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Numerical solutions of Burgers' equation with a large Reynolds number

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In this article the exact solution of Burgers' equation represented as an infinite series is transformed into a simpler form involving the elliptic function $\vartheta_3(v,q)$. To evaluate $\vartheta_3(v,q)$, we use the Jacobi Imaginary Transformation. It is made clear that the solutions obtained by the proposed approach are numerically stable and precise.

Численные решения уравнения Бюргера при большом числе Рейнольдса

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Предложено преобразование точного решения уравнения Бюргера, представленного в виде бесконечного ряда, в более простую форму с использованием эллиптической функции $\vartheta_3(v,q)$. Для вычисления $\vartheta_3(v,q)$ используется мнимое преобразование Якоби. Показано, что полученные таким образом решения являются численно устойчивыми и точными.

1. Introduction

It is known that Burgers' equation: $u_t + uu_x = \nu u_{xx} (= \frac{1}{R} u_{xx})$ has a similiarity to the Navier-Stokes equation. Here, $u = u(x,t)$ denotes the velocity for space and time, and the parameter ν denotes a value which corresponds to an inverse of a Reynolds number R of viscous fluid flow problems. Moreover, the exact solution of Burgers' equation can be expressed as an infinite series due to the so-called Cole-Hopf Transformation [2, 3]. Numerical difficulties, however, have been experienced in evaluating the series when the Reynolds number R is large [1, 4].

In this paper we show how to transform the infinite series into a simple form involving the elliptic function $\vartheta_3(v,q)$. Using the Jacobi Imaginary Transformation for calculating the elliptic function $\vartheta_3(v,q)$, we make clear that numerical solutions with a larger Reynolds number R can be gained with numerical stability.

2. Exact solution of Burgers' equation

We consider the following initial-boundary-value problem of Burgers' equation.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1. \tag{1}$$

Initial condition: $u(x,0) = u^0(x)$, boundary conditions: $u(0,t) = u(1,t) = 0$. For this problem, we apply the Cole-Hopf Transformation: $u(x,t) = -2\nu \frac{1}{\theta}\frac{\partial \theta}{\partial x}$. Then the exact solution can be represented as an infinite series as follows:

$$u(x,t) = \frac{4\pi\nu\sum_{n=1}^{\infty} nI_n\left(\frac{1}{2\nu\pi}\right)\sin(n\pi x)\exp(-n^2\pi^2\nu t)}{I_0\left(\frac{1}{2\nu\pi}\right)+2\sum_{n=1}^{\infty} I_n\left(\frac{1}{2\nu\pi}\right)\cos(n\pi x)\exp(-n^2\pi^2\nu t)} = 4\pi\nu\frac{v(x,t)}{w(x,t)} \tag{2}$$

where I_0 and I_n are the modified Bessel functions. We modify (2) by expressing the numerator $v(x,t)$ and the denominator $w(x,t)$ with the help of an elliptic function.

3. Modification of the exact solution

3.1. Modification for the numerator $v(x,t)$

$$\begin{aligned} v(x,t) &= \frac{1}{2}\int_{-1}^{1} e^{\frac{\cos(\pi\xi)}{2\nu\pi}}\left\{\sum_{n=1}^{\infty} n\cos(n\pi\xi)\sin(n\pi x)\exp(-n^2\pi^2\nu t)\right\}d\xi \\ &= \frac{1}{2}\int_{-1}^{1} e^{\frac{\cos(\pi\xi)}{2\nu\pi}}\left\{\frac{1}{2}\sum_{n=1}^{\infty} n\left\{\sin\left(n\pi(\xi+x)\right)e^{-n^2\pi^2\nu t}+\sin\left(n\pi(\xi-x)\right)e^{-n^2\pi^2\nu t}\right\}\right\}d\xi. \end{aligned}$$

From the definition of the elliptic function $\vartheta_3(v,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2n\pi v$ we see that $\frac{d}{dv}\vartheta_3(v,q) = -4\pi\sum_{n=1}^{\infty} q^{n^2} n\sin(2n\pi v)$. Thus, the numerator $v(x,t)$ of (2) is given as

$$v(x,t) = -\frac{1}{16\pi}\int_{-1}^{1}\exp\left(\frac{\cos(\pi\xi)}{2\nu\pi}\right)\left\{\vartheta_3'\left(\frac{\xi+x}{2},e^{-\pi^2\nu t}\right)+\vartheta_3'\left(\frac{x-\xi}{2},e^{-\pi^2\nu t}\right)\right\}d\xi. \tag{3}$$

Using the two relations $2\frac{d}{d\xi}\vartheta_3\left(\frac{\xi+x}{2}\right) = \vartheta_3'\left(\frac{\xi+x}{2}\right)$, $-2\frac{d}{d\xi}\vartheta_3\left(\frac{x-\xi}{2}\right) = \vartheta_3'\left(\frac{x-\xi}{2}\right)$ and partial integration, (3) yields

$$\begin{aligned} v(x,t) &= -\frac{1}{16\pi}\left\{\exp\left(\frac{\cos\pi\xi}{2\nu\pi}\right)\left\{2\vartheta_3\left(\frac{\xi+x}{2}\right)-2\vartheta_3\left(\frac{x-\xi}{2}\right)\right\}\Bigg|_{-1}^{1}\right. \\ &\quad \left.+\int_{-1}^{1}\frac{1}{2\nu\pi}\pi\sin(\pi\xi)\exp\left(\frac{\cos\pi\xi}{2\nu\pi}\right)\left\{2\vartheta_3\left(\frac{\xi+x}{2}\right)-2\vartheta_3\left(\frac{x-\xi}{2}\right)\right\}d\xi\right\} \\ &= -\frac{1}{16\pi}\left\{4\exp\left(\frac{-1}{2\nu\pi}\right)\left\{\vartheta_3\left(\frac{1+x}{2}\right)-\vartheta_3\left(\frac{x-1}{2}\right)\right\}\right. \\ &\quad \left.+\frac{1}{\nu}\int_{-1}^{1}\sin(\pi\xi)\exp\left(\frac{\cos\pi\xi}{2\nu\pi}\right)\left\{\vartheta_3\left(\frac{\xi+x}{2}\right)-\vartheta_3\left(\frac{x-\xi}{2}\right)\right\}d\xi.\right\} \end{aligned}$$

Here we abbreviate $\vartheta_3(v,q)$ as $\vartheta_3(v)$. Moreover, (3) can be represented in the following form using the periodicity of the elliptic function (as $\vartheta_3(\frac{x+1}{2}) = \vartheta_3(\frac{x-1}{2})$).

$$\begin{aligned} v(x,t) &= \frac{-1}{16\pi\nu}\int_{-1}^{1}\sin(\pi\xi)\exp\left(\frac{\cos\pi\xi}{2\nu\pi}\right)\left\{\vartheta_3\left(\frac{\xi+x}{2}\right)-\vartheta_3\left(\frac{x-\xi}{2}\right)\right\}d\xi \\ &= \frac{1}{16\pi\nu}\left(-\int_{-1}^{1}\sin(\pi\xi)\exp\left(\frac{\cos\pi\xi}{2\nu\pi}\right)\vartheta_3\left(\frac{\xi+x}{2}\right)d\xi\right. \\ &\quad \left.+\int_{-1}^{1}\sin(\pi\xi)\exp\left(\frac{\cos\pi\xi}{2\nu\pi}\right)\vartheta_3\left(\frac{x-\xi}{2}\right)d\xi\right). \end{aligned}$$

With the variable transformation $\xi = -\eta$ it can finally be seen that the first term is the same as the second one so that

$$v(x,t) = \frac{1}{8\pi\nu}\int_{-1}^{1}\sin(\pi\xi)\exp\left(\frac{\cos\pi\xi}{2\nu\pi}\right)\vartheta_3\left(\frac{x-\xi}{2}, e^{-\pi^2\nu t}\right)d\xi. \tag{4}$$

3.2. Modification for the denominator $w(x,t)$

The modification for the denominator $w(x,t)$ of (2) is given as follows:

$$\begin{aligned}
w(x,t) &= \frac{1}{2}\int_{-1}^{1} e^{\frac{\cos(\pi\xi)}{2\nu\pi}}\left\{1+2\sum_{n=1}^{\infty}\cos(n\pi\xi)\cos(n\pi x)\exp(-n^2\pi^2\nu t)\right\}d\xi \\
&= \frac{1}{2}\int_{-1}^{1} e^{\frac{\cos(\pi\xi)}{2\nu\pi}}\left\{1+2\sum_{n=1}^{\infty}\frac{1}{2}\Big\{\cos\big(n\pi(\xi+x)\big)+\cos\big(n\pi(\xi-x)\big)\Big\}e^{-n^2\pi^2\nu t}\right\}d\xi. \\
&= \frac{1}{4}\int_{-1}^{1}\exp\left(\frac{\cos(\pi\xi)}{2\nu\pi}\right)\left\{\vartheta_3\left(\frac{\xi+x}{2}\right)+\vartheta_3\left(\frac{-\xi+x}{2}\right)\right\}d\xi \\
&= \frac{1}{4}\int_{-1}^{1}\exp\left(\frac{\cos(\pi\xi)}{2\nu\pi}\right)\vartheta_3\left(\frac{\xi+x}{2}\right)d\xi+\frac{1}{4}\int_{-1}^{1}\exp\left(\frac{\cos(\pi\xi)}{2\nu\pi}\right)\vartheta_3\left(\frac{-\xi+x}{2}\right)d\xi. \qquad (5)
\end{aligned}$$

In (5) the first term is the same as the second one because of the variable transformation $\xi = -\eta$. Therefore, the denominator of (2) is given as

$$w(x,t) = \frac{1}{2}\int_{-1}^{1}\exp\left(\frac{\cos(\pi\xi)}{2\nu\pi}\right)\vartheta_3\left(\frac{x-\xi}{2}, e^{-\pi^2\nu t}\right)d\xi. \tag{6}$$

3.3. Computable form for the exact solution

Finally, the exact solution $u(x,t)$ of Burgers' equation is represented as follows:

$$\begin{aligned}
u(x,t) &= 4\pi\nu\frac{\frac{1}{8\pi\nu}\int_{-1}^{1}\sin\pi\xi\exp\left(\frac{\cos(\pi\xi)}{2\nu\pi}\right)\vartheta_3\left(\frac{x-\xi}{2}, e^{-\pi^2\nu t}\right)d\xi}{\frac{1}{2}\int_{-1}^{1}\exp\left(\frac{\cos(\pi\xi)}{2\nu\pi}\right)\vartheta_3\left(\frac{x-\xi}{2}, e^{-\pi^2\nu t}\right)d\xi} \\
&= \frac{\int_{-1}^{1}\sin\pi\xi\exp\left(\frac{\cos(\pi\xi)}{2\nu\pi}\right)\vartheta_3\left(\frac{x-\xi}{2}, e^{-\pi^2\nu t}\right)d\xi}{\int_{-1}^{1}\exp\left(\frac{\cos(\pi\xi)}{2\nu\pi}\right)\vartheta_3\left(\frac{x-\xi}{2}, e^{-\pi^2\nu t}\right)d\xi}. \qquad (7)
\end{aligned}$$

4. On the computation of $\vartheta_3(v,q)$

The elliptic function $\vartheta_3(v,q)$ is usually defined as

$$\vartheta_3(v,q) = 1+2\sum_{n=1}^{\infty}q^{n^2}\cos 2n\pi v. \tag{8}$$

However, when q $(= e^{-\pi^2\nu t})$ is near 1, that is, the parameter ν is small, this series converges very slowly. Then it is not effective and efficient for us to compute $\vartheta_3(v,q)$ via (8). To avoid these numerical difficulties, we adopted the Jacobi Imaginary Transformation (abbreviated J.I.T.) given by

$$\begin{aligned}\vartheta_3(v, e^{-\pi^2\nu t}) &= \sqrt{c}e^{-\pi c v^2}\left\{1 + 2\sum_{n=1}^{\infty} e^{-n^2 c\pi}\cosh 2\pi n c v\right\} \\ &= \sqrt{c}e^{-\pi c v^2}\left\{1 + 2(e^{-\pi c}\cosh 2\pi c v + e^{-4\pi c}\cosh 4\pi c v + e^{-9\pi c}\cosh 6\pi c v + \cdots)\right\}\end{aligned} \tag{9}$$

where $c = \frac{-\pi}{\log q} = \frac{1}{\pi\nu t}$. Considering the above series, we can see that the remainder with the exception of the three terms in the brackets of (9) is negligibly small, so that this series converges rapidly as compared with (8). Accordingly, we expect that it is sufficient and efficient for the elliptic function $\vartheta_3(v, e^{-\pi^2\nu t})$ to be approximated as

$$\begin{aligned}\vartheta_3(v, e^{-\pi^2\nu t}) &= \sqrt{\frac{1}{\pi\nu t}}\exp\left(\frac{-v^2}{\nu t}\right) \\ &\times\left\{1 + 2\left(e^{\frac{-1}{\nu t}}\cosh\left(\frac{2v}{\nu t}\right) + e^{\frac{-4}{\nu t}}\cosh\left(\frac{4v}{\nu t}\right) + e^{\frac{-9}{\nu t}}\cosh\left(\frac{6v}{\nu t}\right)\right)\right\}.\end{aligned} \tag{10}$$

5. Numerical experiments

In this section, we discuss some numerical approaches for the exact solution of Burgers' equation. All the computations were done on a workstation Sun-4/10 in double precision arithmetic and IEEE extended precision arithmetic. We tested three numerical approaches: (i) Equation (2) with the Bessel function, (ii) Equation (7) with Equation (8) and (iii) Equation (7) with Equation (10), i.e., J.I.T. In the tables below, we abbreviate these approaches as (i) Bessel, (ii) Equation (8) and (iii) J.I.T., respectively.

In the first approach, we used a mathematical library for the computation of the modified Bessel function $I_n(x)$ in double precision arithmetic. In this case, the range of the variable x is restricted to $|x| < 173$. In the latter two approaches, numerical integration is necessary in (7). Since the integrands are periodic, we took the trapezoidal rule.

In Tables 1 and 2 we show the numerical results of the three approaches for the parameter $\nu = 0.01$ at $t = 0.4$, $t = 1.2$ and $0.2 \le x \le 0.98$. In addition, we show the results obtained with the asymptotic approximations of Cole [2]. When we had to evaluate an infinite series $\sum_{n=1}^{\infty} a_n$ in (i) Bessel and (ii) Equation (8), we adopted as the convergence criterion that $|a_n - a_{n-1}|/|a_n|$ $(= \varepsilon)$ is less than 10^{-12}. Concerning the sample points of the numerical integration, we show the results using 4097 points for (ii) Equation (8) and those using 257 points for (iii) J.I.T. in these tables.

The calculation of (i) Bessel was done in double precision arithmetic because of the limitation of x for the Bessel function $I_n(x)$. The calculation of (ii) and (iii) were done in IEEE extended precision arithmetic. From these tables, it can be seen that the solutions at $t = 1.2$ agree with each other in comparison with those at $t = 0.4$. Moreover, in Table 1, we can observe that the solutions of (i) Bessel do not agree with (ii) Equation (8) and (iii) J.I.T. when x is near 1.

In Table 3, we show the results of (ii) Equation (8) and (iii) J.I.T. for the parameter $\nu = 0.01$ at $t = 0.4$, $x = 0.40$ when the number of sample points of the numerical integration varies. The convergence criterion ε for evaluating the infinite series in (ii) Equation (8) is the

x	(i) Bessel	(ii) Equation (8)	(iii) J.I.T.	Cole
0.20	0.27452386	0.27452390	0.27452386	0.27452
0.40	0.53792160	0.53791933	0.53792160	0.53792
0.60	0.77345464	0.77345467	0.77345464	0.77345
0.80	0.94103638	0.94103653	0.94103631	0.94100
0.90	0.95246557	0.95245234	0.95245224	0.94891
0.92	0.93671097	0.93669749	0.93669806	0.93655
0.94	0.90182451	0.90169973	0.90170253	0.89998
0.96	0.81309686	0.81289557	0.81289599	0.81227
0.98	0.55870490	0.55850821	0.55851721	0.56583

Table 1. Comparison of four methods for the parameter $\nu = 0.01$ at $t = 0.4$ and $0.20 \leq x \leq 0.98$

x	(i) Bessel	(ii) Equation (8)	(iii) J.I.T.	Cole
0.20	0.13092009	0.13092010	0.13092009	0.13092
0.40	0.26128123	0.26128124	0.26128123	0.26128
0.60	0.39043845	0.39043846	0.39043845	0.39044
0.80	0.51752803	0.51752800	0.51752803	0.51753
0.90	0.57781210	0.57781208	0.57781210	0.57781
0.92	0.58472371	0.58472372	0.58472371	0.58472
0.94	0.57778815	0.57778815	0.57778815	0.57779
0.96	0.52523899	0.52523910	0.52523899	0.52524
0.98	0.35060231	0.35060206	0.35060231	0.35061

Table 2. Comparison of four methods for the parameter $\nu = 0.01$ at $t = 1.2$ and $0.20 \leq x \leq 0.98$

Sample	(ii) Equation (8)	(iii) J.I.T.
33	0.53637410	0.53792162
65	0.53714812	0.53792160
129	0.53753493	0.53792160
257	0.53772828	0.53792160
512	0.53782494	0.53792160
1024	0.53787327	0.53792160
2048	0.53789744	0.53792160
4096	0.53791933	0.53792160

Table 3. Numerical results of Equation (8) and J.I.T. for the parameter $\nu = 0.01$ at $t = 0.4$, $x = 0.40$ when the number of sample points of the numerical integration varies

Sample	ε=10^{-8} (105)	ε=10^{-12} (115)	ε=10^{-15} (125)
256	0.41163746	0.41163745	0.41163745
512	0.41178366	0.41178365	0.41178365
1024	0.41184801	0.41184800	0.41184800
2048	0.41188178	0.41188178	0.41188178
4096	0.41205971	0.41205971	0.41205971
8192	0.41194105	0.41194105	0.41194105
16384	0.41189615	0.41189615	0.41189615
32768	0.41186312	0.41186313	0.41186313
65536	0.41187385	0.41187385	0.41187385

Table 4. Numerical solutions by (ii) Equation (8) for the parameter $\nu = 0.001$ at $t = 0.4$, $x = 0.30$ when the convergence criterion ε varies

Sample	(iii) J.I.T.
32	0.387704357344037
64	0.411466334934168
128	0.411686285023018
256	0.411686285023031
512	0.411686285023031
1024	0.411686285023031
2048	0.411686285023031
4096	0.411686285023031

Table 5. Numerical solutions by (iii) J.I.T. for the parameter $\nu = 0.001$ at $t = 0.4$, $x = 0.30$ when the number of the sample points of the numerical integration varies

same as that of Tables 1, 2. From Table 3, we can easily see that the convergence rate of the numerical integration of (ii) Equation (8) is very slow as compared to that of (iii) J.I.T.

In Tables 4 and 5, we show solutions using (ii) Equation (8) and (iii) J.I.T. for the parameter $\nu = 0.001$ at $t = 0.4$, $x = 0.30$. In Table 4, we investigated the relation between the sample points of the numerical integration and the convergence criterion for the infinite series for $\varepsilon = 10^{-8}$, 10^{-12} and 10^{-15}, respectively. The number in brackets is the number of terms of the series necessary to satisfy the convergence criterion. We mention that in this case the solution by (i) Bessel is 0.41235888. From Table 4, it can be observed that the solutions oscillate as the number of the sample points of the numerical integration increases. Therefore, we can see that the approach of (ii) Equation (8) has a numerical instability. On the contrary, Table 5 shows that the proposed approach of (iii) J.I.T. is numerically stable.

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