Error Estimates with Explicit Constants for Sinc Quadrature and Sinc Indefinite Integration over Infinite Intervals*

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Abstract

Sinc quadrature and Sinc indefinite integration are approximation formulas for definite integration and indefinite integration, respectively, which can be applied on any interval by using an appropriate variable transformation. Their convergence rates have been analyzed for typical cases including finite, semi-infinite, and infinite intervals. In addition, for verified automatic integration, more explicit error bounds that are computable have been recently given on a finite interval. In this paper, such explicit error bounds are given in the remaining cases on semi-infinite and infinite intervals.

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1 Introduction

"Sinc quadrature" is an approximation formula for the integral over the whole real line, expressed as

$$\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x \approx h \sum_{k=-M}^{N} F(kh), \tag{1.1}$$

where M, N, h are selected appropriately depending on n. This approximation is also called the (truncated) "trapezoidal formula." It is well known that the formula (1.1) can achieve quite a fast, *exponential* convergence. Furthermore, its optimality is proved for a certain class of functions [4, 6]. Here, there are two important conditions to be satisfied: (i) the interval of integration is $(-\infty, \infty)$, and (ii) $|F(x)|$ decays exponentially as $x \to \pm\infty$. In other cases, an appropriate variable transformation $t = \psi(x)$

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should be employed, i.e., the given integral should be transformed as

$$\int_a^b f(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(\psi(x))\psi'(x)\,\mathrm{d}x,$$

so that the above two conditions are met.

In this regard, Stenger [5] considered the following four typical cases:

1. $[a, b] = [-\infty, \infty]$, and $|f(x)|$ decays algebraically as $x \to \pm\infty$,
2. $[a, b] = [0, \infty]$, and $|f(x)|$ decays algebraically as $x \to \infty$,
3. $[a, b] = [0, \infty]$, and $|f(x)|$ decays (already) exponentially as $x \to \infty$,
4. The interval $[a, b]$ is finite.

He then gave the concrete transformations to be employed in all cases:

$$\begin{aligned}
\psi_{\text{SE1}}(t) &= \sinh t,\\
\psi_{\text{SE2}}(t) &= \mathrm{e}^{t},\\
\psi_{\text{SE3}}(t) &= \operatorname{arcsinh}(\mathrm{e}^{t}),\\
\psi_{\text{SE4}}(t) &= \frac{b-a}{2}\tanh\left(\frac{t}{2}\right) + \frac{b+a}{2},
\end{aligned}$$

which are called the "Single-Exponential (SE) transformations." Takahasi–Mori [7] proposed the following improved transformations:

$$\begin{aligned}
\psi_{\text{DE1}}(t) &= \sinh\big((\pi/2)\sinh t\big),\\
\psi_{\text{DE2}}(t) &= \mathrm{e}^{(\pi/2)\sinh t},\\
\psi_{\text{DE3}\dagger}(t) &= \mathrm{e}^{t-\exp(-t)},\\
\psi_{\text{DE4}}(t) &= \frac{b-a}{2}\tanh\left(\frac{\pi}{2}\sinh t\right) + \frac{b+a}{2},
\end{aligned}$$

which are called the "Double-Exponential (DE) transformations." In addition, in case 3, another DE transformation

$$\psi_{\text{DE3}}(t) = \log(1 + \mathrm{e}^{(\pi/2)\sinh t})$$

was proposed [2] so that its inverse function can be explicitly written with elementary functions (whereas $\psi_{\text{DE3}\dagger}(t)$ cannot).

Error analyses for Sinc quadrature combined with $\psi_{\text{SE1}}(t), \dots, \psi_{\text{SE4}}(t)$ are given [5] in the following form:

$$|\text{Error}| \leq C\,\mathrm{e}^{-\sqrt{2\pi d\mu n}},$$

and for $\psi_{\text{DE1}}(t)$, $\psi_{\text{DE2}}(t)$, $\psi_{\text{DE3}}(t)$, $\psi_{\text{DE4}}(t)$, their error analyses have been given [8] as

$$|\text{Error}| \leq C\,\mathrm{e}^{-2\pi dn/\log(8dn/\mu)}, \tag{1.2}$$

and for $\psi_{\text{DE3}\dagger}(t)$, also given [8] as

$$|\text{Error}| \leq C\,\mathrm{e}^{-2\pi dn/\log(2\pi dn/\mu)}, \tag{1.3}$$

where μ indicates the decay rate of the integrand, d denotes the width of the domain in which the transformed integrand is analytic, and C is a constant independent of n. In view of the inequalities above, the accuracy of the approximation can be guaranteed if the constant C is explicitly given in a computable form. In fact, the explicit form of C was revealed in case 4 (the interval is finite) [3], and the result was used for verified automatic integration [9].

The main objective of this study is to reveal the explicit form of C's in the remaining cases: 1–3 (the interval is not finite), which enables us to bound the errors by computable terms.

As a second objective, this paper improves the DE transformation in case 3. Instead of $\psi_{\mathrm{DE3}}(t)$ or $\psi_{\mathrm{DE3}\dagger}(t)$,

$$\psi_{\mathrm{DE3}\ddagger}(t) = \log(1 + \mathrm{e}^{\pi \sinh t})$$

is introduced in this paper, and it is shown that the error is estimated as

$$|\mathrm{Error}| \leq C\,\mathrm{e}^{-2\pi d n/\log(4dn/\mu)}, \tag{1.4}$$

while clarifying the constant C. The rate of (1.4) is better than (1.2) and (1.3). Furthermore, by using Sugihara's nonexistence theorem [6], it can be shown that $\psi_{\mathrm{DE3}\ddagger}$ is the best among the possible variable transformations in case 3 (although the point is not discussed in this paper and left for another occasion).

In addition to the "Sinc quadrature" described above, similar results can be given for the "Sinc indefinite integration" for indefinite integrals $\int_a^{\xi} f(t)\,\mathrm{d}t$, which is also examined in this paper.

The remainder of this paper is organized as follows. The main results of this paper are stated in Sections 2 and 3; new error estimates for Sinc quadrature are presented in Section 2, and for Sinc indefinite integration in Section 3. Numerical examples are shown in Section 4. All proofs of the presented theorems are given in Section 5.

2 Error Estimates with Explicit Constants for Sinc Quadrature

In this section, after reviewing existing results, new error estimates for Sinc quadrature are stated. First, we introduce necessary notation.

Let $\mathscr{D}_d$ be a strip domain defined by $\mathscr{D}_d = \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < d\}$ for $d > 0$. Furthermore, let $\mathscr{D}_d^{-} = \{\zeta \in \mathscr{D}_d : \operatorname{Re}\zeta < 0\}$ and $\mathscr{D}_d^{+} = \{\zeta \in \mathscr{D}_d : \operatorname{Re}\zeta \geq 0\}$. In all the theorems presented in Sections 2 and 3, d is assumed to be a positive constant with $d < \pi/2$. For a variable transformation ψ, $\psi(\mathscr{D}_d)$ denotes the image of $\mathscr{D}_d$ by the map ψ, i.e., $\psi(\mathscr{D}_d) = \{z = \psi(\zeta) : \zeta \in \mathscr{D}_d\}$. Let $I_1 = (-\infty, \infty)$, $I_2 = I_3 = (0, \infty)$, and let us define the following three functions:

$$E_1(z;\gamma) = \frac{1}{(1+z^2)^{(\gamma+1)/2}},$$

$$E_2(z;\alpha,\beta) = \frac{z^{\alpha-1}}{(1+z^2)^{(\alpha+\beta)/2}},$$

$$E_3(z;\alpha,\beta) = \left(\frac{z}{1+z}\right)^{\alpha-1} \mathrm{e}^{-\beta z}\,.$$

We write $E_i(z;\gamma,\gamma)$ as $E_i(z;\gamma)$ for short.

2.1 Existing and New Error Estimates for Sinc Quadrature with the SE Transformation

Existing error analyses for Sinc quadrature with ψ_{SE1}, ψ_{SE2}, and ψ_{SE3} are written in the following form (Theorems 2.1 and 2.2).

Theorem 2.1 (Stenger [5, Theorem 4.2.6]) *Assume that the function f is analytic in $\psi_{\mathrm{SE1}}(\mathscr{D}_d)$, and there exist positive constants K, α, and β such that*

$$|f(z)| \leq K|E_1(z;\alpha)| \tag{2.1}$$

for all $z \in \psi_{\mathrm{SE1}}(\mathscr{D}_d^-)$, and

$$|f(z)| \leq K|E_1(z;\beta)| \tag{2.2}$$

for all $z \in \psi_{\mathrm{SE1}}(\mathscr{D}_d^+)$. Let $\mu = \min\{\alpha,\, \beta\}$, let h be defined as

$$h = \sqrt{\frac{2\pi d}{\mu n}}, \tag{2.3}$$

and let M and N be defined as

$$\begin{cases} M = n, \quad N = \lceil \alpha n/\beta \rceil & (\text{if } \mu = \alpha), \\ N = n, \quad M = \lceil \beta n/\alpha \rceil & (\text{if } \mu = \beta). \end{cases} \tag{2.4}$$

Then there exists a constant C_1, independent of n, such that

$$\left| \int_{I_1} f(t)\,\mathrm{d}t - h \sum_{k=-M}^{N} f(\psi_{\mathrm{SE1}}(kh))\psi'_{\mathrm{SE1}}(kh) \right| \leq C_1\, \mathrm{e}^{-\sqrt{2\pi d\mu n}}\,. \tag{2.5}$$

Theorem 2.2 (Stenger [5, Theorem 4.2.6]) *The following is true for $i = 2,\, 3$. Assume that f is analytic in $\psi_{\mathrm{SE}i}(\mathscr{D}_d)$, and there exist positive constants K, α, and β such that*

$$|f(z)| \leq K|E_i(z;\alpha,\beta)| \tag{2.6}$$

for all $z \in \psi_{\mathrm{SE}i}(\mathscr{D}_d)$. Let $\mu = \min\{\alpha,\, \beta\}$, let h be defined as (2.3), *and let M and N be defined as* (2.4). *Then there exists a constant C_i, independent of n, such that*

$$\left| \int_{I_i} f(t)\,\mathrm{d}t - h \sum_{k=-M}^{N} f(\psi_{\mathrm{SE}i}(kh))\psi'_{\mathrm{SE}i}(kh) \right| \leq C_i\, \mathrm{e}^{-\sqrt{2\pi d\mu n}}\,. \tag{2.7}$$

This paper explicitly estimates the constant C_i's in (2.5) and (2.7) as follows.

Theorem 2.3 *Let the assumptions in Theorem 2.1 be fulfilled. Furthermore, let $\nu = \max\{\alpha,\, \beta\}$. Then the inequality* (2.5) *holds with*

$$C_1 = \frac{2^{\nu+1}K}{\mu}\left\{ \frac{2}{(1-\mathrm{e}^{-\sqrt{2\pi d\mu}})\{\cos d\}^{\nu}} + 1 \right\}.$$

Theorem 2.4 *Let the assumptions in Theorem 2.2 be fulfilled. Then the inequality* (2.7) *holds with*

$$C_2 = \frac{2K}{\mu}\left\{\frac{2}{(1-\mathrm{e}^{-\sqrt{2\pi d\mu}})\{\cos d\}^{(\alpha+\beta)/2}} + 1\right\},$$

$$C_3 = \frac{2K}{\mu}\left\{\frac{2^{1+(\beta/2)}c_{\alpha,d}}{(1-\mathrm{e}^{-\sqrt{2\pi d\mu}})\{\cos d\}^{(\alpha+\beta)/2}} + 2^{(1-\alpha+|1-\alpha|)/2}\right\},$$

where $c_{\alpha,d}$ *is a constant defined by*

$$c_{\alpha,d} = \begin{cases} \left\{2\left(1+\frac{1}{\cos d}\right)\right\}^{(1-\alpha)/2} & (\textit{if } 0<\alpha<1), \\ 2^{(\alpha-1)/2} & (\textit{if } 1\leq\alpha). \end{cases} \tag{2.8}$$

2.2 Existing and New Error Estimates for Sinc Quadrature with the DE Transformation

Existing error analyses for Sinc quadrature with ψ_{DE1}, ψ_{DE2}, ψ_{DE3}, and $\psi_{\mathrm{DE3\dagger}}$ are written in the following form (Theorems 2.5 and 2.6).

Theorem 2.5 (Tanaka et al. [8, Theorem 3.1]) *The following is true for* $i=1$, 2, 3. *Assume that* f *is analytic in* $\psi_{\mathrm{DE}i}(\mathscr{D}_d)$*, and there exist positive constants* K *and* μ *(with* $\mu\leq 1$ *in case* $i=3$*) such that* $|f(z)|\leq K|E_i(z;\mu)|$ *for all* $z\in\psi_{\mathrm{DE}i}(\mathscr{D}_d)$*. Then there exists a constant* C*, independent of* n*, such that*

$$\left|\int_{I_i} f(t)\,\mathrm{d}t - h\sum_{k=-n}^{n} f(\psi_{\mathrm{DE1}}(kh))\psi'_{\mathrm{DE1}}(kh)\right| \leq C\,\mathrm{e}^{-2\pi dn/\log(8dn/\mu)},$$

where

$$h = \frac{\log(8dn/\mu)}{n}. \tag{2.9}$$

Theorem 2.6 (Tanaka et al. [8, Theorem 3.1]) *Assume that the function* f *is analytic in* $\psi_{\mathrm{DE3\dagger}}(\mathscr{D}_d)$*, and there exist positive constants* K *and* μ *with* $\mu\leq 1$ *such that* $|f(z)|\leq K|E_3(z;\mu)|$ *for all* $z\in\psi_{\mathrm{DE3\dagger}}(\mathscr{D}_d)$*. Then there exists a constant* C*, independent of* n*, such that*

$$\left|\int_{I_3} f(t)\,\mathrm{d}t - h\sum_{k=-n}^{n} f(\psi_{\mathrm{DE3\dagger}}(kh))\psi'_{\mathrm{DE3\dagger}}(kh)\right| \leq C\,\mathrm{e}^{-2\pi dn/\log(2\pi dn/\mu)},$$

where $h=\log(2\pi dn/\mu)/n$*.*

Remark 2.1 *As for Theorem 2.5 with* $i=3$ *and Theorem 2.6, although the condition '*$\mu\leq 1$*' is not assumed (only* $\mu>0$ *is assumed) in the original paper [8], that condition is necessary to avoid the case where* $|z/(1+z)|^{\mu-1}=\infty$ *at* $z=-1$ *(see* $E_3(z;\mu)$*).*

As for case 1 (Theorem 2.5 with $i=1$) and case 2 (Theorem 2.5 with $i=2$), this paper not only explicitly estimates the constant C's, but also generalizes the approximation formula from $\sum_{k=-n}^{n}$ to $\sum_{k=-M}^{N}$ as stated below. Here, x_γ is defined for $\gamma > 0$ by

$$x_\gamma = \begin{cases} \operatorname{arcsinh}\left(\frac{\sqrt{1+\sqrt{1-(2\pi\gamma)^2}}}{2\pi\gamma}\right) & (\text{if } 0 < \gamma < 1/(2\pi)), \\ \operatorname{arcsinh}(1) & (\text{if } 1/(2\pi) \leq \gamma), \end{cases}$$

which is introduced to determine the region of x where $\cosh(x)\,\mathrm{e}^{\pm\pi\gamma\sinh x}$ is monotone (see Okayama et al. [3, Proposition 4.17]).

Theorem 2.7 *Assume that f is analytic in $\psi_{\mathrm{DE1}}(\mathscr{D}_d)$, and there exist positive constants K, α, and β such that (2.1) holds for all $z \in \psi_{\mathrm{DE1}}(\mathscr{D}_d^-)$, and (2.2) holds for all $z \in \psi_{\mathrm{DE1}}(\mathscr{D}_d^+)$. Let $\mu = \min\{\alpha, \beta\}$, let $\nu = \max\{\alpha, \beta\}$, let h be defined as (2.9), and let M and N be defined as*

$$\begin{cases} M = n, \quad N = n - \lfloor \log(\beta/\alpha)/h \rfloor & (\textit{if } \mu = \alpha), \\ N = n, \quad M = n - \lfloor \log(\alpha/\beta)/h \rfloor & (\textit{if } \mu = \beta). \end{cases} \tag{2.10}$$

Furthermore, let n be taken sufficiently large so that $n \geq (\nu\,\mathrm{e})/(8d)$, $Mh \geq x_{\alpha/2}$, and $Nh \geq x_{\beta/2}$ hold. Then it holds that

$$\left| \int_{I_1} f(t)\,\mathrm{d}t - h \sum_{k=-M}^{N} f(\psi_{\mathrm{DE1}}(kh))\psi'_{\mathrm{DE1}}(kh) \right| \leq C_1\,\mathrm{e}^{-2\pi dn/\log(8dn/\mu)},$$

where C_1 is a constant independent of n, expressed as

$$C_1 = \frac{2^{\nu+1}K}{\mu}\left\{ \frac{2}{(1-\mathrm{e}^{-\pi\mu\,\mathrm{e}/4})\{\cos(\frac{\pi}{2}\sin d)\}^{\nu}\cos d} + \mathrm{e}^{\pi\nu/4} \right\}.$$

Theorem 2.8 *Assume that f is analytic in $\psi_{\mathrm{DE2}}(\mathscr{D}_d)$, and there exist positive constants K, α, and β such that (2.6) holds with $i = 2$ for all $z \in \psi_{\mathrm{DE2}}(\mathscr{D}_d)$. Let $\mu = \min\{\alpha, \beta\}$, let $\nu = \max\{\alpha, \beta\}$, let h be defined as (2.9), and let M and N be defined as (2.10). Furthermore, let n be taken sufficiently large so that $n \geq (\nu\,\mathrm{e})/(8d)$, $Mh \geq x_{\alpha/2}$, and $Nh \geq x_{\beta/2}$ hold. Then it holds that*

$$\left| \int_{I_2} f(t)\,\mathrm{d}t - h \sum_{k=-M}^{N} f(\psi_{\mathrm{DE2}}(kh))\psi'_{\mathrm{DE2}}(kh) \right| \leq C_2\,\mathrm{e}^{-2\pi dn/\log(8dn/\mu)},$$

where C_2 is a constant independent of n, expressed as

$$C_2 = \frac{2K}{\mu}\left\{ \frac{2}{(1-\mathrm{e}^{-\pi\mu\,\mathrm{e}/4})\{\cos(\frac{\pi}{2}\sin d)\}^{(\alpha+\beta)/2}\cos d} + \mathrm{e}^{\pi\nu/4} \right\}.$$

As for case 3 (Theorem 2.5 with $i = 3$ and Theorem 2.6), this paper employs the improved variable transformation $\psi_{\mathrm{DE3}\ddagger}$ as described in the introduction, and gives the error estimates in a form similar to Theorems 2.7 and 2.8.

Theorem 2.9 *Assume that f is analytic in $\psi_{\mathrm{DE3}\ddagger}(\mathscr{D}_d)$, and there exist positive constants K, β, and α with $\alpha \leq 1$ such that (2.6) holds with $i = 3$ for all $z \in \psi_{\mathrm{DE3}\ddagger}(\mathscr{D}_d)$. Let $\mu = \min\{\alpha, \beta\}$, let $\nu = \max\{\alpha, \beta\}$, let h be defined as*

$$h = \frac{\log(4dn/\mu)}{n}, \tag{2.11}$$

and let M and N be defined as (2.10)*. Furthermore, let n be taken sufficiently large so that $n \geq (\nu\,\mathrm{e})/(4d)$, $Mh \geq x_{\alpha}$, and $Nh \geq x_{\beta}$ hold. Then it holds that*

$$\left| \int_{I_3} f(t)\,\mathrm{d}t - h \sum_{k=-M}^{N} f(\psi_{\mathrm{DE3}\ddagger}(kh))\psi'_{\mathrm{DE3}\ddagger}(kh) \right| \leq C_{3\ddagger}\,\mathrm{e}^{-2\pi dn/\log(4dn/\mu)},$$

where $C_{3\ddagger}$ is a constant independent of n, expressed as

$$C_{3\ddagger} = \frac{2K}{\mu}\left\{ \frac{2(\tilde{c}_d)^{1-\alpha}}{(1-\mathrm{e}^{-\pi\mu\,\mathrm{e}/2})\{\cos(\frac{\pi}{2}\sin d)\}^{\alpha+\beta}\cos d} + \mathrm{e}^{\pi(1-\alpha+6\nu)/12} \right\},$$

and where $\tilde{c}_d$ is a constant expressed by using $c_d = 1 + \{1/\cos(\frac{\pi}{2}\sin d)\}$ as

$$\tilde{c}_d = \frac{1+\log(1+c_d)}{\log(1+c_d)} c_d. \tag{2.12}$$

3 Error Estimates with Explicit Constants for Sinc Indefinite Integration

Sinc indefinite integration is an approximation formula for the indefinite integral [1], expressed as

$$\int_{-\infty}^{\xi} F(x)\,\mathrm{d}x \approx \sum_{k=-M}^{N} F(kh)J(k,h)(\xi), \quad \xi \in \mathbb{R}. \tag{3.1}$$

Here, the basis function $J(k,h)$ is defined by

$$J(k,h)(x) = h\left\{\frac{1}{2} + \frac{1}{\pi}\,\mathrm{Si}[\pi(x/h-k)]\right\},$$

where $\mathrm{Si}(x)$ is the so-called sine integral, defined by $\mathrm{Si}(x) = \int_0^x \{\sin(\sigma)/\sigma\}\,\mathrm{d}\sigma$. The approximation (3.1) can be combined with the SE transformation or the DE transformation [2] similar to Sinc quadrature (1.1). This section presents the error estimates for those formulas.

3.1 New Error Estimates for Sinc Indefinite Integration with the SE Transformation

This paper gives new error estimates for Sinc indefinite integration with ψ_{SE1}, ψ_{SE2}, and ψ_{SE3} in the following form (Theorems 3.1 and 3.2).

Theorem 3.1 *Assume that the function f is analytic in $\psi_{\mathrm{SE1}}(\mathscr{D}_d)$, and there exist positive constants K, α, and β such that* (2.1) *holds for all $z \in \psi_{\mathrm{SE1}}(\mathscr{D}_d^-)$, and* (2.2) *holds for all $z \in \psi_{\mathrm{SE1}}(\mathscr{D}_d^+)$. Let $\mu = \min\{\alpha, \beta\}$, let $\nu = \max\{\alpha, \beta\}$, let h be defined as*

$$h = \sqrt{\frac{\pi d}{\mu n}}, \tag{3.2}$$

and let M and N be defined as (2.4)*. Then, it holds that*

$$\sup_{\tau \in I_1}\left| \int_{-\infty}^{\tau} f(t)\,\mathrm{d}t - \sum_{k=-M}^{N} f(\psi_{\mathrm{SE1}}(kh))\psi'_{\mathrm{SE1}}(kh)J(k,h)(\psi_{\mathrm{SE1}}^{-1}(\tau)) \right| \leq C_1\,\mathrm{e}^{-\sqrt{\pi d\mu n}},$$

where C_1 is a constant independent of n, expressed as

$$C_1 = \frac{2^{\nu+1}K}{\mu}\left\{\frac{1}{(1-\mathrm{e}^{-2\sqrt{\pi d\mu}})\{\cos d\}^{\nu}}\sqrt{\frac{\pi}{d\mu}} + 1.1\right\}.$$

Theorem 3.2 *The following is true for $i = 2, 3$. Assume that the function f is analytic in $\psi_{\mathrm{SE}i}(\mathscr{D}_d)$, and there exist positive constants K, α, and β such that* (2.6) *holds for all $z \in \psi_{\mathrm{SE}i}(\mathscr{D}_d)$. Let $\mu = \min\{\alpha, \beta\}$, let h be defined as* (3.2)*, and let M and N be defined as* (2.4)*. Then, it holds that*

$$\sup_{\tau\in I_i}\left|\int_0^{\tau} f(t)\,\mathrm{d}t - \sum_{k=-M}^{N} f(\psi_{\mathrm{SE}i}(kh))\psi_{\mathrm{SE}i}'(kh)J(k,h)(\psi_{\mathrm{SE}i}^{-1}(\tau))\right| \leq C_i\,\mathrm{e}^{-\sqrt{\pi d\mu n}},$$

where C_2 and C_3 are constants independent of n, expressed as

$$C_2 = \frac{2K}{\mu}\left\{\frac{1}{(1-\mathrm{e}^{-2\sqrt{\pi d\mu}})\{\cos d\}^{(\alpha+\beta)/2}}\sqrt{\frac{\pi}{d\mu}} + 1.1\right\},$$

$$C_3 = \frac{2K}{\mu}\left\{\frac{2^{1+(\beta/2)}c_{\alpha,d}}{(1-\mathrm{e}^{-2\sqrt{\pi d\mu}})\{\cos d\}^{(\alpha+\beta)/2}}\sqrt{\frac{\pi}{d\mu}} + 1.1\cdot 2^{(1-\alpha+|1-\alpha|)/2}\right\},$$

and where $c_{\alpha,d}$ is a constant defined in (2.8)*.*

Remark 3.1 *This paper addresses the indefinite integration formulas based on* (3.1) *developed by Haber [1]. Haber developed his formula for case 4, but did not develop any formula for cases 1–3.*

Other indefinite integration formulas with ψ_{SE1}, ψ_{SE2}, and ψ_{SE3} were developed by Stenger [5], but error estimates of the formulas are left for future work.

3.2 New Error Estimates for Sinc Indefinite Integration with the DE Transformation

This paper gives new error estimates for Sinc indefinite integration with ψ_{DE1}, ψ_{DE2}, and $\psi_{\mathrm{DE3}\ddagger}$ in the following form (Theorems 3.3–3.5). Let us define $\epsilon_{d,\mu}^{\mathrm{DE}}(n)$ as $\epsilon_{d,\mu}^{\mathrm{DE}}(n) = [\mathrm{e}^{-\pi dn/\log(4dn/\mu)}\log(4dn/\mu)]/n$ for short.

Theorem 3.3 *Assume that the function f is analytic in $\psi_{\mathrm{DE1}}(\mathscr{D}_d)$, and there exist positive constants K, α, and β such that* (2.1) *holds for all $z \in \psi_{\mathrm{DE1}}(\mathscr{D}_d^{-})$, and* (2.2) *holds for all $z \in \psi_{\mathrm{DE1}}(\mathscr{D}_d^{+})$. Let $\mu = \min\{\alpha, \beta\}$, let $\nu = \max\{\alpha, \beta\}$, let h be defined as* (2.11)*, and let M and N be defined as* (2.10)*. Furthermore, let n be taken sufficiently large so that $n \geq (\nu\,\mathrm{e})/(4d)$, $Mh \geq x_{\alpha/2}$, and $Nh \geq x_{\beta/2}$ hold. Then, it holds that*

$$\sup_{\tau\in I_1}\left|\int_{-\infty}^{\tau} f(t)\,\mathrm{d}t - \sum_{k=-M}^{N} f(\psi_{\mathrm{DE1}}(kh))\psi_{\mathrm{DE1}}'(kh)J(k,h)(\psi_{\mathrm{DE1}}^{-1}(\tau))\right| \leq C_1\epsilon_{d,\mu}^{\mathrm{DE}}(n),$$

where C_1 is a constant independent of n, expressed as

$$C_1 = \frac{2^{\nu+1}K}{\mu d}\left\{\frac{1}{(1-\mathrm{e}^{-\pi\mu\,\mathrm{e}/2})\{\cos(\frac{\pi}{2}\sin d)\}^{\nu}\cos d} + \mathrm{e}^{\pi(\alpha+\beta)/4}\right\}.$$

Theorem 3.4 *Assume that the function f is analytic in $\psi_{\mathrm{DE2}}(\mathscr{D}_d)$, and there exist positive constants K, α, and β such that* (2.6) *holds with $i = 2$ for all $z \in \psi_{\mathrm{DE2}}(\mathscr{D}_d)$. Let $\mu = \min\{\alpha, \beta\}$, let $\nu = \max\{\alpha, \beta\}$, let h be defined as* (2.11)*, and let M and N be defined as* (2.10)*. Furthermore, let n be taken sufficiently large so that $n \geq (\nu\,\mathrm{e})/(4d)$, $Mh \geq x_{\alpha/2}$, and $Nh \geq x_{\beta/2}$ hold. Then, it holds that*

$$\sup_{\tau\in I_2}\left|\int_0^{\tau} f(t)\,\mathrm{d}t - \sum_{k=-M}^{N} f(\psi_{\mathrm{DE2}}(kh))\psi'_{\mathrm{DE2}}(kh)J(k,h)(\psi_{\mathrm{DE2}}^{-1}(\tau))\right| \leq C_2\epsilon_{d,\mu}^{\mathrm{DE}}(n),$$

where C_2 is a constant independent of n, expressed as

$$C_2 = \frac{2K}{\mu d}\left\{\frac{1}{(1-\mathrm{e}^{-\pi\mu\,\mathrm{e}/2})\{\cos(\frac{\pi}{2}\sin d)\}^{(\alpha+\beta)/2}\cos d} + \mathrm{e}^{\pi(\alpha+\beta)/4}\right\}.$$

Theorem 3.5 *Assume that the function f is analytic in $\psi_{\mathrm{DE3\ddagger}}(\mathscr{D}_d)$, and there exist positive constants K, β, and α with $\alpha \leq 1$ such that* (2.6) *holds with $i = 3$ for all $z \in \psi_{\mathrm{DE3\ddagger}}(\mathscr{D}_d)$. Let $\mu = \min\{\alpha, \beta\}$, let $\nu = \max\{\alpha, \beta\}$, let h be defined as $h = \log(2dn/\mu)/n$, and let M and N be defined as* (2.10)*. Furthermore, let n be taken sufficiently large so that $n \geq (\nu\,\mathrm{e})/(2d)$, $Mh \geq x_{\alpha}$, and $Nh \geq x_{\beta}$ hold. Then, it holds that*

$$\sup_{\tau\in I_3}\left|\int_0^{\tau} f(t)\,\mathrm{d}t - \sum_{k=-M}^{N} f(\psi_{\mathrm{DE3\ddagger}}(kh))\psi'_{\mathrm{DE3\ddagger}}(kh)J(k,h)(\psi_{\mathrm{DE3\ddagger}}^{-1}(\tau))\right| \leq C_{3\ddagger}\epsilon_{d,2\mu}^{\mathrm{DE}}(n),$$

where $C_{3\ddagger}$ is a constant independent of n, expressed as

$$C_{3\ddagger} = \frac{2K}{\mu d}\left\{\frac{(\tilde{c}_d)^{1-\alpha}}{(1-\mathrm{e}^{-\pi\mu\,\mathrm{e}})\{\cos(\frac{\pi}{2}\sin d)\}^{\alpha+\beta}\cos d} + \mathrm{e}^{\pi(1+5\alpha+6\beta)/12}\right\},$$

and where $\tilde{c}_d$ is a constant defined in (2.12)*.*

Remark 3.2 *The formulas with ψ_{DE1}, ψ_{DE2}, and ψ_{DE3} were originally developed by Muhammad–Mori [2], but no error analysis was done on cases 2 and 3. In case 1, the authors claimed that the formula can achieve* $\mathrm{O}(\exp(-\pi dn/\log(4dn/(\mu-\epsilon))))$*, where ϵ denotes an arbitrary small positive number, under some mild conditions (not specified clearly). In contrast, Theorem 3.3 states the better convergence rate than that claimed by Muhammad–Mori [2], under explicit assumptions.*

4 Numerical Examples

In order to numerically confirm the results presented in Sections 2 and 3, let us consider the following three examples.

Example 4.1 (Case 1 [2]) *Consider the function* $f_1(t) = \sqrt{3}/(2\pi(t^2+t+1))$ *and its definite/indefinite integral on I_1:*

$$\int_{-\infty}^{\infty} f_1(t)\,\mathrm{d}t = 1, \tag{4.1}$$

$$\int_{-\infty}^{\tau} f_1(t)\,\mathrm{d}t = \frac{1}{2} + \frac{1}{\pi}\arctan\left\{\frac{2}{\sqrt{3}}\left(\tau + \frac{1}{2}\right)\right\}. \tag{4.2}$$

The integrand f_1 satisfies the assumptions in Theorems 2.3 and 3.1 with $\alpha = \beta = 1$, $d = 3/4$, and $K = \sqrt{3}\,\mathrm{e}$. In addition, f_1 satisfies the assumptions in Theorems 2.7 and 3.3 with $\alpha = \beta = 1$, $d = \pi/7$, and $K = 8\sqrt{3}/\mathrm{e}$.

Example 4.2 (Case 2 [2]) *Consider the function $f_2(t) = 2/(\pi(1+t^2))$ and its definite/indefinite integral on I_2:*

$$\int_0^\infty f_2(t)\,\mathrm{d}t = 1, \tag{4.3}$$

$$\int_0^\tau f_2(t)\,\mathrm{d}t = \frac{2}{\pi}\arctan(\tau). \tag{4.4}$$

The integrand f_2 satisfies the assumptions in Theorems 2.4 and 3.2 ($i = 2$) with $\alpha = \beta = 1$, $d = \cosh(1)$, and $K = 2/\pi$. In addition, f_2 satisfies the assumptions in Theorems 2.8 and 3.4 with $\alpha = \beta = 1$, $d = 3/2$, and $K = 2/\pi$.

Example 4.3 (Case 3 [7]) *Consider the function $f_3(t) = \mathrm{e}^{-(1+t)}/(1+t)$ and its definite/indefinite integral on I_3:*

$$\int_0^\infty f_3(t)\,\mathrm{d}t = \mathrm{E}_1(1), \tag{4.5}$$

$$\int_0^\tau f_3(t)\,\mathrm{d}t = \mathrm{E}_1(1) - \Gamma(0, 1+\tau), \tag{4.6}$$

where $\mathrm{E}_1(x)$ is the exponential integral, and $\Gamma(s,x)$ is the incomplete gamma function. The integrand f_3 satisfies the assumptions in Theorems 2.4 and 3.2 ($i = 3$) with $\alpha = \beta = 1$, $d = 3/2$, and $K = \mathrm{e}^{-1}$. In addition, f_3 satisfies the assumptions in Theorems 2.9 and 3.5 with $\alpha = \beta = 1$, $d = \log(\pi)$, and $K = \mathrm{e}$.

Numerical results are shown in Figures 1–6. All programs were written in C++ with double-precision floating-point arithmetic, and the GNU Scientific Library was used for computing special functions (for this reason, rounding errors are not considered). In Figure 2, 'maximum error' denotes the maximum value of absolute errors investigated on the following 403 points: $\tau = 0$, $\pm 2^{-100}$, $\pm 2^{-99}$, ..., $\pm 2^{-1}$, $\pm 2^{0}$, $\pm 2^{1}$, ..., $\pm 2^{100}$. Similarly, in Figures 4 and 6, errors were investigated on 201 points (just the positive points of above), and their maximum is plotted in those figures. In each graph, we can see that the error estimate by the presented theorem (dotted line) surely bounds the actual error (solid line).

5 Proofs

5.1 In the Case of the SE Transformation

Let us first have a look at the sketch of the proof by using the Sinc quadrature as an example. Let $F(x) = f(\psi_{\mathrm{SE}i}(x))\psi_{\mathrm{SE}i}'(x)$ (recall that we employ the SE transformation $t = \psi_{\mathrm{SE}i}(x)$). Then, we have to evaluate the following term:

$$\left|\int_{I_i} f(t)\,\mathrm{d}t - h\sum_{k=-M}^{N} f(\psi_{\mathrm{SE}i}(kh))\psi_{\mathrm{SE}i}'(kh)\right| = \left|\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x - h\sum_{k=-M}^{N} F(kh)\right|.$$

For the estimation, the function space defined below plays an important role.

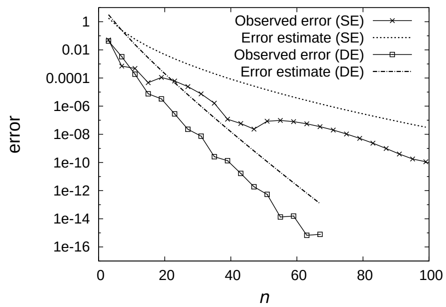

Figure 1: Error of Sinc quadrature for (4.1) and its estimate.

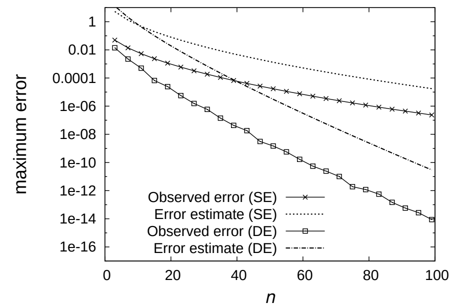

Figure 2: Error of Sinc indefinite integration for (4.2) and its estimate.

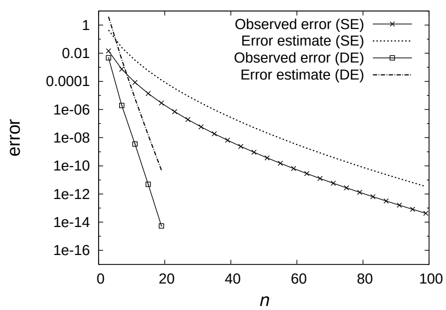

Figure 3: Error of Sinc quadrature for (4.3) and its estimate.

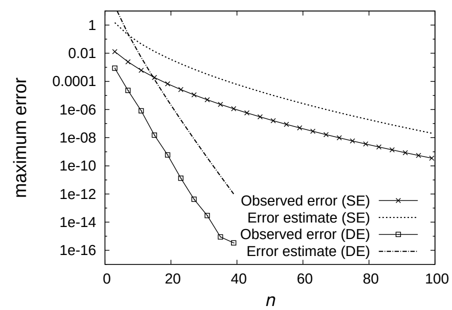

Figure 4: Error of Sinc indefinite integration for (4.4) and its estimate.

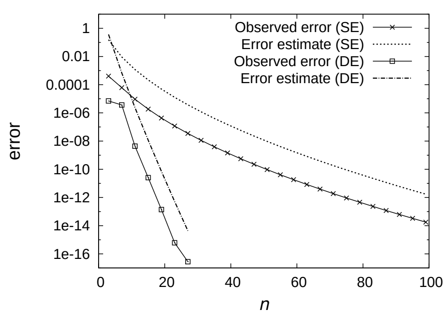

Figure 5: Error of Sinc quadrature for (4.5) and its estimate.

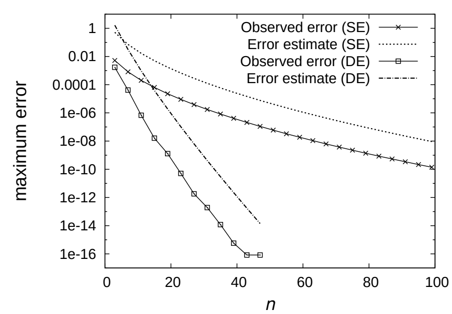

Figure 6: Error of Sinc indefinite integration for (4.6) and its estimate.

Definition 5.1 *Let L, R, α, β be positive constants, and d be a constant where $0 < d < \pi/2$. Then, $\mathbf{L}^{\mathrm{SE}}_{L,R,\alpha,\beta}(\mathscr{D}_d)$ denotes a family of functions F that are analytic on $\mathscr{D}_d$, and for all $\zeta \in \mathscr{D}_d$ and $x \in \mathbb{R}$, satisfy*

$$|F(\zeta)| \leq \frac{L}{|1+\mathrm{e}^{-2\zeta}|^{\alpha/2}|1+\mathrm{e}^{2\zeta}|^{\beta/2}}, \tag{5.1}$$

$$|F(x)| \leq \frac{R}{(1+\mathrm{e}^{-2x})^{\alpha/2}(1+\mathrm{e}^{2x})^{\beta/2}}. \tag{5.2}$$

If F belongs to this function space, the error of the Sinc quadrature is estimated as below. The proof is omitted here because it is quite similar to that of the existing theorem for case 4 [3, Theorem 2.6].

Theorem 5.1 *Let $F \in \mathbf{L}^{\mathrm{SE}}_{L,R,\alpha,\beta}(\mathscr{D}_d)$, let $\mu = \min\{\alpha, \beta\}$, let h be defined as* (2.3)*, and let M and N be defined as* (2.4)*. Then, it holds that*

$$\left|\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x - h\sum_{k=-M}^{N} F(kh)\right| \leq \frac{2}{\mu}\left[\frac{2L}{(1-\mathrm{e}^{-\sqrt{2\pi d\mu}})\{\cos d\}^{(\alpha+\beta)/2}} + R\right]\mathrm{e}^{-\sqrt{2\pi d\mu n}}.$$

This theorem states the desired error estimates with explicit constants for the Sinc quadrature (if $F \in \mathbf{L}^{\mathrm{SE}}_{L,R,\alpha,\beta}(\mathscr{D}_d)$, which is not yet confirmed).

For the Sinc indefinite integration, the next theorem holds. This proof is also omitted because it is quite similar to the proof for case 4 [3, Theorem 2.9].

Theorem 5.2 *Let $F \in \mathbf{L}^{\mathrm{SE}}_{L,R,\alpha,\beta}(\mathscr{D}_d)$, let $\mu = \min\{\alpha, \beta\}$, let h be defined as* (3.2)*, and let M and N be defined as* (2.4)*. Then, it holds that*

$$\sup_{\xi\in\mathbb{R}}\left|\int_{-\infty}^{\xi} F(x)\,\mathrm{d}x - \sum_{k=-M}^{N} F(kh)J(k,h)(\xi)\right|$$
$$\leq \frac{2}{\mu}\left[\frac{L}{(1-\mathrm{e}^{-2\sqrt{\pi d\mu}})\{\cos d\}^{(\alpha+\beta)/2}}\sqrt{\frac{\pi}{d\mu}} + 1.1R\right]\mathrm{e}^{-\sqrt{\pi d\mu n}}.$$

This theorem states the desired error estimates with explicit constants for the Sinc indefinite integration.

In view of Theorems 5.1 and 5.2, our project is completed by checking $F \in \mathbf{L}^{\mathrm{SE}}_{L,R,\alpha,\beta}(\mathscr{D}_d)$ in each case: 1, 2, and 3. Let us check each case one by one.

5.1.1 Proofs in Case 1 (Theorems 2.3 and 3.1)

The claims of Theorems 2.3 and 3.1 follow from the next lemma.

Lemma 5.1 *Let the assumptions in Theorem 2.3 or Theorem 3.1 be fulfilled. Then, the function $F(\zeta) = f(\psi_{\mathrm{SE1}}(\zeta))\psi'_{\mathrm{SE1}}(\zeta)$ belongs to $\mathbf{L}^{\mathrm{SE}}_{L,R,\alpha,\beta}(\mathscr{D}_d)$ with*

$$L = 2^{\nu}K/\{\cos d\}^{(\nu-\mu)/2} \qquad \textit{and} \qquad R = 2^{\nu}K.$$

Proof: First, consider the case $\alpha \leq \beta$. From the inequality (2.1), it follows that

$$|F(\zeta)| \leq K|E_1(\psi_{\mathrm{SE1}}(\zeta);\alpha)||\psi'_{\mathrm{SE1}}(\zeta)| = \frac{K}{|1+\mathrm{e}^{-2\zeta}|^{\alpha/2}|1+\mathrm{e}^{2\zeta}|^{\beta/2}} \cdot 2^{\alpha}|1+\mathrm{e}^{2\zeta}|^{(\beta-\alpha)/2}$$

for $\zeta \in \mathscr{D}_d$ with $\operatorname{Re}\zeta < 0$, and from the inequality (2.2), it follows that

$$|F(\zeta)| \leq K|E_1(\psi_{\mathrm{SE1}}(\zeta);\beta)||\psi'_{\mathrm{SE1}}(\zeta)| = \frac{K}{|1+\mathrm{e}^{-2\zeta}|^{\alpha/2}|1+\mathrm{e}^{2\zeta}|^{\beta/2}} \cdot \frac{2^{\beta}}{|1+\mathrm{e}^{-2\zeta}|^{(\beta-\alpha)/2}}$$

for $\zeta \in \mathscr{D}_d$ with $\operatorname{Re}\zeta \geq 0$. Setting $\zeta = x + \mathrm{i}\,y$ with $x < 0$, we have

$$\begin{aligned} 2^{\alpha}|1+\mathrm{e}^{2\zeta}|^{(\beta-\alpha)/2} &= 2^{\alpha}(1+\mathrm{e}^{2x})^{(\beta-\alpha)/2}\left\{1-\frac{\sin^2 y}{\cosh^2 x}\right\}^{(\beta-\alpha)/4} \\ &\leq 2^{\alpha}(1+\mathrm{e}^{0})^{(\beta-\alpha)/2}\left\{1-0\right\}^{(\beta-\alpha)/4} \\ &= 2^{(\alpha+\beta)/2} \leq 2^{\beta} \leq \frac{2^{\beta}}{\{\cos y\}^{(\beta-\alpha)/2}}. \end{aligned}$$

Furthermore, setting $\zeta = x + \mathrm{i}\,y$ with $x \geq 0$, we have

$$\begin{aligned} \frac{2^{\beta}}{|1+\mathrm{e}^{-2\zeta}|^{(\beta-\alpha)/2}} &= \frac{2^{\beta}}{(1+\mathrm{e}^{-2x})^{(\beta-\alpha)/2}\left\{1-\frac{\sin^2 y}{\cosh^2 x}\right\}^{(\beta-\alpha)/4}} \\ &\leq \frac{2^{\beta}}{(1+0)^{(\beta-\alpha)/2}\left\{1-\frac{\sin^2 y}{\cosh^2 0}\right\}^{(\beta-\alpha)/4}} = \frac{2^{\beta}}{\{\cos y\}^{(\beta-\alpha)/2}}. \end{aligned}$$

Thus, because $\mu = \alpha$ and $\nu = \beta$ in this case, it holds for all $\zeta \in \mathscr{D}_d$ that

$$|F(\zeta)| \leq \frac{K}{|1+\mathrm{e}^{-2\zeta}|^{\alpha/2}|1+\mathrm{e}^{2\zeta}|^{\beta/2}} \cdot \frac{2^{\nu}}{\{\cos d\}^{(\nu-\mu)/2}},$$

and it holds for all $x \in \mathbb{R}$ that

$$|F(x)| \leq \frac{K}{(1+\mathrm{e}^{-2x})^{\alpha/2}(1+\mathrm{e}^{2x})^{\beta/2}} \cdot \frac{2^{\nu}}{\{\cos 0\}^{(\nu-\mu)/2}}.$$

In the case $\alpha > \beta$, the same inequalities hold. This completes the proof. □

5.1.2 Proofs in Case 2 (Theorems 2.4 and 3.2 with $i = 2$)

The claims of Theorems 2.4 and 3.2 ($i = 2$) follow from the next lemma.

Lemma 5.2 *Let the assumptions in Theorem 2.4 or Theorem 3.2 be fulfilled with $i = 2$. Then, the function $F(\zeta) = f(\psi_{\mathrm{SE2}}(\zeta))\psi'_{\mathrm{SE2}}(\zeta)$ belongs to $\mathbf{L}^{\mathrm{SE}}_{L,R,\alpha,\beta}(\mathscr{D}_d)$ with $L = K$ and $R = K$.*

Proof: From the inequality (2.6) with $i = 2$, (5.1) and (5.2) immediately hold with $L = R = K$. □

5.1.3 Proofs in Case 3 (Theorems 2.4 and 3.2 with $i = 3$)

The claims of Theorems 2.4 and 3.2 ($i = 3$) follow from the next lemma.

Lemma 5.3 *Let the assumptions in Theorem 2.4 or Theorem 3.2 be fulfilled with $i = 3$. Then, the function $F(\zeta) = f(\psi_{\mathrm{SE3}}(\zeta))\psi'_{\mathrm{SE3}}(\zeta)$ belongs to $\mathbf{L}^{\mathrm{SE}}_{L,R,\alpha,\beta}(\mathscr{D}_d)$ with $L = 2^{\beta/2}c_{\alpha,d}K$ and $R = 2^{(1-\alpha+|1-\alpha|)/2}K$, where $c_{\alpha,d}$ is the constant defined in* (2.8).

To facilitate the proof, we prepare some useful inequalities (Lemmas 5.4–5.7).

Lemma 5.4 *For all $\zeta \in \overline{\mathscr{D}_{\pi/2}}$, it holds that*

$$\frac{1}{\sqrt{2}}\left|\frac{\mathrm{e}^{\zeta}}{1+\mathrm{e}^{\zeta}}\right| \leq \left|\frac{\operatorname{arcsinh}(\mathrm{e}^{\zeta})}{1+\operatorname{arcsinh}(\mathrm{e}^{\zeta})}\right| \leq \sqrt{2}\left|\frac{\mathrm{e}^{\zeta}}{1+\mathrm{e}^{\zeta}}\right|. \tag{5.3}$$

Furthermore, for all $x \in \mathbb{R}$, it holds that

$$\frac{\operatorname{arcsinh}(\mathrm{e}^{x})}{1+\operatorname{arcsinh}(\mathrm{e}^{x})} \leq \frac{\mathrm{e}^{x}}{1+\mathrm{e}^{x}}. \tag{5.4}$$

Proof: First, consider (5.4), which is proved by showing that $p(t) \leq p(\sinh t)$ for $t \geq 0$, where $p(t) = t/(1+t)$ (just put $x = \log(\sinh t)$). Because $p(t)$ is monotonically increasing, the desired inequality $p(t) \leq p(\sinh t)$ clearly holds. Hence, (5.4) is proved.

Next, consider (5.3), which can be proved by showing that $|g(\zeta)| \leq \sqrt{2}$ and $|1/g(\zeta)| \leq \sqrt{2}$, where

$$g(\zeta) = \frac{\operatorname{arcsinh}(\mathrm{e}^{\zeta})}{1+\operatorname{arcsinh}(\mathrm{e}^{\zeta})}\frac{1+\mathrm{e}^{\zeta}}{\mathrm{e}^{\zeta}}.$$

The functions g and $1/g$ are analytic on $\mathscr{D}_{\pi/2}$ (and continuous on $\overline{\mathscr{D}_{\pi/2}}$). Therefore, by the maximum modulus principle, $|g(\zeta)|$ and $|1/g(\zeta)|$ have their maximum on the boundary of $\mathscr{D}_{\pi/2}$. It is sufficient to consider $z = x + \mathrm{i}\,(\pi/2)$ for $x \in \mathbb{R}$.

First, let us show $1/|g(z)| \leq \sqrt{2}$, which is relatively easy. Setting

$$X = \operatorname{Re}\{\operatorname{arcsinh}(\mathrm{i}\,\mathrm{e}^{x})\} \qquad \text{and} \qquad Y = \operatorname{Im}\{\operatorname{arcsinh}(\mathrm{i}\,\mathrm{e}^{x})\},$$

we have

$$|g(z)|^2 = |g(x+\mathrm{i}\,\pi/2)|^2 = (1+\mathrm{e}^{-2x})\frac{X^2+Y^2}{(X+1)^2+Y^2}.$$

Furthermore, from

$$X = \begin{cases} \log[\mathrm{e}^{x} + \sqrt{\mathrm{e}^{2x}-1}] & (\text{if } x \geq 0), \\ 0 & (\text{if } x < 0), \end{cases}$$

$$Y = \begin{cases} \pi/2 & (\text{if } x \geq 0), \\ \arctan(\mathrm{e}^{x}/\sqrt{1-\mathrm{e}^{2x}}) & (\text{if } x < 0), \end{cases}$$

it holds for $x \geq 0$ that

$$\begin{aligned} \frac{1}{|g(z)|^2} &= \frac{1}{1+\mathrm{e}^{-2x}} \cdot \frac{(1+\log[\mathrm{e}^{x}+\sqrt{\mathrm{e}^{2x}-1}])^2 + (\pi/2)^2}{\log^2[\mathrm{e}^{x}+\sqrt{\mathrm{e}^{2x}-1}] + (\pi/2)^2} \\ &\leq \frac{1}{1+\mathrm{e}^{-2x}} \cdot \frac{(1+\{\sqrt{1+\pi^2}-1\}/2)^2 + (\pi/2)^2}{(\{\sqrt{1+\pi^2}-1\}/2)^2 + (\pi/2)^2} < 2, \end{aligned}$$

and it holds for $x < 0$ that

$$\frac{1}{|g(z)|^2} = \frac{1}{1+\mathrm{e}^{-2x}}\left\{1 + \frac{1}{\arctan^2[\mathrm{e}^x/\sqrt{1-\mathrm{e}^{2x}}]}\right\} \leq \frac{1}{1+\mathrm{e}^{-2x}}\left\{1+\frac{1}{(\mathrm{e}^x)^2}\right\} = 1.$$

Thus, $|1/g(x + \mathrm{i}\,\pi/2)|^2 \leq 2$ holds for all $x \in \mathbb{R}$.

Next, let us show $|g(z)| \leq \sqrt{2}$. It holds for $x \geq 0$ that

$$|g(z)|^2 = (1+\mathrm{e}^{-2x})\frac{\log^2[\mathrm{e}^x + \sqrt{\mathrm{e}^{2x}-1}] + (\pi/2)^2}{(1+\log[\mathrm{e}^x+\sqrt{\mathrm{e}^{2x}-1}])^2 + (\pi/2)^2} \leq (1+1)\cdot 1 = 2.$$

For $x < 0$, we have

$$|g(z)|^2 = (1+\mathrm{e}^{-2x})\frac{\arctan^2[\mathrm{e}^x/\sqrt{1-\mathrm{e}^{2x}}]}{1+\arctan^2[\mathrm{e}^x/\sqrt{1-\mathrm{e}^{2x}}]} = \frac{2s^2}{1+s^2} + \left(\frac{s}{\tan s}\right)^2\frac{1}{1+s^2},$$

where $0 \leq s = \arctan[\mathrm{e}^x/\sqrt{1-\mathrm{e}^{2x}}] < \pi/2$. In the case where $0 \leq s < 1$, it holds that

$$|g(z)|^2 \leq \frac{2\cdot 1^2}{1+1^2} + \left(\frac{s}{\tan s}\right)^2 \frac{1}{1+s^2} \leq \frac{2\cdot 1^2}{1+1^2} + (1)^2\frac{1}{1+0^2} = 2,$$

and in the case where $1 \leq s < \pi/2$, it holds that

$$|g(z)|^2 \leq \frac{2\cdot(\pi/2)^2}{1+(\pi/2)^2} + \left(\frac{s}{\tan s}\right)^2\frac{1}{1+s^2} \leq \frac{2\cdot(\pi/2)^2}{1+(\pi/2)^2} + \left(\frac{1}{\tan 1}\right)^2\frac{1}{1+1^2} < 2.$$

Thus, $|g(x+\mathrm{i}\,\pi/2)|^2 \leq 2$ for all $x \in \mathbb{R}$. This completes the proof. □

Lemma 5.5 *For all $\zeta \in \overline{\mathscr{D}_{\pi/2}}$ and $x \in \mathbb{R}$, we have*

$$\frac{1}{|\mathrm{e}^\zeta + \sqrt{1+\mathrm{e}^{2\zeta}}|} \leq \frac{\sqrt{2}}{|1+\mathrm{e}^\zeta|}, \tag{5.5}$$

$$\frac{1}{\mathrm{e}^x + \sqrt{1+\mathrm{e}^{2x}}} \leq \frac{1}{1+\mathrm{e}^x}. \tag{5.6}$$

Proof: First, consider (5.6), which is proved by showing

$$g(x) = \frac{1+\mathrm{e}^x}{\mathrm{e}^x+\sqrt{1+\mathrm{e}^{2x}}} \leq 1$$

for $x \in \mathbb{R}$. Because

$$g'(x) = -\mathrm{e}^x\left(1 - \frac{\mathrm{e}^x}{\sqrt{1+\mathrm{e}^{2x}}}\right)\left((1+\mathrm{e}^x) - \sqrt{1+\mathrm{e}^{2x}}\right) \leq 0,$$

we have $g(x) \leq \lim_{x\to-\infty}\{g(x)\} = 1$, which is the desired result.

Next, consider (5.5), which is proved by showing $|g(\zeta)| \leq \sqrt{2}$. The function g is analytic on $\mathscr{D}_{\pi/2}$, (and continuous on $\overline{\mathscr{D}_{\pi/2}}$). Therefore, by the maximum modulus principle, $|g(\zeta)|$ has its maximum on the boundary of $\mathscr{D}_{\pi/2}$, i.e., $|\operatorname{Im}\zeta| = \pi/2$. It is sufficient to consider $z = x + \mathrm{i}\,(\pi/2)$ for $x \in \mathbb{R}$. In the case $x \geq 0$, we have

$$|g(z)|^2 = \left|\frac{1}{1+\sqrt{1-\mathrm{e}^{-2x}}} - \mathrm{i}\,\frac{\mathrm{e}^{-x}}{1+\sqrt{1-\mathrm{e}^{-2x}}}\right|^2 = \frac{1+\mathrm{e}^{-2x}}{(1+\sqrt{1-\mathrm{e}^{-2x}})^2} \leq 1+\mathrm{e}^{-2x} \leq 2.$$

In the case $x < 0$, we have

$$|g(z)|^2 = \left|\sqrt{1-\mathrm{e}^{2x}} + \mathrm{e}^{2x} - \mathrm{i}\,(1-\sqrt{1-\mathrm{e}^{2x}})\,\mathrm{e}^x\right|^2 = 1+\mathrm{e}^{2x} \leq 2.$$

This completes the proof. □

Lemma 5.6 *For all $\zeta \in \mathscr{D}_{\pi/2}$, it holds that*

$$\frac{1}{|1+\mathrm{e}^{\zeta}|} \leq \frac{1}{|1+\mathrm{e}^{2\zeta}|^{1/2}}.$$

Proof: Let $x, y \in \mathbb{R}$ with $|y| < \pi/2$, and let $\zeta = x + \mathrm{i}\, y$. Then, we have

$$\left|\frac{1}{1+\mathrm{e}^{2\zeta}}\right| - \left|\frac{1}{1+\mathrm{e}^{\zeta}}\right|^2 = \frac{\mathrm{e}^{-x}}{2(\cosh(x)+\cos(y))}\left(\sqrt{\frac{2(\cosh(x)+\cos(y))^2}{\cosh(2x)+\cos(2y)}} - 1\right).$$

To show that the right-hand side is nonnegative, it suffices to observe

$$\frac{2(\cosh(x)+\cos(y))^2}{\cosh(2x)+\cos(2y)} - 1 = \frac{1+2\cosh(x)\cos(y)}{\cosh^2(x)-\sin^2(y)} \geq 0,$$

because it generally holds that if $t-1 \geq 0$ then $\sqrt{t}-1 = (t-1)/(\sqrt{t}+1) \geq 0$. □

Lemma 5.7 *Let $x, y \in \mathbb{R}$ with $|y| < \pi/2$. Then, it holds that*

$$\sup_{x\in\mathbb{R}}\left|\frac{\cosh^2[(x+\mathrm{i}\, y)/2]}{\cosh(x+\mathrm{i}\, y)}\right| \leq \frac{\cos^2(y/2)}{\cos y} = \frac{1}{2}\left(1+\frac{1}{\cos y}\right).$$

Proof: First, we have

$$\left|\frac{\cosh^2[(x+\mathrm{i}\, y)/2]}{\cosh(x+\mathrm{i}\, y)}\right| = \frac{\cosh(x)+\cos(y)}{2\sqrt{\cosh^2(x)-\sin^2(y)}} \leq \frac{\cosh(x)+\cos(y)}{2\sqrt{\cosh^2(x)\{1-\sin^2(y)\}}},$$

which is then estimated as

$$\frac{\cosh(x)+\cos(y)}{2\sqrt{\cosh^2(x)\{1-\sin^2(y)\}}} = \frac{\cosh(x)+\cos(y)}{2\cosh(x)\cos(y)} \leq \frac{1+\cos(y)}{2\cdot 1\cdot\cos(y)}.$$

This completes the proof. □

Lemma 5.3 is proved as follows.

Proof: From the inequality (2.6) with $i = 3$, it follows that

$$|F(\zeta)| \leq K\left|\frac{\operatorname{arcsinh}(\mathrm{e}^{\zeta})}{1+\operatorname{arcsinh}(\mathrm{e}^{\zeta})}\right|^{\alpha-1}\left|\frac{1}{\mathrm{e}^{\zeta}+\sqrt{1+\mathrm{e}^{2\zeta}}}\right|^{\beta}\left|\frac{1}{1+\mathrm{e}^{-2\zeta}}\right|^{1/2}.$$

First, consider the case $1 \leq \alpha$. From Lemmas 5.4, 5.5, and 5.6, it holds that

$$\begin{aligned}|F(\zeta)| &\leq K\left|\frac{\sqrt{2}}{1+\mathrm{e}^{-\zeta}}\right|^{\alpha-1}\left|\frac{\sqrt{2}}{1+\mathrm{e}^{\zeta}}\right|^{\beta}\left|\frac{1}{1+\mathrm{e}^{-2\zeta}}\right|^{1/2}\\ &\leq 2^{\beta/2}2^{(\alpha-1)/2}K\left|\frac{1}{1+\mathrm{e}^{-2\zeta}}\right|^{(\alpha-1)/2}\left|\frac{1}{1+\mathrm{e}^{2\zeta}}\right|^{\beta/2}\left|\frac{1}{1+\mathrm{e}^{-2\zeta}}\right|^{1/2}\end{aligned}$$

for all $\zeta \in \mathscr{D}_d$. For $x \in \mathbb{R}$, this means that

$$\begin{aligned}|F(x)| &\leq K\left(\frac{1}{1+\mathrm{e}^{-x}}\right)^{\alpha-1}\left(\frac{1}{1+\mathrm{e}^{x}}\right)^{\beta}\left(\frac{1}{1+\mathrm{e}^{-2x}}\right)^{1/2}\\ &\leq K\left(\frac{1}{1+\mathrm{e}^{-2x}}\right)^{(\alpha-1)/2}\left(\frac{1}{1+\mathrm{e}^{2x}}\right)^{\beta/2}\left(\frac{1}{1+\mathrm{e}^{-2x}}\right)^{1/2}.\end{aligned}$$

This completes the proof for $1 \leq \alpha$.

Next, consider the case $0 < \alpha < 1$. From Lemmas 5.4, 5.5, and 5.6, it follows that

$$\begin{aligned}|F(\zeta)| &\leq K \left|\sqrt{2}(1+\mathrm{e}^{-\zeta})\right|^{1-\alpha} \left|\frac{\sqrt{2}}{1+\mathrm{e}^{\zeta}}\right|^{\beta} \left|\frac{1}{1+\mathrm{e}^{-2\zeta}}\right|^{1/2} \\ &= K \left|\frac{2(1+\mathrm{e}^{-\zeta})^2}{1+\mathrm{e}^{-2\zeta}}\right|^{(1-\alpha)/2} \left|\frac{\sqrt{2}}{1+\mathrm{e}^{\zeta}}\right|^{\beta} \left|\frac{1}{1+\mathrm{e}^{-2\zeta}}\right|^{\alpha/2}.\end{aligned}$$

Furthermore, from Lemma 5.7, it holds for $\zeta = x + \mathrm{i}\, y \in \mathscr{D}_d$ that

$$\left|\frac{2(1+\mathrm{e}^{-\zeta})^2}{1+\mathrm{e}^{-2\zeta}}\right| = 4\left|\frac{\cosh^2(\zeta/2)}{\cosh(\zeta)}\right| \leq 2\left(1+\frac{1}{\cos y}\right).$$

Then, using Lemma 5.6, we have

$$|F(\zeta)| \leq 2^{\beta/2} K \left\{2\left(1+\frac{1}{\cos d}\right)\right\}^{(1-\alpha)/2} \left|\frac{1}{1+\mathrm{e}^{2\zeta}}\right|^{\beta/2} \left|\frac{1}{1+\mathrm{e}^{-2\zeta}}\right|^{\alpha/2}$$

for all $\zeta \in \mathscr{D}_d$. For $x \in \mathbb{R}$, it holds that

$$\begin{aligned}|F(x)| &\leq K \left(\sqrt{2}(1+\mathrm{e}^{-x})\right)^{1-\alpha} \left(\frac{1}{1+\mathrm{e}^{x}}\right)^{\beta} \left(\frac{1}{1+\mathrm{e}^{-2x}}\right)^{1/2} \\ &= K \left(\frac{2(1+\mathrm{e}^{-x})^2}{1+\mathrm{e}^{-2x}}\right)^{(1-\alpha)/2} \left(\frac{1}{1+\mathrm{e}^{x}}\right)^{\beta} \left(\frac{1}{1+\mathrm{e}^{-2x}}\right)^{\alpha/2} \\ &\leq K \left(2\left(1+\frac{1}{\cos 0}\right)\right)^{(1-\alpha)/2} \left(\frac{1}{1+\mathrm{e}^{2x}}\right)^{\beta/2} \left(\frac{1}{1+\mathrm{e}^{-2x}}\right)^{\alpha/2}.\end{aligned}$$

This completes the proof for $0 < \alpha < 1$. □

5.2 In the Case of the DE Transformation

We need the following definition in the case of the DE transformation.

Definition 5.2 *Let L, R, α, β be positive constants, and d be a constant with $0 < d < \pi/2$. Then, $\mathbf{L}^{\mathrm{DE}}_{L,R,\alpha,\beta}(\mathscr{D}_d)$ denotes a family of functions F that are analytic on $\mathscr{D}_d$, and for all $\zeta \in \mathscr{D}_d$ and $x \in \mathbb{R}$, satisfy*

$$|F(\zeta)| \leq \frac{(\pi/2)L|\cosh\zeta|}{|1+\mathrm{e}^{-\pi\sinh\zeta}|^{\alpha/2}|1+\mathrm{e}^{\pi\sinh\zeta}|^{\beta/2}}, \tag{5.7}$$

$$|F(x)| \leq \frac{(\pi/2)R\cosh x}{(1+\mathrm{e}^{-\pi\sinh x})^{\alpha/2}(1+\mathrm{e}^{\pi\sinh x})^{\beta/2}}. \tag{5.8}$$

If F belongs to this function space, the errors of the Sinc quadrature and Sinc indefinite integration are estimated as below. The proofs are omitted because they are quite similar to those for the existing theorems for case 4 [3, Theorems 2.14 and 2.16].

Theorem 5.3 *Let $F \in \mathbf{L}^{\mathrm{DE}}_{L,R,\alpha,\beta}(\mathscr{D}_d)$, let $\mu = \min\{\alpha, \beta\}$, let $\nu = \max\{\alpha, \beta\}$, let h be defined as* (2.9)*, and let M and N be defined as* (2.10)*. Furthermore, let n be taken* *sufficiently large so that* $n \geq (\nu\,\mathrm{e})/(8d)$*,* $Mh \geq x_{\alpha/2}$*, and* $Nh \geq x_{\beta/2}$ *hold. Then,*

$$\left|\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x - h\sum_{k=-M}^{N} F(kh)\right|$$
$$\leq \frac{2}{\mu}\left[\frac{2L}{(1-\mathrm{e}^{-\pi\mu\,\mathrm{e}/4})\{\cos(\frac{\pi}{2}\sin d)\}^{(\alpha+\beta)/2}\cos d} + R\,\mathrm{e}^{\pi\nu/4}\right]\mathrm{e}^{-2\pi dn/\log(8dn/\mu)}.$$

Theorem 5.4 *Let* $F \in \mathbf{L}^{\mathrm{DE}}_{L,R,\alpha,\beta}(\mathscr{D}_d)$*, let* $\mu = \min\{\alpha,\,\beta\}$*, let* $\nu = \max\{\alpha,\,\beta\}$*, let* h *be defined as* (2.11)*, and let* M *and* N *be defined as* (2.10)*. Furthermore, let* n *be taken sufficiently large so that* $n \geq (\nu\,\mathrm{e})/(4d)$*,* $Mh \geq x_{\alpha/2}$*, and* $Nh \geq x_{\beta/2}$ *hold. Then,*

$$\sup_{\xi\in\mathbb{R}}\left|\int_{-\infty}^{\xi} F(x)\,\mathrm{d}x - \sum_{k=-M}^{N} F(kh)J(k,h)(\xi)\right|$$
$$\leq \frac{2}{\mu d}\left[\frac{L}{(1-\mathrm{e}^{-\pi\mu\,\mathrm{e}/2})\{\cos(\frac{\pi}{2}\sin d)\}^{(\alpha+\beta)/2}\cos d} + R\,\mathrm{e}^{\pi(\alpha+\beta)/4}\right]\epsilon^{\mathrm{DE}}_{d,\mu}(n).$$

In view of Theorems 5.3 and 5.4, our project is completed by checking $F \in \mathbf{L}^{\mathrm{DE}}_{L,R,\alpha,\beta}(\mathscr{D}_d)$ in each case: 1, 2, and 3. Let us check each case one by one. The next lemma is useful for the proofs.

Lemma 5.8 (Okayama et al. [3, Lemma 4.22]) *Let* $x,\,y \in \mathbb{R}$ *with* $|y| < \pi/2$*, and let* $\zeta = x + \mathrm{i}\,y$*. Then,*

$$\left|\frac{1}{1+\mathrm{e}^{\pi\sinh\zeta}}\right| \leq \frac{1}{(1+\mathrm{e}^{\pi\sinh(x)\cos y})\cos(\frac{\pi}{2}\sin y)},$$
$$\left|\frac{1}{1+\mathrm{e}^{-\pi\sinh\zeta}}\right| \leq \frac{1}{(1+\mathrm{e}^{-\pi\sinh(x)\cos y})\cos(\frac{\pi}{2}\sin y)}.$$

5.2.1 Proofs in Case 1 (Theorems 2.7 and 3.3)

The claims of Theorems 2.7 and 3.3 follow from the next lemma.

Lemma 5.9 *Let the assumptions in Theorem 2.7 or Theorem 3.3 be fulfilled. Then, the function* $F(\zeta) = f(\psi_{\mathrm{DE1}}(\zeta))\psi'_{\mathrm{DE1}}(\zeta)$ *belongs to* $\mathbf{L}^{\mathrm{DE}}_{L,R,\alpha,\beta}(\mathscr{D}_d)$ *with*

$$L = 2^{\nu}K\Big/\left\{\cos\left(\frac{\pi}{2}\sin d\right)\right\}^{(\nu-\mu)/2} \qquad \textit{and} \qquad R = 2^{\nu}K.$$

Proof: First, consider the case $\alpha \leq \beta$. From the inequality (2.1), it follows that

$$|F(\zeta)| \leq \frac{K}{|1+\mathrm{e}^{-\pi\sinh\zeta}|^{\alpha/2}|1+\mathrm{e}^{\pi\sinh\zeta}|^{\beta/2}}\cdot 2^{\alpha}|1+\mathrm{e}^{\pi\sinh\zeta}|^{(\beta-\alpha)/2}$$

for $\zeta \in \mathscr{D}_d$ with $\operatorname{Re}\zeta < 0$, and from the inequality (2.2), it follows that

$$|F(\zeta)| \leq \frac{K}{|1+\mathrm{e}^{-\pi\sinh\zeta}|^{\alpha/2}|1+\mathrm{e}^{\pi\sinh\zeta}|^{\beta/2}}\cdot\frac{2^{\beta}}{|1+\mathrm{e}^{-\pi\sinh\zeta}|^{(\beta-\alpha)/2}}$$

for $\zeta \in \mathscr{D}_d$ with $\operatorname{Re}\zeta \geq 0$. Setting $\zeta = x + \mathrm{i}\,y$ with $x < 0$, we have

$$\begin{aligned}
&2^{\alpha}|1+\mathrm{e}^{\pi\sinh\zeta}|^{(\beta-\alpha)/2}\\
&= 2^{\alpha}(1+\mathrm{e}^{\pi\sinh(x)\cos y})^{(\beta-\alpha)/2}\left\{1-\frac{\sin^2(\pi\cosh(x)\sin y)}{\cosh^2(\pi\sinh(x)\cos y)}\right\}^{(\beta-\alpha)/4}\\
&\leq 2^{\alpha}(1+\mathrm{e}^{0})^{(\beta-\alpha)/2}\{1-0\}^{(\beta-\alpha)/4} = 2^{(\alpha+\beta)/2} \leq 2^{\beta} \leq \frac{2^{\beta}}{\{\cos(\frac{\pi}{2}\sin y)\}^{(\beta-\alpha)/2}}.
\end{aligned}$$

Furthermore, setting $\zeta = x + \mathrm{i}\,y$ with $x \geq 0$ and using Lemma 5.8, we have

$$\begin{aligned}
\frac{2^{\beta}}{|1+\mathrm{e}^{-\pi\sinh\zeta}|^{(\beta-\alpha)/2}} &\leq \frac{2^{\beta}}{(1+\mathrm{e}^{-\pi\sinh(x)\cos y})^{(\beta-\alpha)/2}\{\cos(\frac{\pi}{2}\sin y)\}^{(\beta-\alpha)/2}}\\
&\leq \frac{2^{\beta}}{(1+0)\{\cos(\frac{\pi}{2}\sin y)\}^{(\beta-\alpha)/2}}.
\end{aligned}$$

Thus, because $\mu = \alpha$ and $\nu = \beta$ in this case, it holds for all $\zeta \in \mathscr{D}_d$ that

$$|F(\zeta)| \leq \frac{K}{|1+\mathrm{e}^{-\pi\sinh\zeta}|^{\alpha/2}|1+\mathrm{e}^{\pi\sinh\zeta}|^{\beta/2}} \cdot \frac{2^{\nu}}{\{\cos(\frac{\pi}{2}\sin d)\}^{(\nu-\mu)/2}},$$

and it holds for all $x \in \mathbb{R}$ that

$$|F(x)| \leq \frac{K}{(1+\mathrm{e}^{-\pi\sinh x})^{\alpha/2}(1+\mathrm{e}^{\pi\sinh x})^{\beta/2}} \cdot \frac{2^{\nu}}{\{\cos(\frac{\pi}{2}\sin 0)\}^{(\nu-\mu)/2}}.$$

In the case $\alpha > \beta$, the same inequalities hold. This completes the proof. □

5.2.2 Proofs in Case 2 (Theorems 2.8 and 3.4)

The claims of Theorems 2.8 and 3.4 follow from the next lemma.

Lemma 5.10 *Let the assumptions in Theorem 2.8 or Theorem 3.4 be fulfilled. Then, the function* $F(\zeta) = f(\psi_{\mathrm{DE2}}(\zeta))\psi'_{\mathrm{DE2}}(\zeta)$ *belongs to* $\mathbf{L}^{\mathrm{DE}}_{L,R,\alpha,\beta}(\mathscr{D}_d)$ *with* $L = K$ *and* $R = K$.

Proof: From the inequality (2.6) with $i = 2$, (5.7) and (5.8) immediately hold with $L = R = K$. □

5.2.3 Proofs in Case 3 (Theorems 2.9 and 3.5)

The claims of Theorems 2.9 and 3.5 follow from the next lemma.

Lemma 5.11 *Let the assumptions in Theorem 2.9 or Theorem 3.5 be fulfilled. Then, the function* $F(\zeta) = f(\psi_{\mathrm{DE3}\ddagger}(\zeta))\psi'_{\mathrm{DE3}\ddagger}(\zeta)$ *belongs to* $\mathbf{L}^{\mathrm{DE}}_{L,R,2\alpha,2\beta}(\mathscr{D}_d)$ *with*

$$L = 2(\tilde{c}_d)^{1-\alpha}K \qquad \textit{and} \qquad R = 2(\mathrm{e}^{\pi/12})^{1-\alpha}K,$$

where $\tilde{c}_d$ *is the constant defined in* (2.12).

For the proof, we need the following estimate.

Lemma 5.12 *Let d be a constant with $0 < d < \pi/2$. Then,*

$$\sup_{\zeta\in\mathscr{D}_d}\left|\frac{1+\log(1+\mathrm{e}^{\pi\sinh\zeta})}{\log(1+\mathrm{e}^{\pi\sinh\zeta})}\cdot\frac{1}{1+\mathrm{e}^{-\pi\sinh\zeta}}\right|\leq\tilde{c}_d, \tag{5.9}$$

$$\sup_{x\in\mathbb{R}}\left\{\frac{1+\log(1+\mathrm{e}^{\pi\sinh x})}{\log(1+\mathrm{e}^{\pi\sinh x})}\cdot\frac{1}{1+\mathrm{e}^{-\pi\sinh x}}\right\}\leq\mathrm{e}^{\pi/12}, \tag{5.10}$$

where $\tilde{c}_d$ is a constant defined in (2.12).

Proof: First, consider (5.10), which is proved by showing

$$p(t)=\frac{1+t}{t}(1-\mathrm{e}^{-t})\leq 1+\frac{\log 6}{6}\quad\left(<\mathrm{e}^{\pi/12}\right)$$

for $t\geq 0$ (put $t=\log(1+\mathrm{e}^{\pi\sinh x})$). Let λ be a value with $\log 6<\lambda<\log 7$ so that $p'(\lambda)=(1+\lambda+\lambda^2-\mathrm{e}^{\lambda})/(\mathrm{e}^{\lambda}\lambda^2)=0$. Then, $p(t)\leq p(\lambda)$ clearly holds. Furthermore, using the relation $1+\lambda+\lambda^2=\mathrm{e}^{\lambda}$, we have

$$p(\lambda)=\frac{1+\lambda}{\lambda}\left(\frac{\mathrm{e}^{\lambda}-1}{\mathrm{e}^{\lambda}}\right)=\frac{1+\lambda}{\lambda}\left(\frac{\lambda(1+\lambda)}{\mathrm{e}^{\lambda}}\right)=\frac{(1+\lambda+\lambda^2)+\lambda}{\mathrm{e}^{\lambda}}=1+\lambda\,\mathrm{e}^{-\lambda}.$$

Because the function $1+x\,\mathrm{e}^{-x}$ is monotonically decreasing for $x\geq 1$, $1+\lambda\,\mathrm{e}^{-\lambda}<1+(\log 6)\,\mathrm{e}^{-\log 6}$ holds (note that $\log 6<\lambda$). Hence, (5.10) is proved.

Next, consider (5.9). By the maximum modulus principle, it is proved by showing

$$|p(\xi)|\leq\frac{1+\log(2+\mathrm{e}^{\gamma})}{\log(2+\mathrm{e}^{\gamma})}(1+\mathrm{e}^{\gamma})=\tilde{c}_d, \tag{5.11}$$

where $\xi=\log(1+\mathrm{e}^{\pi\sinh(x+\mathrm{i}\,d)})$ and $\gamma=-\log[\cos(\frac{\pi}{2}\sin d)]$. Here, notice that

$$\operatorname{Re}\xi=\log|1+\mathrm{e}^{\pi\sinh(x+\mathrm{i}\,d)}|\geq\log\left[(1+\mathrm{e}^{\pi\sinh(x)\cos d})\cos(\tfrac{\pi}{2}\sin d)\right]\geq-\gamma$$

holds from Lemma 5.8. Let us bound $|p(\xi)|$ in the two cases: a) $|\xi|\leq\log(2+\mathrm{e}^{\gamma})$ and b) $|\xi|>\log(2+\mathrm{e}^{\gamma})$. For case a), it holds that

$$|p(\xi)|=\left|(1+\xi)\sum_{k=1}^{\infty}\frac{(-\xi)^{k-1}}{k!}\right|\leq(1+|\xi|)\sum_{k=1}^{\infty}\frac{|\xi|^{k-1}}{k!}=\frac{1+|\xi|}{|\xi|}(\mathrm{e}^{|\xi|}-1).$$

Furthermore, because $(1+x)(\mathrm{e}^{x}-1)/x$ is monotonically increasing, we have (5.11). For case b), because $\operatorname{Re}\xi\geq\gamma$, it holds that

$$|p(\xi)|\leq\frac{1+|\xi|}{|\xi|}(1+|\mathrm{e}^{-\xi}|)=\frac{1+|\xi|}{|\xi|}(1+\mathrm{e}^{-\operatorname{Re}\xi})\leq\frac{1+|\xi|}{|\xi|}(1+\mathrm{e}^{\gamma}).$$

Furthermore, because $(1+x)/x$ is monotonically decreasing, we have (5.11). This completes the proof. □

Lemma 5.11 is proved as follows.

Proof: From the inequality (2.6) with $i=3$, it follows that

$$\begin{aligned}|F(\zeta)|&\leq\left|\frac{1+\log(1+\mathrm{e}^{\pi\sinh\zeta})}{\log(1+\mathrm{e}^{\pi\sinh\zeta})}\right|^{1-\alpha}\frac{|\psi'_{\mathrm{DE3\ddagger}}(\zeta)|}{|1+\mathrm{e}^{\pi\sinh\zeta}|^{\beta}}\\&=2\left|\frac{1+\log(1+\mathrm{e}^{\pi\sinh\zeta})}{\log(1+\mathrm{e}^{\pi\sinh\zeta})}\cdot\frac{1}{1+\mathrm{e}^{-\pi\sinh\zeta}}\right|^{1-\alpha}\frac{K(\pi/2)|\cosh\zeta|}{|1+\mathrm{e}^{-\pi\sinh\zeta}|^{\alpha}|1+\mathrm{e}^{\pi\sinh\zeta}|^{\beta}}.\end{aligned}$$

Then, use Lemma 5.12 to obtain the desired result. □

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