Reliable Computing 2 (3) (1996), pp. 265–285

Optimal interval enclosures for fractionally-linear functions, and their application to intelligent control

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One of the main problems of interval computations is, given a function $f(x_1, \dots, x_n)$ and n intervals $\mathbf{x}_1, \dots, \mathbf{x}_n$, to compute the range $\mathbf{y} = f(\mathbf{x}_1, \dots, \mathbf{x}_n)$. This problem is feasible for linear functions f, but for generic polynomials, it is known to be computationally intractable. Because of that, traditional interval techniques usually compute the *enclosure* of $\mathbf{y}$, i.e., an interval that contains $\mathbf{y}$. The closer this enclosure to $\mathbf{y}$, the better. It is desirable to describe cases in which we can compute the *optimal enclosure*, i.e., the range itself.

In this paper, we describe a feasible algorithm for computing the optimal enclosure for *fractionally linear* functions f. Applications of this result to *intelligent control* are described.

Оптимальные интервальные включения для дробно-линейных функций и их приложение к интеллектуальному управлению

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Одна из основных задач интервальных вычислений формулируется следующим образом: дана функция $f(x_1, \dots, x_n)$ и n интервалов $\mathbf{x}_1, \dots, \mathbf{x}_n$; требуется вычислить множество значений $\mathbf{y} = f(\mathbf{x}_1, \dots, \mathbf{x}_n)$. Эта задача имеет смысл для линейных функций f, однако известно, что для обобщенных многочленов она вычислительно неразрешима. Поэтому традиционные интервальные методы, как правило, вычисляют *включение* $\mathbf{y}$, т.е. интервал, содержащий в себе $\mathbf{y}$. Чем ближе это включение к $\mathbf{y}$, тем лучше. Желательно найти случаи, в которых возможно вычислить *оптимальное включение*, т.е. само множество значений.

В работе описан алгоритм, допускающий практическую реализацию, для вычисления оптимального включения *дробно-линейных* функций f. Описаны приложения этого результата в области *интеллектуального управления*.

1. Introduction

One of the main objectives of interval computations is: given a function $y = f(x_1, \dots, x_n)$ of n variables and n intervals $\mathbf{x}_i = [x_i^-, x_i^+]$, $1 \le i \le n$, to estimate the interval $\mathbf{y} = f(\mathbf{x}_1, \dots, \mathbf{x}_n)$ of the possible values of y. A typical application is when f describes how a physical quantity y depends on the physical quantities x_i, intervals $\mathbf{x}_i$ describe possible values of x_i, and the desired interval $\mathbf{y}$ consists of possible values of y. In general, the problem of computing $\mathbf{y}$ is computationally intractable (NP-hard) [8]. Crudely speaking, this means that if we can solve this problem in feasible time (i.e., in time that does not exceed a polynomial of the length of the input), then we would be able to solve practically all discrete problems in polynomial time, which is usually believed to be impossible.

In view of this negative result, traditional algorithms of interval computations (e.g., *naive interval computations* or more sophisticated methods like a *centered form* [1, 11, 24–28]), provide an *enclosure* of the desired interval, i.e., an interval $\mathbf{Y}$ with the property $\mathbf{y} \subseteq \mathbf{Y}$. For several simple classes of functions, exact ("optimal") enclosure $\mathbf{Y} = \mathbf{y}$ can be computed. E.g., for a linear function $f(x_1, \ldots, x_n) = a_0 + a_1 x_1 + \cdots + a_n x_n$, the desired interval is $\mathbf{y} = a_0 + a_1 \cdot \mathbf{x}_1 + \cdots + a_n \cdot \mathbf{x}_n$, where $a \cdot [x^-, x^+] = [\min(a \cdot x^-, a \cdot x^+), \max(a \cdot x^-, a \cdot x^+)]$.

In many practical applications, the function f is *fractionally linear*, i.e., it is equal to the ratio of two linear functions:

$$y = f(x_1, \ldots, x_n) = \frac{a_0 + a_1 x_1 + \cdots + a_n x_n}{b_0 + b_1 x_1 + \cdots + b_n x_n}$$

(see, e.g., [9, 10, 13, 14, 21, 22, 31]). For such functions, naive interval computations and central form do not lead to the exact enclosure. In this paper, an optimal enclosure for such f will be described.

We also describe a "continuous" version of this enclosure, and its application to intelligent control.

2. Definitions and the main result

Definition 1.

- *By a fractionally linear function, we mean a tuple* $(n, \vec{a}, \vec{b})$, *where:*
  - n *is a positive integer;*
  - $\vec{a} = (a_0, a_1, \ldots, a_n)$ *is a tuple of real numbers;*
  - $\vec{b} = (b_0, b_1, \ldots, b_n)$ *is a tuple of real numbers.*
- *The value of this function for* $\vec{x} = (x_1, \ldots, x_n)$ *is defined as*

$$y = f(x_1, \ldots, x_n) = \frac{a_0 + a_1 x_1 + \cdots + a_n x_n}{b_0 + b_1 x_1 + \cdots + b_n x_n}.$$

- *By a basic interval computation problem for a fractionally linear function* f *(or simply a problem, for short), we mean a tuple* $(f, \mathbf{x}_1, \ldots, \mathbf{x}_n)$, *where* f *is a fractionally linear function, and* $\mathbf{x}_i = [x_i^-, x_i^+]$ *are intervals. We say that an algorithm computes the optimal enclosure for the problem* $(f, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ *if it computes the range* $\mathbf{y} = f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$.
- *We say that a problem* $(f, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ *with* $f = (n, \vec{a}, \vec{b})$ *is non-degenerate if*

$$0 \notin b_0 + b_1 \cdot \mathbf{x}_1 + \cdots + b_n \cdot \mathbf{x}_n.$$

Comment. If a problem is not non-degenerate, then the interval of possible values of the denominator contains 0, and therefore, the set of all possible values of the fractions contains ∞. So, the range is an interval only if the problem is non-degenerate.

Theorem 1. *There exists an algorithm that computes the optimal enclosure for an arbitrary non-degenerate fractionally linear problem in quadratic time (i.e., in computation time $\le Cn^2$).*

Comment. So, the *basic problem of interval computations is feasible for fractionally linear functions.*

Let us describe the algorithm (the proof that it always computes the optimal enclosure is given in Section 5).

Algorithm.

- **Step 1: making a denominator positive.** If $b_0 + \sum b_i x_i^- < 0$, change the signs of all the coefficients, i.e., set $a_i := -a_i$ and $b_i := -b_i$.
- **Step 2: making the coefficients in the denominator non-negative.** For all $i = 1, \ldots, n$, if $b_i < 0$, replace x_i with the new variable $y_i = -x_i$, change the signs of the coefficients a_i and b_i, and change $\mathbf{x}_i = [x_i^-, x_i^+]$ to $\mathbf{y}_i = [-x_i^+, -x_i^-]$.
- **Step 3: eliminating degenerate variables.** If $a_i/b_i = a_j/b_j$ for some $i \neq j$, and $|b_i| \ge |b_j|$, replace variables x_i and x_j with a single variable y_i, for which b_i and a_i stay the same as before, but for which $\mathbf{y}_i = \mathbf{x}_i + (b_j/b_i)\mathbf{x}_j$.
- **Step 4: ordering the variables.** Order the variables x_i in the increasing order of the corresponding ratios a_i/b_i, so that:

$$\frac{a_1}{b_1} < \frac{a_1}{b_2} < \cdots < \frac{a_n}{b_n}.$$

- **Final step: computing $y^{\pm}$.** Compute $y^+ = \max(y_0^+, y_1^+, \ldots, y_n^+)$, where

$$y_k^+ = \frac{a_0 + a_1 x_1^- + \cdots + a_k x_k^- + a_{k+1} x_{k+1}^+ + \cdots + a_n x_n^+}{b_0 + b_1 x_1^- + \cdots + b_k x_k^- + b_{k+1} x_{k+1}^+ + \cdots + b_n x_n^+}.$$

  Compute $y^- = \min(y_0^-, y_1^-, \ldots, y_n^-)$, where

$$y_k^+ = \frac{a_0 + a_1 x_1^+ + \cdots + a_k x_k^+ + a_{k+1} x_{k+1}^- + \cdots + a_n x_n^-}{b_0 + b_1 x_1^+ + \cdots + b_k x_k^+ + b_{k+1} x_{k+1}^- + \cdots + b_n x_n^-}.$$

Example. Let $n = 2$, $\mathbf{x}_1 = \mathbf{x}_2 = [1, 2]$,

$$f(x) = \frac{1 + x_1 + x_2}{1 + x_1 - 4x_2}.$$

In this case, the interval $1 + [1, 2] - 4[1, 2] = [-6, -1]$ does not contain 0, so, the problem is non-degenerate. Let us apply the above algorithm:

- **Step 1.** Since $b_0 + \sum b_i x_i^- = 1 + 1 - 4 = -2 < 0$, we change the signs of all the coefficients. As a result, we arrive at the following problem: $\mathbf{x}_1 = \mathbf{x}_2 = [1, 2]$,

$$f(x) = \frac{-1 - x_1 - x_2}{-1 - x_1 + 4x_2}.$$

- **Step 2.** The coefficient b_i is negative for $i = 1$, so for this i, we introduce the new variable, and correspondingly change the coefficients a_1, b_1 and the interval $\mathbf{x}_1$. As a result, we get the following problem: $\mathbf{x}_1 = [-2, -1]$, $\mathbf{x}_2 = [1, 2]$,
$$f(x) = \frac{-1 + x_1 - x_2}{-1 + x_1 + 4x_2}.$$
- **Step 3.** The values $a_1/b_1 = 1$ and $a_2/b_2 = -1/4$ are different, so, we do nothing at this step.
- **Step 4.** Since $a_1/b_1 > a_2/b_2$, we change the order of the variables. As a result, we get the following problem: $\mathbf{x}_1 = [x_1^-, x_1^+] = [1, 2]$, $\mathbf{x}_2 = [x_2^-, x_2^+] = [-2, -1]$,
$$f(x) = \frac{-1 - x_1 + x_2}{-1 + 4x_1 + x_2}.$$
- **Final step.** We compute y^+ as $\max(y_0^+, y_1^+, y_2^+)$, where:
$$\begin{aligned} y_0^+ &= \frac{-1 - 2 + (-1)}{-1 + 8 + (-1)} = \frac{-4}{6} = -\frac{2}{3}, \\ y_1^+ &= \frac{-1 - 1 + (-1)}{-1 + 4 + (-1)} = \frac{-3}{2} = -\frac{3}{2}, \\ y_2^+ &= \frac{-1 - 1 + (-2)}{-1 + 4 + (-2)} = \frac{-4}{1} = -4. \end{aligned}$$

  Hence, $y^+ = -(2/3)$. We also compute y^- as $\min(y_0^-, y_1^-, y_2^-)$, where:
$$\begin{aligned} y_0^- &= \frac{-1 - 1 + (-2)}{-1 + 4 + (-2)} = \frac{-4}{1} = -4, \\ y_1^- &= \frac{-1 - 2 + (-2)}{-1 + 8 + (-2)} = \frac{-5}{5} = -1, \\ y_2^- &= \frac{-1 - 2 + (-1)}{-1 + 8 + (-1)} = \frac{-4}{6} = -\frac{2}{3}. \end{aligned}$$

  So, $y^- = -4$, and $\mathbf{y} = [-4, -(2/3)]$.

3. Continuous version of this algorithm

Let us describe a version of this algorithm that can be used for a *continuous* version of the above problem, in which, instead of n intervals $\mathbf{x}_i$, we have an *interval-valued function*. The motivation for this case will be given in Section 4.

Definition 2.

- *We say that a function $f : R \to R$ is piecewise-continuous if it is continuous everywhere except maybe finitely many points x, in which both one-sided limits $f(x - 0) = \lim_{h \to 0; h > 0} f(x - h)$ and $f(x + 0) = \lim_{h \to 0; h > 0} f(x + h)$ exist (but may be different from each other), and $f(x)$ coincides with one of these limits.*

- *Let real numbers $a < b$ be fixed. By $\mathcal{F}$, we will denote a class of all piecewise-continuous functions $f : [a, b] \to [0, 1]$ for which $f(a) = f(b) = 0$ and $f(x) \neq 0$ for some x.*
- *Let $f \in \mathcal{F}$. By $B_\delta(f)$, we will denote the set of all functions $g \in \mathcal{F}$ for which $g(x) \in [f(x) - \delta, f(x) + \delta]$ for all x.*
- *For a function $f \in \mathcal{F}$, we denote*

$$D(f) = \frac{\int x f(x)\, dx}{\int f(x)\, dx}.$$

By $D(B_\delta(f))$, we denote the range of D, i.e., $\{D(g) \mid g \in B_\delta(f)\}$.

Theorem 2. *For given (a, b), $f(x)$ and δ:*

- *if $f(x) \leq \delta$ for all x, then $D(B_\delta(f)) = (a, b)$;*
- *else, $D(B_\delta(f)) = [u^-, u^+]$, where u^- is the solution of the equation $F^-(u) = 0$, u^+ is the solution of the equation $F^+(u) = 0$,*

$$F^-(u) = \int_a^u (u - x) \min\Big(1, f(y) + \delta\Big)\, dx - \int_u^b (x - u) \max\Big(0, f(x) - \delta\Big)\, dx$$

and

$$F^+(u) = \int_a^u (u - x) \max\Big(0, f(x) - \delta\Big)\, dx - \int_u^b (x - u) \min\Big(1, f(x) + \delta\Big)\, dx.$$

(All the proofs are given in Section 5.)

Comment. Both functions $F^-(u)$ and $F^+(u)$ are increasing, so the values u^- and u^+ can be computed by, e.g., a binary search algorithm. This means that to determine u^+ and u^- with precision ε, we need no more than $C|\log_2(\varepsilon)|$ computational steps. In other words, to get k digits of u^+ and u^-, we need $\leq Ck$ computational steps; this is a quite feasible algorithm.

Remark 1. **Why cannot we just use interval analysis?** For every x from (a, b), possible values of $g(x)$ form an interval $(f(x) - \delta, f(x) + \delta)$. Therefore, we can apply the general methodology of interval analysis (initiated in [23]; see, e.g., [1, 24, 28]) to get the interval estimates for $\int x g(x)\, dx$ and $\int g(x)\, dx$, and then apply the interval division rule to get an interval that contains $[u^-, u^+]$.

The main reason why we did not apply this approach is that the resulting interval is larger than $[u^-, u^+]$, and we will see in Section 4, in the intended applications, overestimating ("overshoot") can diminish the reliability and quality of the resulting control.

Let us give a simple example where simple interval computations really "overshoots": take $f(x) = x$ for $0 \leq x \leq 1$, $f(x) = 2 - x$ for $1 \leq x \leq 2$, and $f(x) = 0$ else. Such functions are among the most commonly used in fuzzy control (they are called *triangular* because their graphs form a triangle). For this function, the above-described interval estimate leads to an interval $[1 - 4\delta + o(\delta), 1 + 4\delta + o(\delta)]$ (computations are given in Section 3), while the actual bounds (computed by using the above Theorem) are $[1 - \delta + o(\delta), 1 + \delta + o(\delta)]$. So, for a reasonable membership function, and for small δ, we have a 4 times "overshoot."

Remark 2. **The above simple interval estimates can be reasonably improved** if we use a continuous version of what is called a *centered from* in interval mathematics (see, e.g., [1, 11, 24–28]), namely, if we first represent $D(g)$ in the form $D(f)+A(g-f)/B(g-f)$, and then apply the interval computations technique. As a result (for details see Section 5), we get an interval

$$\Big[D(f)-\delta\big(D(f)(a+b)-1/2(a^2+b^2)\big)/\Big(\int f(x)\,dx\Big)+o(\delta),$$
$$D(f)+\delta\big(D(f)(a+b)-1/2(a^2+b^2)\big)/\Big(\int f(x)\,dx\Big)+o(\delta)\Big].$$

This estimate gives the right asymptotics for small δ, but still gives an overshoot. And since we explained why overestimate is dangerous, it is much better to use a precise estimate from Theorem 1.

4. Applications to intelligent control

4.1. What is intelligent control. Successes of intelligent control methodology

Traditional control theory is not always applicable. In case we do not have the precise knowledge of a controlled system, we are unable to apply traditional control theory. Such situations occur, e.g., when we are devising a control for a future Martian rover, or a control for any other space mission into the unknown. In such cases, we often have a skillful operator who is good at making control decisions in uncertain environments. This operator can communicate his skills only in terms of natural-language rules that use words like "small," "medium," etc. So, it is desirable to transform these rules into a precise control. Such a methodology was first outlined by L. Zadeh [5, 34, 35] and experimentally tested by E. Mamdani [20] in the framework of fuzzy set theory [33], therefore the whole area of research is now called *fuzzy control* (it is also called *intelligent*, or *rule-based* control). For the current state of fuzzy control, the reader is referred to the surveys [2, 19, 30].

Let us explain its main ideas on the following simple example. The goal of a thermostat is to keep a temperature T equal to some fixed value T_0, or, in other words, to keep the difference $x=T-T_0$ equal to 0. To achieve this goal, one can switch the heater (or the cooler) on and off and control the degree of cooling or heating. We actually control the rate with which the temperature changes, i.e., in mathematical terms, a derivative $\dot{T}$ of temperature with respect to time. So, if we apply the control u, the behavior of the thermostat will be determined by the equation $\dot{T}=u$. In order to automate this control, we must come up with a function $u(x)$ that describes what control to apply if the temperature difference x is known.

Why can't we extract $u(x)$ from an expert? We are talking about a situation where traditional control theory does not help, so we must use the experience of an expert to determine the control function $u(x)$. Why can't we just ask the expert questions like "suppose that x is 5 degrees; what do you do?", write down the answers, and thus plot $u(x)$? It sounds reasonable at first glance, until you try applying the same idea to a skill in which practically all American adults consider themselves experts: driving a car. If you ask a driver a question like: "you are driving at 55 mph. when the car in 30 ft. in front of you slows down to 47 mph., for how many seconds do you hit the brakes?", nobody will give a precise number. You might install measuring devices into a car or a simulator, and simulate this situation, but what will happen is that the amount of time to brake will be different for different simulations. The problem is not that the expert has some precise number in mind (like 1.453 sec) that he cannot express in words; the problem is that one time it will be 1.3, another time it may be 1.5, etc.

An expert usually expresses his knowledge in words. An expert cannot express his knowledge in precise numeric terms (such as "hit the brakes for 1.43 sec"), but what he can say is "hit the brakes for a while." So the rules that can be extracted from him are "if the velocity is a little bit smaller than maximum, hit the breaks for a while." Let's illustrate the rules on the thermostat example. If the temperature T is close to T_0, i.e., if the difference $x = T - T_0$ is negligible, then no control is needed, i.e., u is also negligible. If the room is slightly overheated, i.e., if x is positive and small, we must cool it a little bit (i.e., $u = \dot{x}$ must be negative and small). If the temperature is a little lower, then we need to heat the room a little bit. In other terms, if x is small negative, then u must be small positive, etc. So we have the following rules:

1) if x is negligible, then u must be negligible;

2) if x is small positive, then u must be small negative;

3) if x is small negative, then u must be small positive;

etc.

Brief description of fuzzy control methodology. First, we combine all the rules into one statement relating x and the control u. If we know x, what control u should we apply? u is a reasonable control if either:

- the first rule is applicable (i.e., x is negligible) and u is negligible; or
- the second rule is applicable (i.e., x is small positive), and u is small negative; or
- the third rule is applicable (i.e., x is small negative), and u is small positive; or
- one of the other rules is applicable.

Summarizing, we can say that u is an appropriate choice for a control if and only if either x is negligible and u is negligible, or x is small positive and u is small negative, etc. If we use the denotations $C(u)$ for "u is an appropriate control," $N(x)$ for "x is negligible," SP for "small positive," SN for "small negative" and use the standard mathematical notations & for "and," $\vee$ for "or" and $\equiv$ for "if and only if," we come to the following informal "formula":

$$C(x) \equiv \big(N(x)\&N(u)\big) \vee \big(SP(x)\&SN(u)\big) \vee \big(SN(x)\&SP(u)\big) \vee \cdots \tag{1}$$

How do we formalize this combined statement: four stages of fuzzy control methodology. In order to formalize statements like the one we just wrote down, we first need to somehow interpret what notions like "negligible," "small positive," "small negative," etc., mean. The main difference between these notions and mathematically precise ("crisp") ones like "positive" is that any value is either positive or not, while for some values it is difficult to decide whether they are negligible or not. Some values are so small that practically everyone would agree that they are negligible, but the bigger is the value, the fewer experts that will say that it is negligible, and the less confident they will be in that statement. For example, if someone is performing a complicated experiment that needs fixed temperature, then for him 0.1 degree is negligible, but 1 degree is not. For another expert ± 5 degrees is negligible.

First stage: describing the degree of confidence. This degree of confidence (also called *degree of belief, degree of certainty, truth value, certainty value*) can take all possible values from "false" to "true." Inside the computer, "false" is usually described by 0, "true" by 1. Therefore it is reasonable to use intermediate values from the interval $(0,1)$ to describe arbitrary degrees of certainty. This idea appeared in *fuzzy logic* [33], and that's why the resulting control is called *fuzzy control*. So the first stage of a fuzzy control methodology is to somehow assign values from the interval $[0,1]$ to different statements like "0.3 is negligible" or "0.6 is small positive." There are several ways to do that [7]. For example [3; 4; 7, IV.1.d; 12] we can take several (N) experts, and ask each of them whether he believes that a statement is true (for example, that 0.3 is negligible). If M of them answer "yes," we take M/N as a desired certainty value. Another possibility is to ask one expert and express his degree of confidence in terms of the so-called subjective probabilities [29].

Second stage: forming membership functions. The procedure described above allows us to get the truth values of, for example, $N(x)$ for different values of x. But even if we spend a lot of time questioning experts, we can only ask a finite amount of questions. Therefore, we will only get the values $N(x)$ for finitely many different values of x: $x_1, x_2, \ldots, x_n$, so we must somehow extrapolate the known truth values of $N(x_i)$ to come out with a function that, for every possible x, gives a value from the interval $[0,1]$ that expresses our degree of confidence that this property is true for x. Such a function is called a *membership function* and is usually denoted by $\mu(x)$. A membership function of the property N is denoted by $\mu_N(x)$, a membership function of the property $SP(x)$ by $\mu_{SP}(x)$, etc.

Third stage: & and $\vee$ operations. After the second stage we are able to assign truth values to the statements $N(x)$, $SP(x)$, etc. Our goal is to describe the possible values of control. In formula (1), control is represented by the statement $C(u)$, meaning "u is an appropriate value of control." To get the truth value of this statement for different values of C, we must somehow interpret the operations "and" and "or" that relate them to the values that we already know.

Suppose that we have already chosen some rules to process & and $\vee$. Namely, we have chosen a procedure that allows us, given the truth values a and b of some statements A and B, to compute the truth values of $A\&B$ and $A \vee B$. Let's denote the resulting truth value of $A\&B$ by $f_\&(a,b)$, and the truth value of $A \vee B$ by $f_\vee(a,b)$. Now we can compute the truth value of $C(x)$ for every x, i.e., a membership function of the property C.

In particular, for our thermostat example the resulting membership function is

$$\mu_C(u) = f_\vee\Big(f_\&\big(\mu_N(x), \mu_N(u)\big), f_\&\big(\mu_{SP}(x), \mu_{SN}(u)\big), f_\&\big(\mu_{SN}(x), \mu_{SP}(u)\big), \ldots\Big).$$

Typical choices for $f_\vee(a,b)$ are $\max(a,b)$ or $a+b$, for $f_\&(a,b)$: $\min(a,b)$ or ab, etc.

Fourth stage: defuzzification. After the first three stages we have the "fuzzy" information about the possible controls: something like "with degree of certainty 0.9 the value $u = 0.3$ is reasonable, with degree of certainty 0.8 the value $u = 0.35$ is reasonable, etc." We want to build an automatic system, so we must choose one value $\bar{u}$. So we must somehow transform a membership function $\mu(u)$ into a single value. Such a procedure is called *defuzzification*. The most commonly used defuzzification is a *centroid* $D(\mu) = (\int x\mu(x)\,dx)/(\int \mu(x)\,dx)$.

Successes of intelligent control. Fuzzy control is applied to control trains, appliances (dishwashers, laundry machines, camcorders, etc.), manufacturing processes, etc. (see, e.g., [2, 19, 30]). Computer simulations show that intelligent control is a very efficient way to control space missions [16, 17].

4.2. The problems of intelligent control, and what is necessary for their solution

The number is generated; with what precision shall we implement it? Suppose that this methodology generates a control value -0.310745. Does this mean that we really need to generate such a precise control? Of course not, because the whole methodology is based on the experts' estimates of membership functions, and they are never that precise. But what precision should we keep? Is -0.3 sufficient, or we need to achieve -0.31?

This is a problem of efficiency. This is not a purely academic question, because if it is sufficient to use -0.3, then we can install a simple controlling devices that can generate controls with a precision 0.1, but if we really need a 1% precision, then we need a much more complicated and more expensive device. For example, it is relatively easy to get a 10^o precision in the orientation of a space station, but to get 1^o or less, one needs to use complicated and expensive super-precise engines, and in addition, be very careful about all possible experiments that would interfere with these engines.

So, if we implement the control in too crude a manner, we will get only a crude approximation to the desired control, and thus loose some of this control's quality. If we try to implement it with too much of a precision, we will waste efforts and money on unnecessary precision. Therefore, we must know with what precision the control should be implemented.

Another efficiency-related issue: Do we need control at all? Another precision-related efficiency problem appears in the situations, when the intelligent control methodology leads to the values that are very close to 0 (here, 0 value of control means that no control is necessary). In many cases, it would be more efficient not to control at all than to apply a very small control. For example, if we are controlling the spaceship, then some fuel and energy is spent just for keeping the control engines working. In these cases, it is better to switch the control off whenever possible. So, we would like to apply 0 control if the methodology prescribes to use a small control (that may be different from 0 just because of the uncertainty in the initial data).

Here again we have a dilemma: if we set this threshold too low, we do many unnecessary control actions, and waste efforts and energy (in case of a spaceship, fuel). If we set this threshold too high, we do not control in the situations where we really need to, and thus we loose stability and/or other desired properties of control. So, ideally we should set this threshold "just right."

Reliability. Intelligent control is not based on our precise knowledge of the controlled system, it is just a translation of the expert operator's control into mathematical terms. Since even the best human operator can err, this control that simulates his behavior may also err. This raises the problem of reliability.

This problem is extremely important for intelligent control, because, e.g., in space applications the doubts in reliability are the main reason why, in spite of the very promising results of computer simulations ([16, 17], etc.), fuzzy control techniques are not yet widely applied to space missions.

How to solve this problem? One way to do it is to take into the consideration the fact that fuzzy control is just a simulation of an expert's control. If we are not sure that an expert is giving us the right advice, what do we do? Ask another expert or experts; if their opinions coincide, we believe in them much more than we believed in a single expert's opinion. If they differ, but there is a strong majority in favor of one of them, we go with a majority.

The similar approach can be applied to fuzzy control: namely, instead of implementing the rules of a single expert controller, let us implement several fuzzy control systems that represent opinions of several experts. These systems can have different sets of rules, use different fuzzy notions. For every situation, each of these systems will generate some control value. If these values coincide, then this is the right value, and this value is much more reliable than in the case when we have only one fuzzy control system. If they differ, but the majority of these systems prompt the same control, we control according to the majority.

This sounds like a reasonable idea, but how to interpret the word "coincides"? Fuzzy control systems produce real values, and it is hardly unprobable that systems with different rules, that use different notions, will produce precisely the same real numbers. So, if the first systems generates the value 0.310987, and the second one the value 0.345901, do these control recommendations coincide or not? It depends on what is the precision of fuzzy control recommendations. If it is 0.1, then these controls are in good accordance, and we must apply it. If the precision is 0.01, then there is a disagreement between them, and we better consult the human operator before actually controlling the spaceship.

In this case, it is also important not to underestimate and not to overestimate this precision. If we underestimate it (e.g., estimate it at 0.01, while in reality it is 0.1), then in many cases, when the control decisions are really in good precision, and we can rely on the automated control system, we would call the human operator. If we overestimate (e.g., estimate it at 0.1, while in reality precision is 0.01), and use this "overshooting" estimate to determine consistency, then we will occasionally erroneously apply the automated control even when the results of several fuzzy systems disagree with each other (e.g., if they are 0.3 and 0.35). In the first case, we do not use the entire potential and ability of the automated control system; in the second case, we loose in reliability. Therefore, it is extremely important to get the precise estimates.

4.3. Interval estimates help to solve the problems of intelligent control

Informal summary of the problem. The above examples show that it is very important that a fuzzy control system would not only generate a number u of the recommended control, but would also generate an interval $[u^-, u^+]$ of possible reasonable controls.

Motivation of the following mathematical formalization. How to estimate these intervals? As we have already mentioned, the main reason why the resulting estimates for a control are not precise is that the entire methodology of intelligent control is based on the experts' estimates of membership functions, and these estimates are not precise. In other words, if we apply the same procedure to the same expert (or the same group of experts) twice, for the same statement we can get different degrees of belief $p \neq p'$. This uncertainty can be easily determined: namely, we apply the same procedure two, three, or more times to the same expert or group of experts, and compute the differences $p - p'$ between the values that were obtained on different repetitions of this procedure. The biggest value $|p - p'|$ is the desired estimate of the experts' uncertainty. Let us denote this estimate by δ.

This number means that when we produce a membership function $f(x)$, for the same experts it could as well be any function $g(x)$ such that $|f(x) - g(x)| \leq \delta$. In the above-described fuzzy control methodology, we first process membership functions, and then use a centroid defuzzification procedure $D(f) = (\int x f(x)\, dx)/(\int f(x)\, dx)$ to produce a single value of control. So, the problem is: knowing that $g(x)$ can be any function such that $|f(x)-g(x)| \le \delta$, what are the possible results $D(g)$ of defuzzifying $g(x)$?

Additional remark: Membership functions must have compact support. We have already formulated the mathematical problem, but we want to add one more remark before we turn to a mathematical formulation. The parameters that we can control are usually bounded by technical reasons: we cannot get arbitrary acceleration, since our abilities are limited by the existing boost engines; we cannot get unlimited rotation speed, etc. In all cases, there exists an interval (a,b) of control values that are technically and/or physically feasible. Values of control outside this interval are simply impossible, therefore, for $x \notin (a,b)$ we must have $f(x)=0$, and $g(x)=0$ for all possible membership functions g.

In mathematical terms, this restriction means that we limit ourselves to functions $g(x)$ with compact support $\mathrm{supp}(f) \subset [a,b]$.

The membership functions are usually assumed to be piecewise continuous and not necessarily everywhere continuous is that we want to include the case, when the knowledge is not fuzzy, and the membership function $f(x)$ is equal to 1 or 0 for all x.

As a result, we can apply Theorem 2 to compute the set of possible values of the control.

4.4. Other defuzzification rules

Centroid of largest area. Centroid is just one of the possible defuzzification rules. Sometimes, it does not work fine, therefore, we need to apply more complicated rules (see, e.g., [15, 32]). In [15] we proved that the reasonable demands select either this centroid rule, or an alternative rule called *centroid of largest area*, where we first restrict a membership function $\mu_C(x)$ for control to some interval I, and only then apply a defuzzification. So, the resulting control is equal to $\bar{u} = D((\mu_C)_{|I})$. In this case, the only step where we go from membership functions to an actual control is also a centroid, therefore, we can apply Theorem 2 to compute the interval of possible controls for this case as well.

Center-of-Maximum. Another frequently used defuzzification rule is as follows:

Definition 3. *By a center-of-maximum defuzzification we mean a mapping that transforms a function $\mu(x)$ from $\mathcal{F}$ into a number $D_{COM}(\mu) = 1/2(m_- + m_+)$, where $m_- = \inf\{x : \mu(x) = \max_y(\mu(y))\}$, and $m_+ = \sup\{x : \mu(x) = \max_y(\mu(y))\}$.*

A natural property of the defuzzification procedure d is that if we know the membership function with better and better accuracy (i.e., if $\delta \to 0$, then the resulting interval of possible values $d(B_\delta(f))$ must converge to the point $d(f)$). In other words, the mapping d must be *continuous* w.r.t. the metric $\rho(f,g) = \sup|f(x)-g(x)|$.

Theorem 3.

- *Centroid defuzzification D is continuous for all f.*
- *Center-of mass defuzzification is not everywhere continuous.*

4.5. Conclusions

When applying fuzzy control, we suggest not only to compute the recommended value, but also to compute the interval $[u^-, u^+]$ of possible values. To compute this interval, we must determine the uncertainty δ with which we can estimate the experts' degrees of belief (see Section 1.3), and then apply the algorithm from Theorem 2. These interval estimates will enable us to do the following:

- **Increase efficiency:** Explain what control to use: any value from u^- to u^+ is fine. This solves the problem of how precisely we must follow the recommendation u, and thus enables to choose the least costly way of following these control recommendations.
- Solves the problem of whether to apply a control or not: if $u^- \le 0 \le u^+$, we do not need to apply any control; else we must. This will enable us to *avoid unnecessary control actions.*
- **Increase reliability:** If we have several fuzzy control systems, and for each of them we have an interval of recommended controls, then:
  - if the intersection of all these intervals is non-empty (i.e., all these intervals have a common point), then we take any control from this intersection as a reliable control value;
  - if there is no point common to all these intervals, but the majority of them have a non-empty intersection, then we choose a control from this intersection.

5. Proofs

Proof of Theorem 1. Let us denote the numerator of the function f by A, and its denominator by B. Let us first prove that Steps 1–4 do not change the problem:

- **Step 1.** If we change the signs of all the coefficients a_i and b_i, then both numerator and denominator will change signs, and the ratio will remain unchanged.
- **Step 2.** If we rename the variable $x_i = -y_i$, then the values a_i and b_i, and the interval of possible values of y_i must be changed accordingly.
- **Step 3.** If $a_i/b_i = a_j/b_j$, then $a_j/a_i = b_j/b_i$. Therefore,

$$a_ix_i + a_jx_j = a_i\Big(x_i + (a_j/a_i)x_j\Big) = a_i\Big(1 + (b_j/b_i)x_j\Big)$$

  and $b_ix_i + b_jx_j = b_i(x_i + (b_j/b_i)x_j)$. Therefore, we can replace the terms $a_ix_i + a_jx_j$ and $b_ix_i + b_jx_j$ that depend on x_i and x_j with a_iy_i and b_iy_i, where the new variable y_i is equal to $y_i = x_i + (b_j/b_i)x_j$. If $x_i \in \mathbf{x}_i$ and $x_j \in \mathbf{x}_j$, then the set $\mathbf{y}_i$ of possible values of the new variable is equal to $\mathbf{y}_i = \mathbf{x}_i + (b_j/b_i)\mathbf{x}_i$.
- **Step 4.** Renaming the variables does not change the problem.

In view of Steps 1–4, we can assume that $b_i \ge 0$, and that the ratio a_i/b_i is increasing as i increases.

After Step 1, we can be sure that the value of the denominator B is positive at least for one combination of $x_i \in \mathbf{x}_i$; since the problem is non-degenerate, the denominator cannot attain 0, and hence, it is always positive.

The function f is a continuous function defined on a compact $\mathbf{x}_1 \times \cdots \times \mathbf{x}_n$. Therefore, its maximum y^+ is attained at some point $(x_1, \ldots, x_n)$.

The function f is smooth; therefore, if the maximum is attained for x_i inside the interval $[x_i^-, x_i^+]$ (i.e., for $x_i \in (x_i^-, x_i^+)$), we will have $\partial f/\partial x_i = 0$. Applying the formula for the derivative of the fraction, we conclude that $a_i \cdot B - b_i \cdot A = 0$, hence, $a_i/b_i = A/B = f$. If we replace x_i with x_i^-, then, the new value B' of the denominator B will be equal to $B + b_i(x_i^- i - x_i)$. The new value A' of A will be $A + a_i(x_i^- - x_i)$. Since $A = bf$ and $a_i = b_i f$, we conclude that $A' = Bf + b_i f(x_i^- - x_i) = f(B + b_i(x_i^- - x_i)) = fA'$. Hence, the new value of the ratio $f' = A'/B'$ is equal to the old value $f = A/B$. Therefore, we can change this x_i to x_i^- without changing the optimal value of f. So, without losing generality, we can assume that for every i, x_i is either equal to x_i^-, or to x_i^+.

If $x_i = x_i^-$, then, since we have achieved the maximum of f for x_i, an increase in x_i can only decrease f. So, the partial derivative $\partial f/\partial x_i$ must be non-negative at the point $(x_1, \ldots, x_n)$. This partial derivative is equal to $(a_i B - b_i A)/B^2$, so, the fact that it is non-negative, means that $a_i B - b_i A \leq 0$, which is equivalent to $a_i B \leq b_i A$. Since $B > 0$ and $b_i > 0$, we can divide both sides of this inequality by $b_i B$, resulting in $a_i/b_i \leq A/B$.

Similarly, if $x_i = x_i^+$, a decrease in x_i can only decrease f. So, we will get $\partial f/\partial x_i \geq 0$, and $a_i/b_i \geq A/B$.

So, for every i, either $x_i = x_i^-$ and $a_i/b_i \leq A/B$, or $x_i = x_i^+$ and $a_i/b_i \geq A/B$. Hence, if $a_i/b_i < A/B$, we have $x_i = x_i^-$, and if $a_i/b_i > A/B$, we have $x_i = x_i^+$. If $a_i/b_i = A/B$, then, as above, we can switch x_i to x_i^- without changing the value of f.

Since after Step 4, the variables are ordered in the order of the ratio a_i/b_i, this means that for all variables $x_1, \ldots, x_k$ up to some k-th one, we have $x_i = x_i^-$, and for the other variables $x_{k+1}, \ldots, x_n$, we have $x_i = x_i^+$. In other words, we conclude that $y^+ = y_k^+$ for some k. Hence,

$$y^+ \leq \max(y_0^+, \ldots, y_n^+).$$

On the other hand, each value y_k^+ is a possible value of the function f and is therefore, not exceeding the maximum y^+ of the function f. So, $y_k^+ \leq y^+$ for all k; hence,

$$\max(y_0^+, \ldots, y_n^+) \leq y^+.$$

From these two inequalities, we conclude that $y^+ = \max(y_0^+, \ldots, y_n^+)$.

In this proof, we used transformations that, strictly speaking, make sense only if $b_i \neq 0$. However, the case when $b_i = 0$ for some i, can be handled in the same manner, if we allow expressions like $a_i/0$ (meaning $+\infty$ or $-\infty$ depending on the sign of a_i).

The proof for y^- is similar.

To complete the proof, we must now show that the algorithm described in Section 2 requires quadratic time. Indeed, initial steps 1 and 2 require the number of operations that is linear in n. Sorting (Step 4) can be done in time $n \log_2(n) \ll n^2$ (see, e.g., [6]), and the Final step requires us to compute $2(n+1)$ expressions $y_k^\pm$ each of which requires $4n+1$ arithmetic operations: $2n$ multiplications, $2n$ additions, and 1 division. Totally, we need $\leq 2(n+1)(4n+1) = O(n^2)$ arithmetic operations. □

Proof of Theorem 2. Let us denote the desired set $D(B_\delta(f))$ by C.

1. Let us first notice that $D(f) \leq b$. Indeed, since $x \leq b$ for $x \in [a, b]$, we have $xf(x) \leq bf(x)$, hence $\int_a^b xf(x)\,dx \leq \int_a^b bf(x)\,dx = b\int_a^b f(x)\,dx$. Dividing both sides of this inequality by $\int_a^b f(x)\,dx$, we conclude that $D(f) \leq b$.

Likewise, one can prove that $D(f) \geq a$.

2. Let us now prove that $D(f) < b$.

Since we have already proved that $D(f) \leq b$, it is sufficient to prove that $D(f) \neq b$. Let us prove it by reduction to a contradiction. Suppose that $D(f) = b$. Then, multiplying both sides by $\int_a^b f(x)\,dx$, we conclude that $\int_a^b xf(x)\,dx = b\int_a^b f(x)\,dx$. Therefore, $\int_a^b xf(x)\,dx = \int_a^b bf(x)\,dx$. Moving all the terms to the right-hand side, we conclude that $\int_a^b (b-x)f(x)\,dx = 0$. We assumed that $f(x)$ is a piecewise-continuous non-negative function; therefore, the function $(b-x)f(x)$ is also non-negative and piecewise-continuous. This means that we can divide the interval (a,b) into finitely many intervals, on each of which $(b-x)f(x)$ is non-negative and continuous. The integral of $(b-x)f(x)$ over (a,b) is equal to the sum of the integrals of this functions over these integrals. Since we are integrating a non- negative function, all these integrals are non-negative. Hence, their sum can be equal to 0 only in one case: if all the terms in the sum (i.e., all the integrals) are equal to 0. But an integral of a continuous non-negative function is equal to 0 only if this function is identically 0. So, $(b-x)f(x) = 0$ for all x, hence, $f(x) = 0$ for all $x \in (a,b)$. Since we demanded that $f(x) = 0$ outside the interval (a,b), we conclude that $f(x)$ is identically 0, which contradicts to our definition of a membership function.

This contradiction proves that our assumption was wrong, and so, $D(f) < b$.

3. Likewise, we can prove that $D(f) > a$. Hence, $C \subset (a,b)$.

4. Now, let us consider the case when $f(x) \le \delta$ for all δ. We have just proved that $C \subset (a,b)$, so, in order to prove that $C = (a,b)$, we must prove that any number c from the interval (a,b) is a possible control.

Indeed, suppose that $c \in (a,b)$. In this case, $c-a > 0$ and $b-c > 0$. Let us denote the smallest of these two numbers by $\Delta = \min(c-a, b-c)$. Let us take $g(x) = \delta$ if $|x-c| < \Delta$, and $g(x) = 0$ else. Then $g(x) \le \delta$, hence $g(x) - f(x) \le g(x) \le \delta$. Likewise, from $f(x) \le \delta$, we conclude that $f(x) - g(x) \le \delta$. So, $|f(x) - g(x)| \le \delta$, and $g(x)$ is possible. Then, $\int_a^b xg(x)\,dx = \int_{c-\Delta}^{c+\Delta} x\delta\,dx = \delta x^2/2\,|_{c-\Delta}^{c+\Delta} = \delta/2((c+\Delta)^2 - (c-\Delta)^2) = 2\delta c\Delta$. The denominator of $D(g)$ is equal to $\int_a^b g(x)\,dx = \int_{c-\Delta}^{c+\Delta} \delta\,dx = \delta((c+\Delta) - (c-\Delta)) = 2\delta\Delta$ therefore, $D(f) = c$.

5. Let us now consider the case when $f(x_0) > \delta$ for some x_0. In this case, if a non-negative function $g(x)$ equals 0 outside (a,b), and satisfies the inequality $|f(x) - g(x)| \le \delta$ for all x, then from $|f(x_0) - g(x_0)| \le \delta$ and $f(x_0) > \delta$ we can conclude that $g(x_0) > 0$, so $g(x)$ is not identically 0.

6. Let us denote by PC the set of all piecewise-continuous non-negative functions on (a,b) that are not identically 0. The expression $\rho(g,h) = \sup_x |g(x) - h(x)|$ defines a metric on PC. The expressions $\int xg(x)\,dx$ and $\int g(x)\,dx$ are both continuous in this metric. Indeed, if $\rho(g,h) = \sup_x |g(x) - h(x)| \le \delta$, then

$$\left|\int g(x)\,dx - \int h(x)\,dx\right| = \left|\int \big(g(x) - h(x)\big)\,dx\right| \le \int |g(x) - h(x)|\,dx \le \int_a^b \delta\,dx = (b-a)\delta$$

and $(b-a)\delta \to 0$ as $\delta \to 0$. Likewise,

$$\left|\int xg(x)\,dx - \int xh(x)\,dx\right| \le \int |x|\delta\,dx = \delta\Big(\int_a^b |x|\,dx\Big) \to 0$$

as $\delta \to 0$. Therefore, the ratio $D(g)$ of these two expressions is also continuous in this metric.

7. Let us now denote the set of all possible functions by $\mathcal{P}$. By definition, $\mathcal{P} \subset \mathcal{PC}$. It is easy to see that $\mathcal{P}$ is a connected set: if $g,h \in \mathcal{P}$, then for every $t \in [0,1]$, their convex combination $g_t(x) = tg(x) + (1-x)h(x)$ also belongs to $\mathcal{P}$, and this family g_t forms a continuous family connecting g and h. Therefore, since D is continuous on $\mathcal{P}$, the set $D(\mathcal{P})$ of all possible values of $D(g)$ is a connected subset of R. Therefore, it is either the entire real line, or a half-line, or an interval (open, close or semi-close).

If we denote $u^- = \inf\{D(g) : g \in \mathcal{P}\}$ and $u^+ = \sup\{D(g) : g \in \mathcal{P}\}$, then we can conclude that the desired set $\mathcal{C}$ coincides with one of the following sets:

$$[u^-, u^+],\ (u^-, u^+),\ [u^-, u^+),\ \text{or}\ (u^-, u^+].$$

8. Let us now find u^+. For every $u \in [a, b]$, let us define a function $f_u^+(x)$ as follows: $f_u^+(x) = \max(0, f(x) - \delta)$ for $x < u$, $f_u^+(x) = \min(1, f(x) + \delta)$ for $x \geq u$, and $f_u^+(x) = 0$ for $x \notin (a, b)$. This is a piecewise-continuous non-negative function, and it is easy to check that $|f(x) - f_u^+(x)| \leq \delta$ for all x. So, in view of 5., $f_u^+(x)$ is a membership function and hence, for all u, it is a possible function.

9. Let us prove that for every possible function g, $D(g) \leq D(f_{D(g)}^+)$. This inequality means that to find a supremum u^+ of $D(g)$ for all possible g, it is sufficient to consider only possible functions $f_u^+(x)$ (In other words, $u^+ = \sup\{D(f_u^+) : u \in [a, b]\}$).

Since $g(x)$ is a possible function, for every x we have $0 \leq g(x) \leq 1$ and $|f(x) - g(x)| \leq \delta$. Therefore, $g(x) \leq f(x) + \delta$ and $g(x) \leq 1$, hence $g(x) \leq \min(1, f(x) + \delta)$. Likewise, $g(x) \geq \max(0, f(x) - \delta)$.

From the definition of $D(g)$, we conclude that $D(g) = (\int_a^b xg(x)\,dx)/(\int_a^b g(x)\,dx)$, hence,

$$\int_a^b xg(x)\,dx = D(g)\int_a^b g(x)\,dx = \int_a^b D(g)g(x)\,dx$$

and so, $\int_a^b (x - D(g))g(x)\,dx = 0$. If $x < D(g)$, then $x - D(g) < 0$, and from $g(x) \geq \max(0, f(x) - \delta)$, we conclude that

$$\big(x - D(g)\big)g(x) \leq \big(x - D(g)\big)\max\big(0, f(x) - \delta\big) = \big(x - D(g)\big)f_u^+(x).$$

For $x \geq D(g)$, we have $x - D(g) \geq 0$, and from $g(x) \leq \min(1, f(x) + \delta)$, we conclude that $(x - D(g))g(x) \leq (x - D(g))\min(1, f(x) + \delta) = (x - D(g))f_u^+(x)$.

So, for all x, we have $(x - D(g))g(x) \leq (x - D(g))f_u^+(x)$. Therefore,

$$\int_a^b \big(x - D(g)\big)g(x)\,dx \leq \int_a^b \big(x - D(g)\big)f_u^+(x)\,dx.$$

But we have proved that $\int_a^b (x - D(g))g(x)\,dx = 0$, hence, $\int_a^b (x - D(g))f_u^+(x) \geq 0$. Therefore, $\int_a^b xf_u^+(x)\,dx \geq D(g)\int_a^b f_u^+(x)\,dx$. Dividing both sides of this inequality by $\int_a^b f_u^+(x)\,dx$, we conclude that $D(f_u^+) = (\int_a^b xf_u^+(x)\,dx)/(\int_a^b f_u^+(x) \geq D(g)\,dx)$. □

10. Let us now prove that the function $u \to D(f_u^+)$ is continuous.

Indeed, the numerator $\int_a^b xf_u^+(x)\,dx$ of the fraction that defines $D(f_u^+)$ can be represented as the sum of two integrals: $I_1(u) + I_2(u)$, where $I_1(u) = \int_a^u x\max(0, f(x) - \delta)\,dx$ and $I_2(u) = \int_a^b x\min(1, f(x) - \delta)\,dx$. Both integrals $I_i(u)$ are integrals of a piecewise-continuous function, and the only dependency on u is that u is one of the integration limits. It is well known that the value of an integral of a piecewise-continuous function continuously depends on its limits, so both functions $I_1(u)$ and $I_2(u)$ are continuous in u. Therefore, their sum (i.e., the numerator) is also continuous. Likewise, we can prove that the denominator is continuous. Therefore, this ratio $D(f_u^+)$ is also a continuous function of u.

11. Since $D(f_u^+)$ is a continuous function of u, the supremum

$$u^+ = \sup\{D(f_u^+) : u \in [a, b]\}$$

is attained for some u: $u^+ = D(f_u^+)$ for some u. According to 9., for $g = f_u^+$, we have $D(g) \le D(f^+_{D(g)})$. Since $D(f_u^+)$ already equals the supremum u^+, we conclude that $D(f^+_{D(g)}) = D(g) = u^+$, so $D(f^+_{u^+}) = u^+$. Hence, the desired supremum u^+ satisfies the equation $D(f_u^+) = u$.

12. Let us prove that the desired supremum satisfies also the equation $F^+(u) = 0$ from the formulation of the Theorem.

Indeed, if we substitute the definitions of $D(g)$ and f_u^+ into the equation $D(f_u^+) = u$, we conclude that

$$\left(\int_a^u x \max\left(0, f(x) - \delta\right) dx + \int_u^b x \min\left(1, f(x) + \delta\right) dx\right)$$
$$\Big/\left(\int_a^u \max\left(0, f(x) - \delta\right) dx + \int_u^b \min\left(1, f(x) + \delta\right) dx\right) = u.$$

Multiplying both sides by the denominator, and moving all terms to the left-hand side, we conclude that $F(u) = 0$, where we denoted

$$\begin{aligned} F(u) &= \int_a^u x \max\left(0, f(x) - \delta\right) dx + \int_u^b x \min\left(1, f(x) + \delta\right) dx \\ &\quad - u \int_a^u \max\left(0, f(x) - \delta\right) dx - u \int_u^b \min\left(1, f(x) + \delta\right) dx. \end{aligned}$$

Combining integrals from a to u and from u to b, we conclude that $F(u) = F_1(u) + F_2(u)$, where

$$\begin{aligned} F_1(u) &= \int_a^u x \max\left(0, f(x) - \delta\right) dx - u \int_a^u \max\left(0, f(x) - \delta\right) dx \\ &= \int_a^u x \max\left(0, f(x) - \delta\right) dx - \int_a^u u \max\left(0, f(x) - \delta\right) dx \\ &= \int_a^u (x - u) \max\left(0, f(x) - \delta\right) dx \end{aligned}$$

and, likewise, $F_2(u) = \int_u^b (x - u) \min(1, f(x) + \delta)\, dx$. One can easily see that $F(u) = F_1(u) + F_2(u) = -F^+(u)$, so the equation $F(u) = 0$ is equivalent to the equation $F^+(u) = 0$ from the formulation of the Theorem. Therefore, u^+ really satisfies the equation $F^+(u) = 0$.

13. Let us now prove that the equation $F^+(u) = 0$ has only one solution, and therefore, this solution coincides with the desired supremum u^+.

We have $F^+(u) = -F(u) = -F_1(u) - F_2(u)$. In the first integral $F_1(u)$, the integrated function is always non-negative, hence the whole integral is non-negative; likewise, $F_2(u) \ge 0$. So, $F^+(u) = |F_1(u)| - F_2(u)$. We will prove that $|F_1(u)|$ is non-decreasing, and that $F_2(u)$ is decreasing, and from that we will conclude that the equation $F^+(u) = 0$ has only one solution.

14. Let us first prove that a function $|F_1(u)| = \int_a^u (u - x) \max(0, f(x) - \delta)\, dx$ is non-decreasing in u. Indeed, suppose that $u < v$. Then, $u - x < v - x$, hence for all $x \ge v$, $(u - x) \max(0, f(x) - \delta) \le (v - x) \max(0, f(x) - \delta)$. Therefore,

$$\int_a^v (u - x) \max\left(0, f(x) - \delta\right) \le \int_a^v (v - x) \max\left(0, f(x) - \delta\right) = |F_1(v)|.$$

But $|F_1(u)| = \int_a^u (u - x) \max(0, f(x) - \delta) \le \int_a^v (u - x) \max(0, f(x) - \delta)$, hence $|F_1(u)| \le |F_1(v)|$. So, the function $|F_1(u)|$ is non-decreasing.

15. Let us now prove that a function $F_2(u) = \int_u^b (x - u) \min(1, f(x) + \delta)\, dx$ is strictly decreasing. Indeed, suppose that $u < v$. Then, $x - u > x - v$, and hence $(x - u) \min(1, f(x) +$

$\delta) \geq (x-v)\min(1, f(x)+\delta)$ for all $x \geq v$. From this, we conclude that $\int_v^b (x-u)\min(1, f(x)+\delta) > \int_v^b (x-v)\min(1, f(x)+\delta) = F_2(v)$. Since $\min(1, f(x)+\delta) \geq \min(1,\delta) = \delta > 0$, we have $\int_u^v (u-x)\min(1, f(x)+\delta)\,dx > 0$, hence

$$\begin{aligned} F_2(u) &= \int_v^b (x-u)\min\Big(1, f(x)+\delta\Big) + \int_u^v (u-x)\min\Big(1, f(x)+\delta\Big)\,dx \\ &> \int_v^b (x-u)\min\Big(1, f(x)+\delta\Big) \geq F_2(v), \qquad \text{and} \qquad F_2(u) > F_2(v). \end{aligned}$$

16. For $u < v$, from $|F_1(u)| \leq |F_1(v)|$ and $F_2(u) > F_2(v)$, we conclude that $|F_1(u)| - F_2(u) < |F_1(v)| - F_2(v)$, i.e., that $F^+(u) < F^+(v)$, and so, F^+ is strictly increasing. Therefore, there can be only one value u, for which $F^+(u) = 0$, and hence, this value must coincide with u^+.

We have proved the formula for u^+ The fact that $F^+(u)$ is strictly increasing, explains that we can find u^+ by using a bisection method.

17. For u^-, the proof is similar, but instead of the functions f_u^+ we must consider the functions f_u^- that are equal to 0 outside (a,b), are equal to $\min(1, f(x)+\delta)$ for $a < x \leq u$, and to $\max(0, f(x)-\delta)$ for $u < x < b$. Then we prove:

- that $D(g) \geq d(f^-_{D(g)})$, and therefore, while looking for the infimum u^-, it is sufficient to consider only functions of the type f_u^-;
- that $u \to D(f_u^-)$ is continuous, and therefore the maximum is attained for some u;
- that for this u, $D(f_u^-) = u$.

From the last equation, we conclude that $F^-(u) = 0$, and prove that $F^-(u)$ is strictly increasing and hence, the equation $F^-(u) = 0$ has a unique solution. This solution thus coincides with u^-. □

Applying simple interval methods to estimate the uncertainty.

Let us apply simple interval computations to the function f that is defined as follows: $f(x) = x$ for $0 \leq x \leq 1$, $f(x) = 2-x$ for $1 \leq x \leq 2$, and $f(x) = 0$ else. For this function, $D(f) = \int_0^2 xf(x)\,dx / \int_0^2 f(x)\,dx = 1/1 = 1$. For the numerator $\int_0^2 xg(x)\,dx$, the inequality $|f(x) - g(x)| \leq \delta$ leads to a conclusion that

$$\left|\int xf(x)\,dx - \int xg(x)\,dx\right| \leq \int x|f(x)-g(x)|\,dx \leq \delta\int_0^2 x\,dx = \delta x^2/2|_0^2 = 2\delta.$$

Therefore, the interval of possible values of a numerator is contained in $[1-2\delta, 1+2\delta]$: For a denominator, we obtain the similar estimate:

$$\left|\int f(x)\,dx - \int g(x)\,dx\right| \leq \int |f(x)-g(x)|\,dx \leq \delta\int_0^2 dx = 2\delta$$

so the interval of possible values of the denominator is also contained in $[1-2\delta, 1+2\delta]$. Dividing these two intervals, we conclude that the interval of possible values of $D(g)$ is contained in $[(1-2\delta)/(1+2\delta), (1+2\delta)/(1-2\delta)]$. For small δ, these interval bounds are asymptotically equal to $[1-4\delta+o(\delta), 1+4\delta+o(\delta)]$.

Interval estimates for the centered form.

In this approach, we represent $D(g)$ as $D(f) + A(g)/B(g)$, where $A(g) = \int(x - D(f))g(x)\,dx$ and $B(g) = \int g(x)\,dx$, and then apply interval estimates to this form. Then, for $A(g)$, we get an estimate

$$\begin{aligned} |A(g)| \le \int_a^b \delta|x - D(f)|\,dx &= \delta\Big(\int_a^{D(f)} \big(D(f) - x\big)\,dx + \int_{D(f)}^b \big(x - D(f)\big)\,dx\Big) \\ &= \delta\Big(\big(x^2/2 - xD(f)\big)\Big|_a^{D(f)} + \big(D(f)x - x^2/2\big)\Big|_{D(f)}^b\Big) \\ &= \delta\big(D(f)(a+b) - 1/2(a^2+b^2)\big) \end{aligned}$$

hence the interval for $A(g)$ is

$$\Big[-\delta\big(D(f)(a+b) - 1/2(a^2+b^2)\big), \delta\big(D(f)(a+b) - 1/2(a^2+b^2)\big)\Big].$$

For $B(g)$, we get the interval $[B(f) - \delta(b-a), B(f) + \delta(b-a)]$. So, the resulting interval for $D(g) = D(f) + A(g)/B(g)$ is

$$\begin{aligned} &\Big[D(f) - \delta\big(D(f)(a+b) - 1/2(a^2+b^2)\big)\big/\big(B(f) - (b-a)\delta\big), \\ &\quad D(f) + \delta\big(D(f)(a+b) - 1/2(a^2+b^2)\big)\big/\big(B(f) - (b-a)\delta\big)\Big]. \end{aligned}$$

For small δ, we get

$$\begin{aligned} &\Big[D(f) - \delta\big(D(f)(a+b) - 1/2(a^2+b^2)\big)\big/B(f) + o(\delta), \\ &\quad D(f) + \delta\big(D(f)(a+b) - 1/2(a^2+b^2)\big)\big/B(f) + o(\delta)\Big]. \end{aligned}$$

Proof of Theorem 3.

1. Let us first prove that D is a continuous mapping. For that, it is sufficient to prove that the mappings $\mu \to \int \mu(x)\,dx$ and $\mu \to \int x\mu(x)\,dx$ are continuous, then D_C will be continuous as a ratio of two continuous mappings. Indeed, if $|\mu(x) - \mu'(x)| \le \delta$ for all $x \in [a,b]$, then $-\delta \le \mu(x) - \mu'(x) \le \delta$, hence $-\delta(b-a) \le \int(\mu(x) - \mu'(x))\,dx = \int \mu(x)\,dx - \int \mu'(x)\,dx \le (b-a)\delta$. Therefore, $|\int \mu(x)\,dx - \int \mu'(x)\,dx| \le (b-a)\delta$, and we can easily prove continuity with $\delta = \varepsilon/(b-a)$. Likewise, $|\int x\mu(x)\,dx - \int x\mu'(x)\,dx| \le \delta(\int_a^b |x|\,dx)$, so this expression is also continuous.

2. Let us now prove that center-of-maximum is not continuous. Indeed, let us take a trapezoidal function $\mu(x)$ that is equal to 0 for $|x| > 2$, to 1 for $|x| \le 1$, to $2 - |x|$ for $1 \le |x| \le 2$. Then $m_- = -1$, $m_+ = 1$, hence $D_{COM}(\mu) = 0$. For every δ, we can define a new function $\mu_\delta(x) = \mu(x)(1 - (\delta/3)(1 - |x|))$. For this new function, the maximum (equal to 1) is attained in only one point $x = 1$, so $D_{COM}(\mu_\delta) = 1$.

Since we are considering only the values from -1 to 2, we have $|1 - |x|| \le 3$, hence $(\delta/3)(1 - |x|) \le \delta$, and $|\mu(x) - \mu'(x)| \le \delta$. So, for $\varepsilon = 1/2$, no matter how small $\delta > 0$ we take, we can always find a new function μ_δ such that $\rho(\mu, \mu_\delta) \le \delta$, but $D_{COM}(\mu_\delta) - D_{COM}(\mu) = 1 > 1/2$. So D_{COM} is not continuous. □

Acknowledgments

This work was supported by a NSF Grant No. CDA–9015006, NASA Research Grants No. 9–482 and No. 9–757, and the Institute for Manufacturing and Materials Management grant.

The authors are greatly thankful to Lotfi A. Zadeh, Hung T. Nguyen, and Andrew Bernat for valuable discussions, and to the anonymous referees whose suggestions enabled us to formulate our results in more general interval terms.

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Received: May 25, 1992
Revised version: July 3, 1995

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