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Ordering events: Intervals are sufficient, more general sets are usually not necessary

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Traditionally, an interval is used to describe incomplete knowledge about a moment of time when an event occured. In principle, more general sets are sometimes needed to describe our knowledge. In this paper, we show that if we are only interested in the ordering of events, then intervals are sufficient. This result provides one more justification for the use of the intervals.

Для упорядочивания событий достаточно интервалов, более общие виды множеств, как правило, не нужны

А. ПРОВЕТТИ

Как правило, для представления неполного знания о моменте времени, в который произошло некоторое событие, используется интервал. В принципе, иногда для представления знаний требуются более общие виды множеств. В работе показано, что для решения задачи упорядочивания событий достаточно интервалов. Таким образом, дается еще одно обоснование использования интервалов.

1. Informal introduction

1.1. Complete knowledge about the dating of events can be described by real numbers

In physics, real numbers are used to describe the moments of time when different events occur. Real numbers correspond to the case when we have a complete knowledge about the dating of an event.

1.2. To describe partial knowledge, we need sets of real numbers

In real-life situations, our information about the time of different events comes from two sources: from measurements and from expert estimates. Measurements are never absolutely precise; therefore, after each measurement, there are usually many different moments of time that are consistent with it. For example, if the measured time is 15 sec, and the accuracy of the measurement is ± 1 sec, this means that the actual time could take any value from the interval $[14, 16]$. Similarly, expert estimates are also not precise, and therefore, instead of a single moment of time, they describe a *set* of possible values of time.

1.3. Intervals are most frequently used to describe partial knowledge

As we have just mentioned, the most natural description of such a set is an interval that contains it. The use of intervals for describing time (and other physical quantities) dates back to Norbert Wiener [13, 14]. Intervals are actively used to describe uncertainty in measurements (see, e.g., [7–9]).

In particular, starting from the fundamental papers of J. Allen [1–4], intervals are one of the basic formalisms in reasoning about time. In Allen's Interval Logic and in interval variations of event calculus (see, e.g., [5]), intervals are used for two different goals: to describe *duration* of "long" events, and to describe *uncertainty* with which we know the moment of time of an instantaneous ("short") event. The basis application of intervals in describing uncertainty (i.e., in dating) is as follows: if we know that some "long event" (e.g., an AI class) took place during a certain interval of time $[t^-, t^+]$ (e.g., from 9 a.m. to 10 a.m.), and we know that some "short event" E occured during this long event (e.g., a student turned in her paper during that class), then, if this is the only information we have about the event E, we can only say that the (unknown) actual time of the event E belongs to the interval $[t^-, t^+]$.

To describe the *duration* of a continuous process, we clearly need an interval. The natural question is: for dating "short" events (i.e., for describing *uncertainty*), *are intervals sufficient, or we need more general sets?*

1.4. In general, to describe uncertainty, we need sets that are more general than intervals. Case study: declarative planning in robotics

The reason for the above question is as follows:

Allen introduced Interval Logic as a means of formalizing commonsense reasoning about the temporal ordering of different events. The ultimate goal of such a formalization is to design a computer-based intelligent agent that would be able to reason about events in time; in particular, this agent must be able to do *declarative planning*. Usually, planning algorithms (described in *operation research*) are *procedural* in the sense that they assume that the planning problem belong to a certain class (e.g., a class of graph flow problems), for which the planning algorithm is already known. The goal of declarative planning is to design a plan in a general situation, i.e., design a plan based on the *declarative* (as opposed to operational) knowledge about the possible consequences of different actions. One of the main intended applications of such an activity is the design of intelligent *robots*.

An efficient robot must not only design a plan and follow it, this robot must constantly *adapt* this plan to the changing environment. For that, the robot must constantly measure the (dynamically changing) characteristics of the environment in which it operates. The intelligent agent must *process* these values in order to update the plan.

Standard data processing algorithms of numerical mathematics simply process the measured values as if they were absolutely precise. In reality, however, measurements are never absolutely precise; due to the measurement imprecision, the measured value can differ from the actual value. For example, if we measure time with an accuracy 1 sec, and the measured value is 15, then the actual value could be 15.0 or 14.2, or 15.8. It is natural to require that the result of processing the measured values should not depend (or at least, depend as little as possible) on whether we input 15.0, 14.2, or 15.8. In more general terms, we want a small modification of the input not to lead to a major change in the plan. This requirement is called *elaboration tolerance* (see, e.g., [6, 11]).

In view of this requirements, it is necessary to take into consideration that after each measurement (in particular, after each measurement of time), we know not the exact value of the measured quantity, but a *set* of possible values of this quantity.

What kind of set can it be? It is definitely true that for a measuring device to be meaningful, its manufacturer must *guarantee* some accuracy of the resulting measurements. In other words, the manufacturer must provide us with a value $\Delta > 0$ such that the difference $\Delta t = \tilde{t} - t$ between the measured and the actual times cannot exceed Δ: $|\Delta t| \le \Delta$. Indeed, if such a Δ did not exist, then after "measuring" $\tilde{t}$, we would not be able to conclude anything at all about the actual time t.

After we have measured time, we know the measured value $\tilde{t}$, and we know the accuracy Δ of our measurement. Therefore, the set of possible *actual* values of the moment of time must be a subset of the interval $[\tilde{t} - \Delta, \tilde{t} + \Delta]$. In some cases, we also know probabilities of different values from this intervals, but for sensors used in robots, probabilities are usually not known.

The fact that the actual values must belong to the interval given above does not mean that *any* number from this interval is a possible actual date of the measured event. It could happen that the measuring device that we are using is so designed that it either underestimates or overestimates but never gives the precise values.

Let us illustrate this possibility on an example of a time-measuring device on board of a robot. To make robotic movements predictable and sufficiently precise, robots are usually equipped with "discrete" ("step") electric motors, i.e., motors that at any given moment of time can be in one of the finitely many states; e.g., in the simplest case, f ("forward"), b ("back"), and s ("stop"). The magnetic field generated by an engine influences the sensors, and is thus one of the potential sources of measurement errors. Since the engine can be in only finitely many states, the resulting error can take only finitely many possible values. For example, in case of the above-mentioned three states, it can only takes the values e_f, e_b, and e_s that correspond to these states. Therefore, if this magnetic field is the major source of error, we have only two possible values of error. Hence, after having measured $\tilde{t}$, we conclude that the actual value of t can take only three values: $\tilde{t} - e_f$, $\tilde{t} - e_b$, and $\tilde{t} - e_s$, but *not* the values in between.

This is not a typical situation in measurements, because, as shown in [10], under some reasonable assumptions, possible values of the measured quantity do form an interval. However, in several realistic situations, the set of possible values is different from an interval.

Similarly, evidential knowledge can also lead to sets that are different from intervals: e.g., we may say that a murder occured either between 9 and 10, or between 11 and 12, excluding the time when the witness was passing by and saw nothing suspicious.

1.5. Do we need sets different from intervals when describing ordering of events?

Of course, if we are interested in the actual date of each event, then we have to consider the actual set of possible date of each event, and thus, the answer to the above questions is a trivial "yes": yes, we have to consider sets that are more general than intervals (e.g., the set $[9, 10] \cup [11, 12]$ from the above example).

In many situations, however, we are only interested in the *order* of the events (e.g., these were the situations analyzed by Allen in his pioneer work). For this situations, the above question is not so trivial: *if we are only interested in describing the ordering of events, are intervals sufficient, or we need more general sets to date events?* In this paper, we will show that, in contrast to the general case, for orderings, intervals are sufficient.

2. Definitions and the main result

Denotations.

- *In this paper, we will consider arbitrary closed bounded sets as representing our uncertainty. These sets will be denoted by capital letters ($A, B, \ldots$), and their elements (i.e., real numbers that describe the dates) by lowercase letters ($a, b, \ldots$).*
- *The supremum $\sup A$ of a set A will be denoted by A^+, and its infimum $\inf A$ by A^-.*

Motivation of the following definition. In order to formulate our main result, we will describe what we mean by ordering of events A and B. This description will be done is three steps:

- If we know the exact dates of both events, i.e., when the moment of time that corresponds to each event is a real number: $A = \{a\}$ and $B = \{b\}$, then we have the following basic ordering relations $a \odot b$: $a < b$, $a \leq b$, $a > b$, or $a \geq b$.
- If we know the exact timing of one of the events, i.e., if $A = \{a\}$, then, for each of the four ordering relations $\odot$ between numbers, we can describe two different ordering relations $\bigodot$ between a and B, by postulating either of the two:
  1) *it is possible* that a is in relation $\odot$ with the actual moment $b \in B$, i.e., that $\exists b \in B\,(a \odot b)$;
  2) *it is necessary* that a is in relation $\odot$ with the actual moment $b \in B$, i.e., that $\forall b \in B\,(a \odot b)$.
- Similarly, for each of the eight possible ordering relations $\bigodot$ between a number a and a set B, we can describe two different ordering relations $\bigcirc$ between A and B, again by postulating either of the two:
  1) *it is possible* that the actual moment of time $a \in A$ is in relation $\bigodot$ with the set B, i.e., that $\exists a \in A\,(a \bigodot B)$;
  2) *it is necessary* that a is in relation $\bigodot$ with B, i.e., that $\forall a \in A\,(a \bigodot B)$.

We have thus defined sixteen *basic* ordering relations between the sets. It can so happen that our knowledge consists of *several* basic relations; in this case, we can define a *generic* ordering relation $A \circ B$ between the sets as a propositional combination of basic ordering relations.

Let us describe this idea as a precise mathematical definition.

Definition 1.

- *A basic ordering relation $a \odot b$ between real numbers a and b is one of the following four relations: $a < b$, $a \leq b$, $a > b$, $a \geq b$.*

- *A basic ordering relation $a \bigodot B$ between a real number a and a set B is a relation of the type $\exists b \in B\,(a \odot b)$ or $\forall b \in B\,(a \odot b)$, where $\odot$ is a basic ordering relation between real numbers.*
- *A basic ordering relation $A \bigcirc B$ between sets A and B, is a relation of the type $\exists a \in A\,(a \bigodot B)$ or $\forall a \in A\,(a \bigodot B)$, where $\bigodot$ is a basic ordering relation between a real number and a set.*
- *An ordering relation $A \circ B$ between sets A and B, is an arbitrary propositional combination of basic ordering relations, i.e., any relation that can be obtained from basic ordering relations $A \bigcirc B$ by using propositional connectives & ("and"), $\vee$ ("or"), and $\neg$ ("not").*

Theorem 1. *If A and B are bounded closed sets, and $\circ$ is an ordering relation (in the sense of the above definition), then*

$$A \circ B \Longleftrightarrow [A^-, A^+] \circ [B^-, B^+].$$

Comments.

1. This theorem shows that for a given ordering relation $\circ$, and for any two sets A and B, A and B are in a given ordering relation if and only if the same relation holds between the two intervals: the smallest interval $[A^-, A^+]$ that contains A and the smallest interval $[B^-, B^+]$ that contains B. Therefore, if we are only interested in ordering of the events, we can consider intervals $[A^-, A^+]$ and $[B^-, B^+]$ instead of the sets A and B. In other words, *if we are only interested in describing the ordering of events, then intervals are sufficient, and we do not need more general sets to date events.*
2. If we are interested not only in ordering, but also in other relations between events, then intervals may no longer be sufficient. For example, describing equality leads to the following relations:
   - $a = b$;
   - $\exists b \in B\,(a = b)$ (meaning $a \in B$); and
   - $\forall a \in A\ \exists b \in B\,(a = b)$ that is equivalent to $\forall a \in A\,(a \in B)$, i.e., to $A \subseteq B$.

   The final relation is no longer equivalent to a similar relation between intervals $[A^-, A^+]$ and $[B^-, B^+]$; e.g., for $A = [0, 1]$ and $B = \{0, 1\}$:
   - $A = [0, 1] \not\subseteq B = \{0, 1\}$, while
   - $[A^-, A^+] = [0, 1] \subseteq [0, 1] = [B^-, B^+]$.

3. Proof

To prove our theorem, let us show that all basic ordering relations between a number and a set and between two sets can be reformulated in terms of the infimum and supremum of these sets.

1°. Let us first prove that $\exists b \in B\,(a < b)$ is equivalent to $a < B^+$.

Indeed, since we only consider closed bounded sets, our set B contains its own supremum and infimum ($B^- \in B$ and $B^+ \in B$). So, if $a < B^+$, then, due to $B^+ \in B$, we have $\exists b \in B\,(a < b)$.

Vice versa, if there exists a $b \in B$ such that $a < b$, then due to $b \leq B^+$, we can conclude that $a < b \leq B^+$ and $a < B^+$.

2°. $\exists b \in B\,(a \leq b) \leftrightarrow a \leq B^+$.

Indeed, if $a \leq B^+$, then due to $B^+ \in B$, we get $\exists b \in B\,(a \leq b)$.

Vice versa, if there exist a $b \in B$ such that $a \leq b$, then from $a \leq b \leq B^+$, we can conclude that $a \leq B^+$.

Similarly, we can prove the following equivalences $3^o - 8°$:

3°. $\exists b \in B\,(a > b) \leftrightarrow a > B^-$.

4°. $\exists b \in B\,(a \geq b) \leftrightarrow a \geq B^-$.

5°. $\forall b \in B\,(a < b) \leftrightarrow a < B^-$.

6°. $\forall b \in B\,(a \leq b) \leftrightarrow a \leq B^-$.

7°. $\forall b \in B\,(a > b) \leftrightarrow a > B^+$.

8°. $\forall b \in B\,(a \geq b) \leftrightarrow a \geq B^+$.

9°. Similarly, when we add a quantifier over $a \in A$ to one of these relation, we can replace the relation with an equivalent one that connects $B^\pm$ with $A^\pm$ and does not contain any quantifier.

For example, to describe $\exists a \in A\, \exists b \in B\,(a < b)$ in this manner, we proceed to:

- represent $\exists b \in B\,(a < b)$ in terms of the bounds, obtaining $a < B^+$, and, by substitution into the initial formula, $\exists a \in A\,(a < B^+)$;
- invert $a < B^+$ to $B^+ > a$, thus getting $\exists a \in A\,(B^+ > a)$, and use one of the above formulas (actually, formula 3°) to represent this relation as $B^+ > A^-$.

10°. Since each basic ordering relation is described only in terms of the bounds $A^\pm$ and $B^\pm$, an arbitrary ordering relation $\circ$, defined as a propositional combination of the basic ones, can also be expressed solely in terms of the bounds. Therefore, for any other two sets $\tilde{A}$ and $\tilde{B}$ with the same bounds (i.e., with $\tilde{A}^- = A^-$, $\tilde{A}^+ = A^+$, $\tilde{B}^- = B^-$, and $\tilde{B}^+ = B^+$), we obtain $A \circ B$ iff $\tilde{A} \circ \tilde{B}$. In particular, this is true for the intervals $\tilde{A} = [A^-, A^+]$ and $\tilde{B} = [B^-, B^+]$. □

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