Reliable Computing 2 (3) (1996), pp. 287–297

If we measure a number, we get an interval. What if we measure a function or an operator?

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Assume that we measure a physical quantity x with a measuring device whose accuracy is δ (i.e., whose producers guarantee that the difference $x - \tilde{x}$ between the actual value x and the measured value $\tilde{x}$ does not exceed δ). If the result of this measurement is $\tilde{x}$, then possible values of x form an interval $[\tilde{x} - \delta, \tilde{x} + \delta]$.

Suppose now that we know that a physical quantity y is a function of the physical quantity x (in other words, we know that $y = f(x)$ for some function $f(x)$), but we do not know f. How to determine f? We can measure only finitely many values, with finite precision, so, after finitely many measurements, we get a set of possible functions $f(x)$. This set can be called a *function interval* (function intervals were first analyzed by R. Moore himself).

The situation can become even more complicated. For example, if we analyze how physical fields evolve, then in addition to functions, we must describe *operators*, i.e., mappings that transform a function (current value $f(\vec{x})$ of a physical field) into a function (predicted future value of this field). Again, since we can perform only finitely many measurements, at any moment of time, our measurement results are consistent with the whole bunch of different operators. So, at any moment of time, we have a set of operators; we can call it an *operator interval*.

One can apply different ideas to describe function intervals, operator intervals, etc. But it is desirable to develop a general formalism that would cover all these cases. In this paper, we propose and justify such a formalism.

Измерив число, получим интервал. Что получим, измерив функцию или оператор?

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Предположим, что измеряется физическая величина x с помощью измерительного устройства, дающего погрешность δ (т. е. изготовители этого устройства гарантируют, что разность $x - \tilde{x}$ между реальным значением x и измеренным значением $\tilde{x}$ не превышает δ). Если результат измерений равен $\tilde{x}$, возможные значения x лежат в интервале $[\tilde{x} - \delta, \tilde{x} + \delta]$.

Предположим, что известна физическая величина y, являющаяся функцией от физической величины x (другими словами, мы знаем, что $y = f(x)$ для некоторой функции $f(x)$), но сама функция f неизвестна. Как определить f? Мы можем измерить только конечное число значений с ограниченной точностью, так что после конечного числа измерений мы получим множество возможных функций $f(x)$. Это множество называют *функциональным интервалом* (такие интервалы были впервые исследованы Р. Муром).

Ситуация может стать еще более сложной. Например, если мы исследуем изменение физического поля, то, кроме функций, мы должны описать *операторы*, т. е. отображения, которые переводят одну функцию (текущее значение поля $f(\vec{x})$) в другую (предсказанное будущее значение

This work was supported by NSF grant No. CDA–9015006, NASA Research Grant No. 9–482, and a grant from the Materials Research Institute. The authors also wish to thank the participants of Leningrad seminar on mathematical logic and constructive mathematics, especially N. A. Shanin, for valuable discussions.

этого поля). И опять, из-за того, что мы можем выполнить только конечное число измерений, в каждый момент времени результаты наших измерений согласуются с целой группой различных операторов. Таким образом, в каждый момент времени мы имеем множество операторов, которое можно назвать *операторным интервалом*.

Можно применять различные подходы для описания функциональных интервалов, операторных интервалов и т. д. Но желательно разработать обобщенную формальную систему, приложимую ко всем этим случаям. В работе предлагается и обосновывается такая формальная система.

1. Introduction

When we measure a quantity that is characterized by a real number, intervals are appropriate for describing measurement results. Suppose that we measure a physical quantity x (e.g., length l). The actual value x of this quantity is a real number. The result $\tilde{x}$ that is produced by a real measuring device is always approximate. The producers of measuring devices supply them with the accuracy estimates. In other words, they give a value δ, and they guarantee that the absolute value of the difference $x - \tilde{x}$ between the actual value x and the measured value $\tilde{x}$ does not exceed δ. So, if we apply a measuring device, and get $\tilde{x}$ as a result, then the possible values of the physical quantity x form an interval $[\tilde{x} - \delta, \tilde{x} + \delta]$ (see, e.g., [5–7]).

What if we want to determine an unknown function experimentally? Suppose now that we know that a physical quantity y is a function of a physical quantity x (in other words, we know that $y = f(x)$ for some function $f(x)$), but we do not know this function. How to determine f? Again, since we can measure only finitely many values, with finite precision, so, after finitely many measurements, we get a set of possible functions $f(x)$. This set can be called a *function interval*. Function intervals were first considered by R. Moore (see, e.g., [5, Section 5.1; 6, Section 2.5]).

A more complicated case: how to describe the uncertainty with which we know an operator? The situation can become even more complicated. For example, if we analyze how physical fields evolve, then in addition to functions, we must describe *operators*, i.e., mappings that transform a function (current value $f(\vec{x})$ of the physical field) into a function (predicted future value of this field). Again, since we can perform only finitely many measurements, at any moment of time, our measurement results are consistent with the whole bunch of different operators. So, at any moment of time, we have a set of operators: an *operator interval*.

Physical examples in which these problems are important. These problems are especially important for quantum mechanics, where to describe even a single particle, we need a field $\psi(\vec{x})$ (called a *wave function*).

Even more complicated mathematical structures appear in quantum field theory and in quantum theory of space-time.

Formulation of the problem. One can apply different ideas to describe function intervals, operator intervals, etc. But *it is desirable to develop a general formalism* that would allow us, given a natural definition of an interval for the sets X and Y, to design an appropriate definition of an interval for the set Y^X of all the functions from X to Y.

What we are planning to do. In this paper, we propose such a general definition, and show that it is physically natural.

As a basis of our definition, we take a theory of *semantic domains* developed by Dana Scott to describe efficiency in mathematics and computer science [8, 9].

Some preliminary results of this paper appeared in [2].

2. What is a semantic domain?

Main idea. The theory of semantic domains was developed by D. Scott [8, 9] to provide a semantics for programming languages.

Its main idea is as follows. Let's assume that we are analyzing a class X of objects. These objects can be real numbers, or programs given as "black boxes" (so that we can use (i.e., call) them, but we have no access to their source codes), or real-life objects.

In all these cases, at any given moment of time, we have only a finite information about an object (finite in the sense that it can be represented inside a computer as a finite sequence of 0's and 1's). This information can be obtained from the measurements (if we consider real-life objects), from experts, from computer experiments (if we are talking about programs), etc. In the majority of the cases, this information I does not determine an unknown object x uniquely. In other words, the set $X(I)$ of all the objects from X that are consistent with this information consists of more than one element of X.

There are denumerably many different finite sequences of 0 and 1. Therefore, there are only denumerably many different informations. So, we arrive at the following structure: $(X, \{X(I)\})$, where X is a set that is called a *domain*, and $\{X(I)\}$ is a denumerable sequence of subsets of X. There is a set X, and a denumerable sequence of its subsets $X(I) \subseteq X$. The sets $X(I)$ are called *approximations*.

Intervals: an example of a semantic domain. Let's consider the case when we are measuring a physical quantity that is characterized by a real number. We usually have several different measuring devices for measuring a quantity. Very often, the only information that we have about each of these devices is the guaranteed total accuracy.

In this case, the only possible information about an actual value x comes from the measuring devices. Suppose that we have performed measurements with n devices. The accuracy of i-th device is δ_i, the result of i-th measurement is $\tilde{x}_i$. From the fact that the result of i-th measurement is $\tilde{x}_i$, we conclude that x belongs to an interval $[\tilde{x}_i - \delta_i, \tilde{x}_i + \delta_i]$. After n measurements, we can conclude that x belongs to n such intervals. Therefore, the set $X(I)$ of possible values of x is the intersection of these intervals, i.e., an interval

$$X(I) = [\max(\tilde{x}_i - \delta_i), \min(\tilde{x}_i + \delta_i)].$$

So, here, *approximations are intervals*. Let's show that not all intervals are approximations. Indeed, modern measuring devices are hooked up to computers: they generate a measuring result as a binary fixed-point number. In other words, a binary representation of a number $\tilde{x}_i$ is a finite sequence of 0's and 1's (e.g., 0.1010011). All these numbers have the form $p/2^q$ for some integers p and q. Such numbers are called *binary-rational*.

The accuracy δ_i of a measuring device is also estimated by a computer (as a result of automated testing), so it is also a binary-rational number. Therefore, both endpoints of the above-described interval $X(I)$ are binary rational.

In this case, we have a semantic domain in which the domain X is the set of all real numbers R, and approximations are arbitrary intervals with binary-rational endpoints.

How is a domain of functions defined? Suppose that we have already defined semantic domains that correspond to domains X and Y. In other words, we have defined approximations $X(I)$ and $Y(J)$ for both domains. What if we now consider as a domain Z the set Y^X of all possible functions from X to Y: what approximations to use?

For D. Scott, the main objects of interest were programs. In this case, a function $f : X \to Y$ is a program that calls $x \in X$, and as a result, generates a program $y \in Y$ (this is possible in many programming languages, including standard PASCAL). Hence, we can get an approximation to this program f by observing what it does for different x.

For any $x \in X$, during a finite time interval, this algorithm f can generate only a finite information about $f(x)$. In other words, after a finite interval of time, this algorithm will actually produce an approximation Y_1 to a program $f(x)$. As an input data, this algorithm can use only a finite information about x. In other words, it uses only some approximation X_1 that contains x. Since this algorithm uses only this information X_1, for all other values $x \in X_1$, it will produce the same approximation Y_1. In other words, if $x \in X_1$, then $f(x) \in Y_1$. We can also rewrite this condition as $f(X_1) \subseteq Y_1$ (where $f(X_1)$ denotes an image of X_1 under f).

So, after the first observation, the only information about f that we have is that $f(X_1) \subseteq Y_1$ for some X_1 and Y_1. We can repeat this experiment several times, with different x. For each experiment, we get a pair of approximations (X_i, Y_i). After n experiments, we know that $f(X_i) \subseteq Y_i$ for all $i = 1, 2, \ldots, n$. So, we arrive at the following definition:

We can define approximations on Z as follows: an *information* I is a finite sequence of pairs of approximations $(X_i, Y_i), 1 \leq i \leq n$, and $Z(I)$ is a set of all functions $f : X \to Y$ such that $f(X_i) \subseteq Y_i$ for $i = 1, 2, \ldots, n$.

Functionals, operators, etc. If we apply this construction once again, we can define the notion of an approximation for the set of all functions from Y^X to X (i.e., for the set of all *functionals*), or for the set of all functions from Y^X to Y^X (i.e., for the set of all *operators*).

In the next section, we will apply this idea to the case when the initial domains X and Y are interval domains, and show that the resulting definition is physically meaningful.

3. What do we know about a function after finitely many measurements?

In many cases, it is necessary to determine a function experimentally. If we know that a physical quantity y is a function of another physical quantity x ($y = f(x)$), but we do not know f, then we have to determine $f(x)$ experimentally. The only way to do that is to measure both x and y in different situations.

Example. Suppose that we know that for some conductor, the voltage V is a function of a current I: $V = f(I)$ for an unknown f. In the majority of the cases, Ohm's law $f(I) = RI$ is a good approximation, but there are also many *non-Ohmic* materials for which f is non-linear and unknown. To determine f, we measure voltages and currents in several situations, and try to reconstruct f from the resulting data.

Measurement results. As a result of each measurement, we get two values: $\tilde{x}_i$ and $\tilde{y}_i$. Taking into consideration the accuracies δ_i^x and δ_i^y of these measurements, we conclude that the actual value of x belongs to an interval $X_i = [\tilde{x}_i - \delta_i^x, \tilde{x}_i + \delta_i^x]$, and the actual value of y belongs to an interval $Y_i = [\tilde{y}_i - \delta_i^y, \tilde{y}_i + \delta_i^y]$. After n measurements, we get n pairs of intervals (X_i, Y_i).

Two different situations. For this problem, two different situations are possible:

1) *We know from some theoretical considerations that y is functionally dependent on x. From these same theoretical considerations, we may also have some additional knowledge of f*: e.g., we may know

that f is monotonic, that f is smooth (or even analytical), that it may satisfy some integral inequalities, etc.

In this situation, if as a result of a measurement we got X_i and Y_i, this means that the actual value x was such that $x \in X_i$ and $f(x) \in Y_i$. In other words, a function f is such that:

- for every i, there exists an x such that $x \in X_i$ and $f(x) \in Y_i$ (i.e., the graph of f has a point in common with the set $X_i \times Y_i$), and
- this function f must satisfy some theoretically motivated additional conditions.

Since these theoretical conditions can be very complicated, the resulting description of a "function interval" (set of all possible functions f) can be very complicated, and in the present paper, we will not analyze it.

In this paper, we will consider a situation that is simpler to analyze: namely, a situation when there is no preliminary theoretical knowledge of the relationship between x and y.

2) *We have no preliminary knowledge of the relationship between x and y.* In this case, the only information that we have consists of the measurement results. So, the only possibility to conclude that y is functionally dependent on x is to make this conclusion based on the measurement results.

How can this be done? How, e.g., can we arrive at a conclusion that V is functionally dependent on I? We repeat measurements several times; in several different experiments we have the same value of current, say, 1 A. If we notice that in all these cases, the value of the voltage is also the same (e.g., 2 V), then we conclude that whenever the actual value of the current is consistent with the measurement result 1 A, the actual value of the voltage will be ≈ 2 V.

We can now formulate a general hypothesis that voltage V is a function of current I. To check this hypothesis, we can analyze other cases in which I was the same. If in all such situations, equal values of I lead to equal values of V, then our hypothesis is confirmed, and we can conclude that y (in this case, voltage) is indeed functionally dependent on x (in this case, on the current).

This means that whenever we have a pair of intervals (X_i, Y_i) as a result of a measurement, we usually have not only one, but several measurements. For each of these measurements, the actual value x was inside X_i, and the actual value of $y = f(x)$ was inside Y_i. After observing all these measurement results, we make a general conclusion: whenever x is in X_i, we have $f(x) \in Y_i$. In other words, we conclude that $f(X_i) \subseteq Y_i$.

Comments.

1. This is not a mathematically valid deduction-type conclusion, but this is a typical example of what physicists call *induction*: extracting general laws from examples. This is a typical way how laws of physics are obtained from the experimental data.
2. The same definition can be applied to the case when X and Y are not necessarily the sets of real numbers, but other sets, for which the notion of an interval is already defined.

As a result, we arrive at the following definition of a function interval.

4. Definition of a function interval, its relation to semantic domains, and algorithms that handle these function intervals

Definition 1. *Suppose that we have two sets X and Y. In each of these sets, a family of subsets is chosen; subsets from these families will be called intervals. By a measurement information I (that corresponds to a function from X to Y), we mean a finite list of pairs of intervals (X_i, Y_i), $1 \le i \le n$. For every I, we can define the set $Z(I)$ of all the functions $f : X \to Y$, for which $f(X_i) \subseteq Y_i$ for $i = 1, 2, \ldots, n$. For $Z = Y^X$, by an interval, we will understand a set $Z(I) \subseteq Z$ for some measurement information I.*

Comment. One can easily see that this definition is exactly the one given by D. Scott in his theory of semantic domains! So, our previous section actually provides a physical justification of that definition.

Important case: function intervals. If we take $X = Y = R$, and actual intervals $[a, b]$ with binary-rational endpoints as "interval" subsets of X and Y, then we arrive at the definition of a *function interval* as a set

$$Z(I) = \{f : R \to R \mid f(X_i) \subseteq Y_i \text{ for all } i = 1, 2, \ldots, n\}$$

for some sequences of intervals X_i and Y_i.

How to handle these function intervals? To define a function interval is half of the task. We are actually interested in processing them. So, let us show how, given such an interval (i.e., the sequence of pairs (X_i, Y_i)), we can algorithmically find answers to natural questions about an unknown function.

Namely, we are interested in the following questions: *can this function f be constant?* (i.e., is there a constant function in $Z(I)$?) *can it be monotonic? If it is not monotonic, then how many local extrema can it have and where are they located?* In this paper, we will present fast algorithms to solve these questions.

Comment. For the case when measurements of x are absolutely precise (i.e., the error in x_i is negligible), these questions have been studied in our previous papers [3, 4, 10]. Algorithms that we present here are thus generalizations of the ones presented in those papers. The existence of these generalizations does not mean, however, that the original algorithms are now useless: these algorithms have been designed for a special case, and for that special case they are faster than our more general ones.

First stage: pre-processing a function interval. The fact that we do not know f means that for every $x \in X$, we know only an interval of possible values $f(x)$. If x belongs to only one interval I_i, then the interval of possible values of $f(x)$ is Y_i. If x belongs to several intervals $I_i, I_j, \ldots$, then for such x, we have $f(x) \in Y_i$, $f(x) \in Y_j$, etc. So, the set of possible values of $f(x)$ is an intersection $Y_i \cap Y_j \cap \cdots$. Before we start processing intervals, let us first compute these intersections for all $x \in X$.

The necessity for this "pre-processing" appears when the intervals X_i have a non-empty intersection. So, after pre-processing, we will have a new sequence of intervals $\bar{X}_j$ that can have at most one point in common.

Definition 2. *We say that a function interval $Z(I)$, where $I = \{(X_j, Y_j)\}, 1 \le i \le n$, where $X_j = [x_j^-, x_j^+]$ and $Y_j = [y_j^-, y_j^+]$, is pre-processed if $x_j^+ \le x_{j+1}^-$.*

Definition 3. *We say that function intervals $Z(I)$ and $Z(\bar{I})$ are equivalent, if they contain the same sets of functions (i.e., if $Z(I) = Z(\bar{I})$).*

Theorem 1. *There exists a quadratic-time algorithm that transforms an arbitrary function interval into an equivalent pre-processed one.*

Comment. An algorithm is called *quadratic-time* (see, e.g., [1]), if there exists a constant C such that for every n, the number of elementary computational steps [1] that this algorithm requires for input $I = \{(X_i, Y_i)\}$, $1 \le i \le n$, does not exceed Cn^2.

Algorithm. First, order $2n$ endpoints of n intervals X_i into an increasing sequence $x_1 < x_2 < \cdots < x_m$, $m \le 2n$. This ordering can be done in $O(n \log_2 n)$ computational steps (see, e.g., [1]).

For each j, all the values x from (x_j, x_{j+1}) belong to the same intervals I_i. If there are no intervals I_i for these x, then the set of possible values of $f(x)$ is the entire real line. If there are such I_i, then the interval of possible values of $f(x)$ is equal to the intersection of Y_i for all i such that $x \in X_i$. To compute these intersections, for each of $m \le 2n$ intervals (x_j, x_{j+1}), we must check whether this interval belongs to each of n intervals I_i (for each interval X_i, it takes 2 steps to compare, so totally, we need $2n$ steps for each j), and then compute the min and max of endpoints of x_i to get the endpoints of an intersection ($\le 2n$ steps). Totally, we need $\le m(2n + 2n) \le 8n^2$ computational steps.

After this pre-processing, we have a new sequence of intervals $[x_j, x_{j+1}]$ and the corresponding $y-$intervals $[y_j, y_{j+1}]$ (these values y_j can be $\pm\infty$) such that $f([x_j, x_{j+1}]) \subseteq [y_j, y_{j+1}]$. If we delete the intervals for which $y_j = \pm\infty$, we end up with the sequence of intervals $\bar{X}_j = [x_j^-, x_j^+]$ and $\bar{Y}_j = [y_j^-, y_j^+]$, such that $f(\bar{X}_i) \subseteq \bar{Y}_i$, and $x_j^+ \le x_{j+1}^-$. In other words, we have a pre-processed function interval.

Comment. In the following text, we will assume that the function interval is already given in this pre-processed form.

Theorem 2. *There exists a linear-time algorithm that, given a pre-processed function interval $Z(I)$, returns "yes" if and only if this interval contains a constant function.*

Algorithm. Compute $M = \min y_i^+$ and $m = \max y_j^-$. If $m \le M$, then answer "yes."

Comment. For reader's convenience, proofs are given in the next section.

Theorem 3. *There exists a linear-time algorithm that given a pre-processed function interval $Z(I)$, returns "yes" if and only if this interval contains a monotone non-decreasing function.*

Algorithm. Set $M := y_1^-$. Then, for $j = 2, \ldots, n$, do the following: check whether $y_j^+ \ge M$, and compute the new value $M := \max(y_j^-, M)$.

If for all j, the checked inequality is true, return "yes," else return "no."

Theorem 4. *There exists a linear-time algorithm that given a pre-processed function interval $Z(I)$, returns "yes" if and only if this interval contains a monotone non-increasing function.*

Algorithm. Set $m := y_1^+$. Then, for $j = 2, \ldots, n$, do the following: check whether $y_j^- \le m$, and compute the new value $m := \min(y_j^+, m)$.

If for all j, the checked inequality is true, return "yes," else return "no."

Comment. If a function is not monotonic, this means that it has local maxima or mimima. In many areas (radioastronomy, spectroscopy, particle physics, etc.), it is important to know the locations of these maxima [10].

Definition 4. *We say that a function $f(x)$ has a local maximum on an interval (x^-, x^+) if $f(x^-) < \sup f(x) > f(x^+)$, where* sup *is take over all $x \in [x^-, x^+]$. Likewise, we say that a function $f(x)$ has a local minimum on an interval (x^-, x^+) if $f(x^-) > \inf f(x) < f(x^+)$.*

Definition 5. *Suppose that a function interval $Z(I)$ is given. We say that an interval I locates a local maximum if any function $f \in Z(I)$ has a local maximum on I. We say that an interval I locates a local minimum if any function $f \in Z(I)$ has a local minimum on I. We say that an interval I locates a local maximum precisely, if I locates a local maximum, and no proper subinterval $I' \subset I$ locates it.*

Theorem 5. *There exists a linear-time algorithm that for a given pre-processed function interval, locates all local maxima and all local minima precisely.*

Algorithm. This algorithm consists of 3 different phases, between which we'll switch, and there will be a special variable s that indicates on what phase we are now. Possible values of s are -1, 0, or 1. These values have the following meaning:

- $s = 0$ means that the data that we have already processed is still consistent with the hypothesis that f is constant;
- $s = 1$ means that we are now in an interval on which f can be monotone non-decreasing;
- $s = -1$ means that we are now in an interval on which f can be monotone non-increasing.

The algorithm itself is as follows:

First, set $s := 0$, read the first interval Y_1 and set $m := y_1^+$ and $M := y_1^-$. Then, read all other intervals $Y_j, j = 2, 3, \ldots, n$ one by one and depending on the value of s do the following:

If $s = 0$, then check whether $M \le y_j^+$ and whether $y_j^- \le m$, and compute $M := \max(M, y_j^-)$ and $m := \min(m, y_j^+)$. If both checked inequalities are true, then leave s unchanged. Else, if the first inequality is false (i.e., $M > y_j^+$), set $s := -1$. If the second inequality is false (i.e., $y_j^- > m$), set $s := 1$.

If $s = 1$, then check the inequality $M \le y_j^+$. If it is true, compute $M := \max(M, y_j^-)$ and leave s unchanged. If it is false, then do the following:

i) for $k = j-1, j-2, \ldots$ compare y_k^+ with M until we find the value k, for which $y_k^+ < M$;

ii) for this k, output the interval (x_k^+, x_j^-) as an interval that locates a local maximum;

iii) set $s := -1$, $m := y_j^+$.

If $s = -1$, then check the inequality $m \ge y_j^-$. If it is true, compute $m := \min(m, y_j^+)$ and leave s unchanged. If this inequality is false, then do the following:

i) for $k = j-1, j-2, \ldots$ compare y_k^- with m until we find the value k for which $y_j^- > m$;

ii) for this k, output the interval (x_k^+, x_j^-) as an interval that locates a local minimum;

iii) set $s := 1$, $M := y_j^-$.

5. Proofs

Theorem 1 is proved in the main text.

Proof of Theorem 2. If a constant function $f(x) = c$ belongs to $Z(I)$, then $y_j^- \le c \le y_j^+$ for all j. Therefore, $m \le c \le M$, and $m \le M$.

Vice versa, if $m \le M$, then one can easily check that a function $f(x) = c = (m + M)/2$ belongs to $Z(I)$.

Proof of Theorem 3. Let us prove that this algorithm produces the correct result. In this algorithm, the value M_i of M after we have processed j intervals $Y_1, \ldots, Y_j s$, is $M_j = \max(y_1^-, y_2^-, \ldots, y_j^-)$.

If $Z(I)$ contains a non-decreasing function f, then for $k < j$, $f(x_k) \le f(x_j)$. Therefore, $y_k^- \le f(x_k) \le f(x_j) \le y_j^+$. So, $y_j^+ \ge y_k^-$ for all $k < j$, and hence $y_j^+ \ge \max(y_1^-, \ldots, y_{j-1}^-) = M_{j-1}$. So, in this case, our algorithm will answer "yes."

Vice versa, if an algorithm answers "yes," let us take define f as follows: for every $x < x_n^+$, we find the biggest j for which $x_j^- \le x$, and take $f(x) = M_j$. For $x = x_n$, we take $f(x_n) = y_n^+$. It is easy to check that this f is monotonic non-decreasing, and that it belongs to $Z(I)$ (i.e., that if $x \in X_j$, then $f(x) \in Y_j$).

Theorem 4 is proved similarly.

Proof of Theorem 5 (this proof is similar to the one in [10]).

1) Let us first prove that if this algorithm returns an interval (x_k^+, x_j^-) as containing a local maximum, then every function f from $Z(I)$ has a local maximum on this interval.

Indeed, our algorithm returns such an interval if $y_k^+ < M_{j-1}$ (here, we use the denotation M_j from the proof of Theorem 3), and $y_j^+ < M_{j-1}$.

By definition, M_{j-1} is the biggest of several consequent values y_l^-. Let us prove that $M_{j-1} = y_l^-$ for some l between k and j.

Indeed, suppose that it is not so. Then, $y_l^- < M_{i-1}$ for all $l = k+1, \ldots, j-1$. For $l = k$ and $l = j$, we have $y_l^+ < M_{j-1}$, hence $y_l^- < M_{j-1}$. According to the algorithm, the only way to increase M_j is to encounter the value y_l that is greater than M_{l-1}; in this case, the new value M_l is equal to y_l. So, if M_l was ever increased on one of the steps from k to j, we would have $M_{j-1} = y_l^-$ for that step l. Since this is not true, it means that for all these l, the value M_l did not increase. Hence, $M_{j-1} = M_{j-2} = \cdots = M_k = M_{k-1}$. So, $M_{j-1} = M_{k-1}$, and from $y_k^+ < M_{j-1}$, we conclude that $y_k^+ < M_{k-1}$.

But this inequality, according to our algorithm, would mean that we switched to phase $s = -1$ on the value y_k and would not have waited until y_j (as we did). This contradiction shows that our assumption that $M_{j-1} > y_l$ for all $l = k, k+1, \ldots, j$ is false. Therefore, there exists an l such that $k < l < j$, and $y_l^- = M_{j-1}$.

Now, let $f \in Z(I)$. This means, in particular, that $f(x_k^+) \le y_k^+$, so from $y_k^+ < M_{j-1}$, we conclude that $f(x_k^+) < M_{j-1}$. Likewise, $f(x_j^-) < M_{j-1}$. From $y_l^- = M_{j-1}$ and $f(x_l^-) \ge y_l^-$, we likewise conclude that $f(x_l^-) \ge M_{j-1}$. Therefore, $\sup f(x) \ge M_{j-1}$. So, $f(x_k^+) < \sup f(x)$, and $f(x_j^-) < \sup f(x)$. Therefore, f has a local maximum on (x_k^+, x_j^-).

A similar prove shows that intervals about which the algorithm claims that they locate a local minimum actually locate it.

2) Let us now prove that this algorithm locates all local maxima, and that it locates them precisely. The idea of this proof is as follows: let us write down the sequence of intervals that this algorithm generates. According to an algorithm, in this sequence, after each interval that

locates a local maximum, the next one locates a local minimum, and vice versa. Let us choose a point t_k on each of the intervals that locate a local maximum, and a point s_k on each interval that locates a local minimum. We want to design a function whose only local maxima are t_k.

If we succeed in this construction, we will thus prove that our algorithm locates all local maxima, and locates them precisely. Indeed, since this function f has no local maxima outside the intervals generated by our algorithm, it proves that we enumerated all intervals that necessarily contains local maxima. The fact that an arbitrary point t_k from each interval is (thus) a local maximum for some $f \in Z(I)$, proves that our intervals cannot be diminished, i.e., that we have located them precisely.

The construction of this f consists of two steps. First, for those values t_k and s_k that do not coincide with x_j^-, x_j^+, we add them to our list of endpoints. If this new value belongs to an interval that locates a local maximum, then as the corresponding value y_j, we take the value of M at the moment when we have formed this interval. Likewise, if this new x_i belongs to an interval that locates a local minimum, then we take $y_j = m$.

After this, the entire interval $[x_1, x_n]$ is divided into subintervals, that either go from s_k to the nearest t_k, or from t_k to the nearest s_k. One can see that on each subinterval $[x_m, x_n]$, where x_m is one of the points s_k, and x_n is one of the points t_k, the algorithm from Theorem 3 will generate "yes." We can, therefore, use the construction from the proof of Theorem 3 to design a non-decreasing function f that satisfies the conditions $f(X_j) \subseteq Y_j$ for all j. The values of f in the endpoints are $f(x_m) = y_m^-$ and $f(x_n) = y_n^+$, and f attains its biggest value y_n^+ only in one point: x_n.

A similar construction applies to a subinterval that starts with the left endpoint x_1 and ends in a point t_1.

For intervals that start with t_k and end with s_k (or with an endpoint), we can similarly construct a non-increasing function, for which the maximum is attained in only one point.

These functions agree on endpoints. Therefore, we can combine them into a single continuous function whose only local maxima are the points t_k.

3) A similar construction proves that our algorithm enumerates all local minima, and enumerates them precisely. □

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Received: April 2, 1995
Revised version: July 12, 1995

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