Double bubble minimizes: Interval computations help in solving a long-standing geometric problem

Двойной пузырь минимизирует объем: интервальные вычисления помогают при решении старой геометрической задачи

Main result. It is well known that of all surfaces surrounding an area with a given volume V, the sphere has the smallest area. This result explains, e.g., why a soap bubble tends to become a sphere. More than a hundred years ago, the Belgian physicist J. Plateaux asked a similar question: what is the least area surface enclosing two equal volumes? Physical experiments with bubbles seem to indicate that the desired least area surface is a *double bubble*, a surface formed by two spheres (separated by a flat disk) that meet along a circle at an angle of 120°. However, until 1995, it was not clear whether this is really the desired least area surface. Several other surfaces ("torus bubbles") have been proposed whose areas are pretty close to the area of the double bubble.

The theorem that double bubble really minimizes was recently proven by Joel Hass from Department of Mathematics, University of California at Davis (email `hass@math.ucdavis.edu`) and Roger Schlafly from the Real Software Co. (`rschlafly@attmail.com`). First, they proved that the desired surface is either a double bubble or a torus bubble, and then used interval computations (as well as other ingenious numerical techniques) to prove that for all possible values of parameters, the area of the torus bubble exceeds the area of the double bubble described above.

This result was mentioned in a popular magazine *Discover* as one of the main scientific achievements of the year.

This application of interval mathematics not only provides a solution to a long-standing mathematical problem; the authors also describe potential practical applications, one of the them: to the design of the lightest possible double fuel tanks for rockets.

The paper is not yet published. A preprint is available from the authors.

Brief technical description of the main idea. The intervals are used in the proof as follows: First, it is proven that a torus bubble is uniquely determined by two parameters: $\theta_1 \in [0^\circ, 90^\circ]$ and $h_0 \in [-8, -2]$. These two parameters have explicit geometric meaning: θ_1 is the angle of one of the arcs, and h_0 is the mean curvature of the lower surface. To show that torus bubbles are not optimal, the authors check that no area minimizing torus bubble can enclose equal volumes, i.e., that the enclosed volumes V_1 and V_2 are different.

The desired volumes V_1 and V_2 can be computed by a sequence of explicit integrations; these integrations start with expressions that contain θ_1 and h_0 as parameters. To show that $V_1 \neq V_2$ for all θ_1 and h_0, i.e., that the difference $\Delta V(\theta_1, h_0) = V_1(\theta_1, h_0) - V_2(\theta_1, h_0) \neq 0$ for all $\theta_1 \in [0^\circ, 90^\circ]$ and $h_0 \in [-8, -2]$, the authors show that $0 \notin [\Delta V]([0, 90], [-8, -3])$ for an appropriate interval enclosure $[\Delta V]$.

For given intervals $[\theta_1]$ and $[h_0]$, we can compute the naive interval enclosure for ΔV, if we replace step-by-step each elementary operation ($+$, $-$, $\times$, $/$, $\sqrt{\ }$, etc.) with a corresponding interval operation, and replace the integral with an interval formed by its lower and upper Riemann sums:

$$\int [f(x)]\,dx \subseteq [I^-, I^+]$$

where

$$\begin{aligned} I^- &= \sum f^-([x_i, x_{i+1}]) \cdot (x_{i+1} - x_i), \\ I^+ &= \sum f^+([x_i, x_{i+1}]) \cdot (x_{i+1} - x_i) \end{aligned}$$

and $f^{\pm}$ are the bounds of the naive interval enclosure $[f]$ of the function f: $[f] = [f^-, f^+]$.

If we apply this procedure to the initial (wide) intervals $[0, 90]$ and $[-8, 3]$, then the resulting interval for ΔV is an overestimation that contains 0. To avoid this overestimation, the authors divide each interval into subintervals (of length 1 for degrees and of length 0.1 for curvatures), and apply the naive algorithm described above to each pair of resulting subintervals. If for one of the pairs, the resulting interval $[\Delta V]$ still contains 0, we divide both subintervals $[\theta_1]$ and $[h_0]$ into two equal parts, and repeat the computation for each of the four resulting pairs. If for one of these pairs, we get $0 \in [\Delta V]$, we bisect these subintervals again, etc.

It turned out that all these bisections eventually stopped. As a result, the original rectangle $[0, 90] \times [-8, 3]$ is covered by $\approx 23{,}000$ rectangles, and for each of these rectangles, $0 \notin [\Delta V]$. Hence, $\Delta V \neq 0$ for all $\theta_1 \in [0, 90]$ and $h_0 \in [-8, 3]$ and therefore, the torus bubble with the smallest area cannot enclose two equal volumes. Thus, the surface that minimizes the area enclosing the two equal volumes cannot be a torus bubble and must, therefore, be a double bubble.

Some technical details. Many innovative parts of this paper are not in interval computations themselves, but in transforming the problem to the form in which interval computations can be applied; this includes the results that the desired surface is either the torus bubble or the double bubble; that a torus bubble is uniquely determined by two parameters; that the values of these parameters belong to certain intervals, and that the volumes enclosed by the torus bubble can be described in terms of integrals. The original integrals diverge for $\theta_1 \to 0$, so, for $\theta_1 \approx 0$, a different expression has to be used.

Several ideas are also used on the interval computations stage, to speed up the computations:

- First, the authors prove that some inequalities must hold for the surface with the smallest area; the quantities involved in these inequalities are easier to compute than ΔV. Because of that, for each rectangle, first, these inequalities are checked. If they are not satisfied, then this rectangle is rejected without actually computing ΔV.
- Second, it turns out that for $h_0 \leq -3$, an interval of width > 0.1 is often sufficient in showing that $0 \notin [\Delta V]$ and thus, we do not need to analyze narrower subintervals. Hence, to reduce the total computation time, the authors start with intervals of length 0.1 for $h_0 \geq -3$ and with intervals of length 0.2 for $h_0 \leq -3$.

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