac{\binom{r+y}{r}}{\binom{k}{r}}, \tag{17}$$

where $|r + y| = k$ with $y \in \mathbb{N}^n$.

5.2 Determination of the Bernstein form over the standard simplex

We consider now the determination of the minimum and the maximum Bernstein coefficients of a polynomial p given by (1) with $k = l'$ by using Proposition 5.1 and Remark 5.1. Similarly as in [15] in the tensorial case we employ the following tests:

- **Uniqueness**: If a component of x, e.g., x_1, appears in only one monomial term of p, then p can be divided into two polynomials g and h, i.e., $p = g + h$, where g is the monomial term containing x_1 and h contains all the other monomial terms. By Remark 5.1, the Bernstein coefficients of g are monotone with respect to i_1. The Bernstein patch of h can be calculated from (9) by only computing the subpatch $b_{0,i_2,\ldots,i_n}(h)$; then the remaining Bernstein coefficients $b_{i_1,i_2,\ldots,i_n}(h)$ are equal to the corresponding coefficients $b_{0,i_2,\ldots,i_n}(h)$.

  After having added the two Bernstein patches of g and h, then by Remark 5.1 the maximum Bernstein coefficient of p is attained at $i_{\max}$ with $|i_{\max}| = k$ if g is increasing, i.e., the coefficient of g is positive. Define the function f of the variable $y \in \mathbb{N}^n$ with $|y| = k$

  $$f(y) := \sum_{\nu \leq y} \frac{\binom{y}{\nu}}{\binom{k}{\nu}} a_\nu. \tag{18}$$

  If y^* is a maximizer of f, then $i_{\max} = y^*$. Whereas the minimum Bernstein coefficient is attained at $i_{\min}$ with $i_{\min_1} = 0$. If g is decreasing, i.e., the coefficient of g is negative, then $i_{\min} = y^*$ and so $|i_{\min}| = k$ and $i_{\max_1} = 0$.

- **Monotonicity**: In extension of the uniqueness test let x_s be contained in t' monomial terms of p and the Bernstein coefficients of all these terms are likewise monotone with respect to x_s. Then $i_{\min_s} = 0$ and $i_{\max} = y^*$, i.e., $|i_{\max}| = k$, if all these terms are increasing and $i_{\min} = y^*$, i.e., $|i_{\min}| = k$, and $i_{\max_s} = 0$ if all these terms are decreasing, where y^* is a maximizer of f in (18).

- **Dominance**: Otherwise, let all the terms containing x_s be partitioned into two sets, depending on whether they are increasing or decreasing with respect to x_s. Then the following theorem constitutes the dominance test, cf. [15].

Theorem 5.1 (Location of the minimum Bernstein coefficient under dominance) *Let the polynomial p be given by* (1)*. Let p_1 and p_2 be the polynomials that contain all the terms of p such that the Bernstein coefficients of these terms are increasing and decreasing with respect to x_s, $s \in \{1, \ldots n\}$, respectively. Denote the Bernstein coefficients of p_1 and p_2 by $b_i(p_1)$ and $b_i(p_2)$, respectively.*

If for all $i, |i| < k,$
$$b_{i_1,\ldots,i_s+1,\ldots,i_n}(p_1) - b_{i_1,\ldots,i_s,\ldots,i_n}(p_1) > b_{i_1,\ldots,0,\ldots,i_n}(p_2) - b_{i_1,\ldots,k-\sum\limits_{\substack{r=1,\\ r\neq s}}^{n} i_r,\ldots,i_n}(p_2) \tag{19}$$

then $i_{\min s} = 0$.

If for all $i, |i| < k,$
$$b_{i_1,\ldots,i_s,\ldots,i_n}(p_2) - b_{i_1,\ldots,i_s+1,\ldots,i_n}(p_2) > b_{i_1,\ldots,k-\sum\limits_{\substack{r=1,\\ r\neq s}}^{n} i_r,\ldots,i_n}(p_1) - b_{i_1,\ldots,0,\ldots,i_n}(p_1) \tag{20}$$

then $i_{\min s} = k - \sum\limits_{\substack{r=1,\\ r\neq s}}^{n} i_r$.

Proof: We present the proof only for the first statement (19); the proof of the second one (20) is entirely analogous. For all i, $|i| < k$, we obtain by the linearity of the Bernstein coefficients, the monotonicity of the $b_i(p_2)$ with respect to x_s, and (19)

$$\begin{aligned}
b_{i_1,\ldots,i_s+1,\ldots,i_n}(p) &= b_{i_1,\ldots,i_s+1,\ldots,i_n}(p_1) + b_{i_1,\ldots,i_s+1,\ldots,i_n}(p_2)\\
&\geq b_{i_1,\ldots,i_s+1,\ldots,i_n}(p_1) + b_{i_1,\ldots,k-\sum\limits_{\substack{r=1,\\ r\neq s}}^{n} i_r,\ldots,i_n}(p_2)\\
&> b_{i_1,\ldots,i_s,\ldots,i_n}(p_1) + b_{i_1,\ldots,0,\ldots,i_n}(p_2)\\
&\geq b_{i_1,\ldots,i_s,\ldots,i_n}(p_1) + b_{i_1,\ldots,i_s,\ldots,i_n}(p_2)\\
&= b_{i_1,\ldots,i_s,\ldots,i_n}(p).
\end{aligned}$$

Thus the Bernstein coefficients of p are increasing with respect to x_s, and the claim follows. □

In other words, the statement of Theorem 5.1 means the following: If the width of the Bernstein form of one set (treated as the polynomial comprising its terms) is smaller than the minimum difference between the Bernstein coefficients of the terms of the other set, then the first set can make no contribution to the determination of

$i_{\min_s}$ and the monotonicity test applies.

We demonstrate the application of the three tests for the determination of the minimum Bernstein coefficient, denoted by $b_{i_{\min}}$, by the following example. The determination of the maximum Bernstein coefficient is similar. We make use of the fact [18] that the Bernstein coefficients of p over an ν-dimensional face of Δ, where $1 \leq \nu \leq n-1$, are the same as the Bernstein coefficients that are located at the corresponding ν-dimensional face of the Bernstein patch of p over Δ.

Example 5.1 *Consider the polynomial*

$$\begin{aligned} p(x_1,x_2,x_3,x_4): &= x_1x_2^2x_3^2 - x_1^2x_2^2x_4 + 104x_1^2x_2 - x_1x_2^2 + x_2^2x_3 + 105x_1 \\ &+ 105x_2. \end{aligned}$$

The degree of p is $l = (2,2,2,1)$. We choose $k = l' = 5$. The number of the Bernstein coefficients of p is $\binom{k+n}{n} = \binom{9}{4} = 126$. An estimate for an upper bound on the number of the arithmetic operations required for the computation of the Bernstein coefficients is of magnitude 104, see [18]. We observe the following:

The component x_3 appears in the first and fifth term. Since the coefficients of these terms are positive we conclude that x_3 satisfies the monotonicity test and that $i_{\min_3} = 0$. The Bernstein coefficients with $i_3 = 0$ are just the Bernstein coefficients of p over the face of Δ given by $x_3 = 0$, i.e., these coefficients are located at a three-dimensional face of Δ. Hence these Bernstein coefficients are identical with the ones of the polynomial

$$f(x_1,x_2,x_4) := -x_1^2x_2^2x_4 + 104x_1^2x_2 - x_1x_2^2 + 105x_1 + 105x_2,$$

which is obtained by substituting $x_3 = 0$ in p. In f, the component x_4 appears only in the first term; therefore, x_4 satisfies the uniqueness test. Since the coefficient of this term is negative we conclude that $i_{\min_4} = k - i_1 - i_2$. The Bernstein coefficients of f with $i_4 = k - i_1 - i_2$ are just the Bernstein coefficients of f when $x_4 = 1 - x_1 - x_2$, which are the Bernstein coefficients of p over the two-dimensional face of Δ given by $x_3 = 0, x_4 = 1 - x_1 - x_2$. These coefficients are the Bernstein coefficients of

$$g(x_1,x_2) := -x_1^2x_2^2 + x_1^3x_2^2 + x_1^2x_2^3 + 104x_1^2x_2 - x_1x_2^2 + 105x_1 + 105x_2,$$

which is obtained by substituting $x_4 = 1-x_1-x_2$ in f. We note that g is divided into the two polynomials p_1 and p_2, where $p_1(x_1,x_2) := x_1^3x_2^2 + x_1^2x_2^3 + 104x_1^2x_2 + 105x_1 + 105x_2$ and $p_2(x_1,x_2) := -x_1^2x_2^2 - x_1x_2^2$. Their Bernstein patches are as follows (rounded to five decimal places)

$$B(\Delta,p_1) = \begin{bmatrix} 0 & 21 & 42 & 63 & 84 & 105 \\ 21 & 42 & 63 & 84 & 105 & \\ 42 & 66.46667 & 90.93333 & 115.5 & & \\ 63 & 94.4 & 125.9 & & & \\ 84 & 125 & & & & \\ 105 & & & & & \end{bmatrix},$$

$$B(\Delta,p_2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.03333 & -0.10000 & -0.20000 & \\ 0 & 0 & -0.10000 & -0.30000 & & \\ 0 & 0 & -0.20000 & & & \\ 0 & 0 & & & & \\ 0 & & & & & \end{bmatrix}.$$

We note that all the Bernstein coefficients of p_1 and p_2 are nonnegative and nonpositve, respectively. Inequality (19) *is fulfilled for x_1 and x_2. Therefore, we may conclude that $i_{\min} = (0, 0, 0, 5)$.*

Alternatively, after we have (only) verified that (19) *is fulfilled for x_1 we may conclude that the Bernstein coefficients of g with $i_1 = 0$ are just the Bernstein coefficients of p over the edge of Δ provided by $x_1 = x_3 = 0, x_4 = 1 - x_1 - x_2$. These coefficients are the Bernstein coefficients of*

$$h(x_2) := 105x_2$$

over $[0, 1]$. *So in order to determine $b_{\min}$, it suffices to compute only the two Bernstein coefficients of h and then to take the smallest one. Since $b_0(h) = h(0) = 0$ and $b_1(h) = h(1) = 105$ we find again $i_{min} = (0, 0, 0, 5)$.*

Since the Bernstein coefficients at the vertices of $B(\Delta)$ are the values of p at the respective vertices of Δ, see, e.g., [7, Proposition 3.2 (ii)] we obtain

$$\min_{x \in \Delta} p(x) = p(0, 0, 0, 1) = 0.$$

6 Determination of the Bernstein Form for Rational Functions

6.1 Tensorial Bernstein form

We present our results on the determination of a range enclosure of a rational function over a box. In this section we assume that p and q are polynomials in n variables with Bernstein coefficients $b_i(p)$ and $b_i(q)$, $0 \leq i \leq l$, respectively, over a box which is contained in a single orthant of $\mathbb{R}^n$. We also assume that all Bernstein coefficients $b_i(q)$ have the same strict sign (and without loss of generality we may assume that all of them are positive). We use the notation

$$b_i(f) := \frac{b_i(p)}{b_i(q)}, \quad 0 \leq i \leq l, \tag{21}$$

and call these quantities the *Bernstein coefficients of* the rational function $f := \frac{p}{q}$ (*of degree l*) *over $\boldsymbol{x}$*. The interval spanned by the minimum and the maximum of the Bernstein coefficients of f provides an enclosure for the range of f over $\boldsymbol{x}$ [10]. For properties of this form see [4].

Now we extend the three tests given in Section 5 to the rational case. We consider here only the determination of the minimum Bernstein coefficient; the determination of the maximum Bernstein coefficient is analogous. Also we do not consider the uniqueness test since this test is included in the monotonicity test.

- **Monotonicity**: Assume that the Bernstein coefficients of all monomial terms containing x_s in p are likewise monotone with respect to x_s and those in q are monotone in opposite sense. Then,
  - if the Bernstein coefficients of p are increasing and those of q are decreasing with respect to x_s, then $i_{\min_s}(f) = 0$,

 – if the Bernstein coefficients of p are decreasing and those of q are increasing with respect to x_s, then $i_{\min_s}(f) = l_s$.

- **Dominance**: Assume that all the terms containing x_s in both p and q can be partitioned into two sets, depending on whether they are increasing or decreasing with respect to x_s such that the Bernstein coefficients of two polynomials are likewise monotone.

Theorem 6.1 (Location of the minimum tensorial Bernstein coefficient under dominance for rational functions) *Let polynomials p and q of maximum degree l be given and let p_1 and p_2 be the polynomials that comprise all the terms of p such that the Bernstein coefficients of these terms are increasing and decreasing with respect to x_s, respectively. Then for $0 \le i \le l$, $i_s \ne l_s$, the following statements are true:*
If for p

$$\begin{aligned} b_{i_1,\dots,i_s+1,\dots,i_n}(p_1) \quad - \quad & b_{i_1,\dots,i_s,\dots,i_n}(p_1) \\ & > b_{i_1,\dots,0,\dots,i_n}(p_2) - b_{i_1,\dots,l_s,\dots,i_n}(p_2) \end{aligned} \tag{22}$$

is satisfied and for q the inequality

$$b_{i_1,\dots,i_s,\dots,i_n}(q) > b_{i_1,\dots,i_s+1,\dots,i_n}(q), \tag{23}$$

is fulfilled then $i_{\min s}(f) = 0$.
If for p

$$\begin{aligned} b_{i_1,\dots,i_s,\dots,i_n}(p_2) \quad - \quad & b_{i_1,\dots,i_s+1,\dots,i_n}(p_2) \\ & > b_{i_1,\dots,l_s,\dots,i_n}(p_1) - b_{i_1,\dots,0,\dots,i_n}(p_1) \end{aligned} \tag{24}$$

is satisfied and for q the inequality

$$b_{i_1,\dots,i_s+1,\dots,i_n}(q) > b_{i_1,\dots,i_s,\dots,i_n}(q), \tag{25}$$

is fulfilled then $i_{\min s}(f) = l_s$.

Proof: We will present the proof only for the first statement; the proof of the second one is entirely analogous. For all $i = 0, \dots, l, i_s \ne l_s$, it follows similarly as in the proof of Theorem 5.1 that

$$b_{i_1,\dots,i_s+1,\dots,i_n}(p) > b_{i_1,\dots,i_s,\dots,i_n}(p), \tag{26}$$

and by (23) we may conclude that

$$b_{i_1,\dots,i_s+1,\dots,i_n}(f) > b_{i_1,\dots,i_s,\dots,i_n}(f). \tag{27}$$

□

6.2 Simplicial Bernstein form

The tests for the tensorial rational case carry over to the simplicial case with only minor modifications. We assume that p and q are polynomials in n variables with Bernstein coefficients $b_i(p)$ and $b_i(q)$, $|i| \leq k$, respectively, over Δ. We assume again that all Bernstein coefficients $b_i(q)$ are positive. For the rational function $f := \frac{p}{q}$ we use the notation

$$b_i(f) := \frac{b_i(p)}{b_i(q)}, \text{ for all } i, |i| \leq k. \tag{28}$$

The interval spanned by the minimum and the maximum of these quantities provides an enclosure for the range of f over Δ [10]. For properties of this form see [19].

We employ the same tests as in Subsection 6.1. Then Theorem 6.1 remains in force with the necessary changes to be made if we replace therein the reference to (22) and (24) by the reference to (19) and (20), respectively.

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