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An estimate of the absolute value and width of the solution of a linear system of equations with tridiagonal interval matrix by the interval sweep method

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We consider linear systems of algebraic equations $Su = \mathbf{f}$ with tridiagonal interval matrix $\mathbf{S}$ and interval vector $\mathbf{f}$ An interval version of the sweep method allows us to find an interval vector $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n)^\top$ that contains the united set of solutions of the system. In the paper we present estimates of the absolute value and the width of the intervals $\mathbf{u}_i$, $i = 1, 2, \dots, n$ under certain assumptions on the elements of the matrix $\mathbf{S}$ that do not include the traditional condition of diagonal dominance. The width estimates are three orders of magnitude narrower, and the assumptions on the system's coefficients are weaker than those in works published so far.

Оценка модуля и ширины решения системы линейных уравнений с трехдиагональной интервальной матрицей методом интервальной прогонки

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Рассматривается система линейных алгебраических уравнений $\mathbf{S}u = \mathbf{f}$ с трехдиагональной интервальной матрицей $\mathbf{S}$ и интервальным вектором $\mathbf{f}$. Интервальная версия метода прогонки позволяет отыскать интервальный вектор $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n)^\top$, содержащий объединенное множество решений этой системы. В работе при некоторых ограничениях на элементы матрицы $\mathbf{S}$, не предполагающих традиционного условия диагонального преобладания, даются оценки абсолютного значения и ширины интервалов $\mathbf{u}_i$, $i = 1, 2, \dots, n$. Оценки ширины на три порядка меньше, а ограничения на коэффициенты системы слабее, чем в ранее известных публикациях.

Introduction

In many problems of computational mathematics, we need to solve a linear system of algebraic equations

$$Su = f \tag{1}$$

which has the tridiagonal matrix $S = \mathrm{tridiag}(a_i, b_i, c_i)$, $i = 1, 2, \dots, n$. (We formally set $a_1 = c_n = 0$.) Under the condition of so called *strict diagonal dominance*,

$$|a_i| + |c_i| \le |b_i| - \varepsilon, \quad \varepsilon > 0, \ i = 1, 2, \dots, n \tag{2}$$

the system (1) can be solved by the tridiagonal version of the Gauss algorithm, called the *sweep method*. This method involves sequential application of the three recursion relations:

$$\begin{array}{lllllll} x_0 &=& 0, & x_i &=& -c_i/(b_i+a_ix_{i-1}), & a_1 = c_n = 0, \\ y_0 &=& 0, & y_i &=& (f_i-a_iy_{i-1})/(b_i+a_ix_{i-1}), & i = 1,2,\dots,n, \\ u_n &=& y_n, & u_i &=& y_i+x_iu_{i+1}, & i = n-1,n-2,\dots,1. \end{array} \tag{3}$$

When condition (2) is satisfied, the sweep method is reasonably well-studied [2, 4, 7].

For various reasons, the coefficients of the matrix S and the vector f in (1) should be considered imprecise. Then, instead of the system (1) we have to examine the system

$$\mathbf{S}u = \mathbf{f} \tag{4}$$

which has the tridiagonal interval matrix $\mathbf{S} = \operatorname{tridiag}(\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i)$ and the interval vector $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n)^\top$. The set

$$\mathrm{SOL}(\mathbf{S},\mathbf{f}) = \{\, u \in I\!R^n \mid (\exists S \in \mathbf{S})(\exists f \in \mathbf{f})(\, Su = f\,)\,\} \tag{5}$$

is called the *united solution set* of the system (4). The structure of this set may be very unusual. For instance, even for $n = 2$ this set may be nonconvex [3]. It is obvious that finding and describing such sets presents serious difficulties and is unlikely to be easy. Within the scope of interval computations, there emerges the problem of finding an interval vector $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n)^\top$ such that

$$\mathbf{u} \supseteq \mathrm{SOL}(\mathbf{S},\mathbf{f}). \tag{6}$$

Let us introduce some notation. Let $\mathbf{a} = [\underline{a}, \overline{a}]$ be the interval $(\,\underline{a} \le \overline{a}\,)$. Then

$\langle \mathbf{a} \rangle$ is the closest distance from the points of the interval $\mathbf{a}$ to zero;

$d(\mathbf{a}) = \overline{a} - \underline{a}$ is the *width* of the interval $\mathbf{a}$;

$$\langle \mathbf{a} \rangle_0 = \begin{cases} \langle \mathbf{a} \rangle, & \text{if } \underline{a} \ge 0, \text{ or } 0 \in \mathbf{a}, \\ -\langle \mathbf{a} \rangle, & \text{if } \overline{a} < 0. \end{cases}$$

Supposing that the matrix $\mathbf{S}$ of the system (4) satisfies the condition

$$0 \notin \mathbf{b}_i, \quad i = 1,2,\dots,n$$

we denote

$$\begin{array}{llllll} \mathbf{a}'_i = [\underline{a}'_i, \overline{a}'_i] = \mathbf{a}_i/\langle \mathbf{b}_i \rangle_0, & \mathbf{b}'_i = \mathbf{b}_i/\langle \mathbf{b}_i \rangle_0, & \mathbf{c}'_i = [\underline{c}'_i, \overline{c}'_i] = \mathbf{c}_i/\langle \mathbf{b}_i \rangle_0, \\ \mathbf{f}'_i = \mathbf{f}_i/\langle \mathbf{b}_i \rangle_0, & f = \max|\mathbf{f}|, & i = 1,2,\dots,n. \end{array} \tag{7}$$

Note that

$$\langle \mathbf{b}'_i \rangle = \underline{b}'_i = 1. \tag{8}$$

In [5], the sweep method under the condition of *strict interval diagonal dominance*

$$|\mathbf{a}'_i| + |\mathbf{c}'_i| \le 1 - \delta, \quad \delta \in (0,1),\ i = 1,2,\dots,n \tag{9}$$

has been extended to the interval case (4). It was shown that under the conditions (9) the following sequence of interval recursion relations is realizable (that is, no division by intervals containing zero occurs):

$$\mathbf{x}_0 = 0, \quad \mathbf{x}_i = -\mathbf{c}_i/(\mathbf{b}_i + \mathbf{a}_i\mathbf{x}_{i-1}), \quad \mathbf{a}_1 = \mathbf{c}_n = 0, \tag{10}$$

$$\mathbf{y}_0 = 0, \quad \mathbf{y}_i = (\mathbf{f}_i - \mathbf{a}_i\mathbf{y}_{i-1})/(\mathbf{b}_i + \mathbf{a}_i\mathbf{x}_{i-1}), \quad i = 1,2,\dots,n, \tag{11}$$

$$\mathbf{u}_n = \mathbf{y}_n, \quad \mathbf{u}_i = \mathbf{y}_i + \mathbf{x}_i\mathbf{u}_{i+1}, \quad i = n-1, n-2\dots,1. \tag{12}$$

This is called in [5] the *interval analytic sweep method*. Besides, the interval vector $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n)^\top$ whose components are computed from (10)–(12) satisfies the condition (6). If the additional condition

$$|\mathbf{a}'_i| \le |\mathbf{c}'_i| + \delta \tag{13}$$

is satisfied, then [5] gives an estimate $d(\mathbf{u}_i)$ which does not depend on n, and is reducible to the form

$$d(\mathbf{u}_i) \le 3f\lambda/\delta^7 \tag{14}$$

where

$$\lambda = \max\{d(\mathbf{a}'_i), d(\mathbf{b}'_i), d(\mathbf{c}'_i), d(\mathbf{f}'_i)\}, \quad i = 1, 2, \dots, n. \tag{15}$$

It should be emphasized that the estimate (14) is not an estimate of $\mathrm{SOL}(\mathbf{S}, \mathbf{f})$ itself. Formula (14) presents an estimate of the vector $\mathbf{u}$ which has been found using (10)–(12) and which, as was noted, contains $\mathrm{SOL}(\mathbf{S}, \mathbf{f})$.

In this paper we weaken the conditions (9) and (13) and find some other conditions sufficient for the independence of the absolute value $|\mathbf{u}_i|$ and width $d(\mathbf{u}_i)$ on the order n of the system. The estimate of width $d(\mathbf{u}_i)$ in our work has the order $O(1/\delta^4)$, i.e. is three orders of magnitude smaller than the estimate (14).

1. Sufficient conditions for boundedness of an interval continued fraction

The recurrence relation (10) generates an interval continued fraction

$$\mathbf{x}_i = \cfrac{-\mathbf{c}_i}{\mathbf{b}_i - \cfrac{\mathbf{a}_i\mathbf{c}_{i-1}}{\mathbf{b}_{i-1} - \ddots\, \mathbf{b}_2 - \cfrac{\mathbf{a}_2\mathbf{c}_1}{\mathbf{b}_1}}}. \tag{16}$$

That is why it deserves a separate consideration. If we use (7), we can write down (10) as

$$\mathbf{x}_0 = 0, \quad \mathbf{x}_i = -\mathbf{c}'_i/(\mathbf{b}'_i + \mathbf{a}'_i\mathbf{x}_{i-1}), \quad \mathbf{a}'_1 = \mathbf{c}'_n = 0, \quad i = 1, 2, \dots, n. \tag{17}$$

Theorem 1. *Let the coefficients* $\mathbf{a}'_i, \mathbf{b}'_i \not\ni 0$, $i = 1, 2, \dots, n$, $\mathbf{c}_i \ne 0$, $i = 1, 2, \dots, n-1$ *in (17) satisfy the condition*

$$\begin{cases} r^2|\mathbf{a}'_i| + |\mathbf{c}'_i| \le r, & i = 1, 2, \dots, n-1, \\ |\mathbf{a}'_n| < 1/r \end{cases} \tag{18}$$

for some $r > 0$. *Then*

1^o $\underline{g}'_i > 0$, *where* $\mathbf{g}'_i = [\underline{g}'_i, \overline{g}'_i] = \mathbf{b}'_i + \mathbf{a}'_i\mathbf{x}_{i-1}$,

2^o $\mathbf{x}_i \subseteq [-r, r]$, $\quad i = 1, 2, \dots, n,$ (19)

3^o *the estimating interval* $[-r, r]$ *in (19) and the condition (18) are the best, i.e.*

(*a*) *there exists a recurrece relation of the form (17) satisfying (18) for which the estimate (19) is achievable, i.e.* $\mathbf{x}_i = [-r, r]$ *for some* $i \le n-1$, (*b*) *for any pair of intervals* $(\mathbf{a}', \mathbf{c}')$, $\mathbf{c}' \neq 0$ *not satisfying the condition (18) it is possible to find a formula of the form (17) such that for all* $i = 1, 2, \ldots, k-1 < n-1$ *the condition (18) is satisfied and for* $\mathbf{a}'_k = \mathbf{a}'$, $\mathbf{c}'_k = \mathbf{c}'$ *the inclusion (19) does not hold.*

Proof. For $i = 1$, from (8), (17), and (18) imply

$$|\mathbf{x}_1| = |\mathbf{c}'_1/\mathbf{b}'_1| = |\mathbf{c}'_1|/\langle \mathbf{b}'_1 \rangle = |\mathbf{c}'_1| \leq r.$$

Let $\mathbf{x}_{i-1} \subseteq [-r, r]$. Then

$$\mathbf{x}_i \subseteq \mathbf{c}'_i/(\mathbf{b}'_i + [-r, r]\mathbf{a}'_i). \tag{20}$$

From (8) and the theorem's hypothesis we have

$$\underline{g}'_i \geq \underline{b}'_i - |\mathbf{a}'_i||\mathbf{x}_{i-1}| \geq 1 - |\mathbf{a}'_i| r \geq |\mathbf{c}'_i|/r > 0$$

which proves 1°. From (20) we have

$$|\mathbf{x}_i| \leq |\mathbf{c}'_i|/\langle \mathbf{b}'_i + [-r, r]\mathbf{a}'_i \rangle = |\mathbf{c}'_i|/(1 - r|\mathbf{a}'_i|).$$

This, together with (18), implies

$$|\mathbf{x}_i| \leq r|\mathbf{c}'_i|/(r - r^2|\mathbf{a}'_i|) \leq r|\mathbf{c}'_i|/|\mathbf{c}'_i| = r$$

which proves 2°.

To prove 3°(a) let us take (17) with

$$\begin{array}{llllll} \mathbf{a}'_1 = 0, & \mathbf{a}'_i \subseteq (-1/r, 1/r), & & i = 2, 3, \ldots, n, \\ \mathbf{c}'_n = 0, & \mathbf{c}'_i = [-r + r^2|\mathbf{a}'_i|,\ r^2 - r|\mathbf{a}'_i|], & \mathbf{b}'_i = 1, & i = 1, 2, \ldots, n-1. \end{array} \tag{21}$$

Then $r^2|\mathbf{a}'_i| + |\mathbf{c}'_i| = r$, i.e. the condition (18) is satisfied. It is easy to see that $\mathbf{x}'_i = [-r, r]$, $i = 1, 2, \ldots, n-1$.

Now let us prove 3°(b). For $i = 1, 2, \ldots, k-1 \leq n-2$, the coefficients $\mathbf{a}'_i$, $\mathbf{b}'_i$, $\mathbf{c}'_i$ can be borrowed from (21). Let's take $\mathbf{a}'_k = \mathbf{a}'$, $\mathbf{b}'_k = 1$, $\mathbf{c}'_k = \mathbf{c}'$. Then

$$\mathbf{x}_k = -\mathbf{c}'/(1 + \mathbf{a}'[-r, r]).$$

If $1 - r|\mathbf{a}'| < 0$ then $0 \in 1 + \mathbf{a}'[-r, r]$, and (19) is evidently not satisfied. Let $1 - r|\mathbf{a}'| > 0$. Then $|\mathbf{x}_k| \leq |\mathbf{c}'|/(1 - r|\mathbf{a}'|)$, and condition (18) is necessary for 2° to hold. □

Comment 1. Condition (18) can be interpreted geometrically as follows. An interval "point" $(\mathbf{a}', \mathbf{b}')$ can be represented in the coordinate system $a'Oc'$ by a rectangle with its vertices at the points $(\underline{a}', \underline{b}')$, $(\underline{a}', \overline{b}')$, $(\overline{a}', \underline{b}')$, $(\overline{a}', \overline{b}')$. Consider the rhombus $\nu(r)$ defined by the inequality

$$r^2|a'| + |c'| \leq r.$$

Condition (18) defines the interval points $(\mathbf{a}'_i, \mathbf{c}'_i)$ that lie entirely inside $\nu(r)$. Condition (9) for $1 - \delta = r$ defines interval points that lie entirely inside the square $\mu(r)$ given by the inequality $|a'| + |c'| \leq r$. For $r < 1$ we have $\mu(r) \subseteq \nu(r)$ and $|\mathbf{x}_i| \leq r$, $i = 1, 2, \ldots, n-1$. Thus, condition (18) is a generalization of (9). Note that the area of the rhombus $\nu(r)$ does not depend on r, and equals 2, whereas the area of the square $\mu(r)$ equals $2r^2 < 2$.

Comment 2. For $r_1 \neq r_2$ none of the rhombi $\nu(r_1)$, $\nu(r_2)$ lies inside another. Consider two such rhombi, for example, $\nu(0.5)$ and $\nu(0.7)$. It is easy to imagine a set of interval points $(\mathbf{a}'_i, \mathbf{c}'_i)$ that lie in $\nu(0.5)$ but not in $\nu(0.7)$. Then, for the corresponding formula (17), Theorem 1 does not allow us to suppose that all $|\mathbf{x}_i| \leq 0.7$. On the other hand, however, it is this Theorem that guarantees that all $|\mathbf{x}_i| \leq 0.5$.

Comment 3. We may consider the reciprocal problem for the rhombus $\nu(r)$: Find the condition to apply to the set $\{(\mathbf{a}'_i,\ \mathbf{c}'_i) \mid i = 1, 2, \dots, n-1\}$ which is necessary and sufficient for all $(\mathbf{a}'_i,\ \mathbf{c}'_i)$ to belong to a rhombus $\nu(r)$, and then estimate $\inf r$ in the class of formulae of the form (17) satisfying the condition found. The solution to this problem can be follows from material in [6].

Analogously to Theorem 1, we can prove the following two theorems.

Theorem 2. *Let the coefficients* $\mathbf{a}'_i, \mathbf{b}'_i \not\ni 0$, $i = 1, 2, \dots, n$, $\mathbf{c}'_i \neq 0$, $i = 1, 2, \dots, n-1$ *in (17) satisfy the condition*

$$\begin{cases} \mathbf{c}'_i \subseteq [0, r], \\ r^2 \overline{a}'_i + \overline{c}'_i \leq r, \quad i = 1, 2 \dots, n-1, \\ \mathbf{a}'_n \subseteq (-\infty, 1/r) \end{cases} \tag{22}$$

for some $r > 0$. *Then*

$$\begin{aligned} &1^o\ \underline{g}'_i > 0, \text{ where } \mathbf{g}'_i = \left[\underline{g}'_i, \overline{g}'_i\right] = \mathbf{b}'_i + \mathbf{a}'_i \mathbf{x}_{i-1}, \\ &2^o\ \mathbf{x}_i \subseteq [-r, 0], \quad i = 1, 2 \dots, n, \end{aligned} \tag{23}$$

3^o *the estimating interval* $[-r, 0]$ *in (23) and the condition (22) are the best (just as in Theorem 1).*

Theorem 3. *Let the coefficients* $\mathbf{a}'_i$, $\mathbf{b}'_i \not\ni 0$, $i = 1, 2, \dots, n$, $\mathbf{c}'_i \neq 0$, $i = 1, 2, \dots, n-1$ *in (17) satisfy the condition*

$$\begin{cases} \mathbf{c}'_i \subseteq [-r, 0], \\ -r^2 \underline{a}'_i - \underline{c}'_i \leq r, \quad i = 1, 2, \dots, n-1, \\ \mathbf{a}'_n \subseteq (-1/r, \infty) \end{cases} \tag{24}$$

for some $r > 0$. *Then*

$$\begin{aligned} &1^o\ \underline{g}'_i > 0, \text{ where } \mathbf{g}'_i = \left[\underline{g}'_i, \overline{g}'_i\right] = \mathbf{b}'_i + \mathbf{a}'_i \mathbf{x}_{i-1}, \\ &2^o\ \mathbf{x}_i \subseteq [0, r], \quad i = 1, 2, \dots, n, \end{aligned} \tag{25}$$

3^o *the estimating interval* $[0, r]$ *in (25) and the condition (24) are the best.*

2. Boundedness conditions and estimation of the width of the components of the vector $\mathbf{u}$ produced by the interval analytical sweep method

Theorem 4. *Let the coefficients* $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{c}_i$ *of the matrix* S *of the system (4) satisfy the condition of one of the Theorems 1–3 (see (7)).*

Then

> 1^o *the principal minors of any matrix* $S \in \mathbf{S}$ *never become zero;*
> 2^o *the interval vector* $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n)^\top$ *that is found using (10)–(12) satisfies condition (6).*

Proof. It is easy to show using induction on i that the principal minors Δ_i of the numerical matrix $S = \operatorname{tridiag}(a_i, b_i, c_i) \in \mathbf{S}$ satisfy the relation

$$\Delta_0 = 1, \quad \Delta_1 = b_1, \quad \Delta_i = b_i\Delta_{i-1} - a_i c_{i-1}\Delta_{i-2}, \quad i = 2, 3, \dots, n. \tag{26}$$

Suppose that

$$\Delta_{i-1} = (b_{i-1} + a_{i-1}x_{i-2})\Delta_{i-2} \tag{27}$$

where x_i is defined from (3). Note that $b_{i-1} + a_{i-1}x_{i-2} \neq 0$ from one of the Theorems 1–3. From (26), (27), and (3) mathematical induction gives

$$\Delta_i = b_i\Delta_{i-1} - a_i c_{i-1}\frac{\Delta_{i-1}}{b_{i-1} + a_{i-1}x_{i-2}} = (b_i + a_i x_{i-1})\Delta_{i-1}, \quad i = 1, 2, \dots, n.$$

Hence, again applying induction, we obtain 1^o.

Let us prove 2^o. By virtue of the principal theorem of interval arithmetic [1, 5], if x_1, y_1 are found from (3) then $x_1 \in \mathbf{x}_1$, $y_1 \in \mathbf{y}_1$. Suppose that $x_{i-1} \in \mathbf{x}_{i-1}$, $y_{i-1} \in \mathbf{y}_{i-1}$. Then, by the same theorem, from (3), (10), (11) and induction we have $x_i \in \mathbf{x}_i$, $y_i \in \mathbf{y}_i$, $i = 1, 2, \dots, n$. In particular, $u_n = y_n \in \mathbf{y}_n = \mathbf{u}_n$. Further, from the principal theorem of interval arithmetic mathematical induction gives $u_i \in \mathbf{u}_i$, $i = n, n-1, \dots, 1$. □

We will use the notation λ from (15). For the value of r from Theorems 1–3, we set $r = 1 - \delta$ for some $\delta \in (0, 1)$. Applying additional constraints in the hypotheses of Theorems 1–3, we obtain the following three theorems.

Theorem 5. *Let the following conditions on the interval coefficients* $\mathbf{a}_i$, $\mathbf{b}_i \not\ni 0$, $i = 1, 2, \dots, n$, $\mathbf{c}_i$, $i = 1, 2, \dots, n-1$ *of the system (4) be satisfied (see (7)):*

$$(1-\delta)^2|\mathbf{a}_i'| + |\mathbf{c}_i'| \le 1 - \delta, \tag{28}$$

$$|\mathbf{a}_i'| \le a_\delta := \frac{1-\delta}{1 + (1-\delta)^2}, \quad i = 1, 2, \dots, n. \tag{29}$$

Then the following estimates hold for the components of the interval vector $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n)^\top$ *that is found from (10)–(12):*

$$|\mathbf{u}_i| \le 2f/\delta^2, \tag{30}$$

$$d(\mathbf{u}_i) \le \left(\frac{20f}{\delta^4} + \frac{12f}{\delta^3} + \frac{2}{\delta^2}\right)\lambda, \quad i = 1, 2, \dots, n. \tag{31}$$

These estimates do not depend on n.

Proof. From (28) and Theorem 1 it follows that $|\mathbf{x}_i| \le 1 - \delta$ and $\underline{g}_i' > 0$, $i = 1, 2, \dots, n$. Using the properties of the function d [1], we obtain a recurrence relation for $d(\mathbf{x}_i)$. We have

$$\begin{aligned}
d(\mathbf{x}_i) &\le d\left(\frac{-\mathbf{c}'_i}{\mathbf{b}'_i+\mathbf{a}'_i\mathbf{x}_{i-1}}\right) \le d(\mathbf{c}'_i)\left|\frac{1}{\mathbf{g}_i}\right| + |\mathbf{c}'_i|\, d\left(\frac{1}{\mathbf{g}_i}\right)\\
&\le \frac{\lambda}{\langle\mathbf{g}_i\rangle} + |\mathbf{c}'_i|\frac{(1+|\mathbf{x}_{i-1}|)\lambda + |\mathbf{a}'_i| d(\mathbf{x}_{i-1})}{\langle\mathbf{g}_i\rangle^2}\\
&\le \frac{\lambda}{\langle\mathbf{g}_i\rangle} + \frac{|\mathbf{c}'_i|}{\langle\mathbf{g}_i\rangle}\cdot\frac{(1+|\mathbf{x}_{i-1}|)\lambda}{\langle\mathbf{g}_i\rangle} + \frac{|\mathbf{c}'_i|}{\langle\mathbf{g}_i\rangle}\cdot\frac{|\mathbf{a}'_i| d(\mathbf{x}_{i-1})}{\langle\mathbf{g}_i\rangle}\\
&= \frac{\lambda}{\langle\mathbf{g}_i\rangle} + |\mathbf{x}_i|\frac{1+|\mathbf{x}_{i-1}|}{\langle\mathbf{g}_i\rangle}\lambda + |\mathbf{x}_i|\frac{|\mathbf{a}'_i|}{\langle\mathbf{g}_i\rangle} d(\mathbf{x}_{i-1}),\\
d(\mathbf{x}_i) &\le \frac{3-3\delta+\delta^2}{\langle\mathbf{g}_i\rangle}\lambda + \frac{(1-\delta)|\mathbf{a}'_i|}{\langle\mathbf{g}_i\rangle} d(\mathbf{x}_{i-1}), \quad i = 1,2,\ldots,n-1. \qquad (32)
\end{aligned}$$

Now we obtain recurrence relations for $|\mathbf{y}_i|$ and $d(\mathbf{y}_i)$. From (11), we have

$$|\mathbf{y}_i| \le \frac{f}{\langle\mathbf{g}_i\rangle} + \frac{|\mathbf{a}'_i|}{\langle\mathbf{g}_i\rangle}|\mathbf{y}_{i-1}|, \quad i = 1,2,\ldots,n. \qquad (33)$$

Further,

$$\begin{aligned}
d(\mathbf{y}_i) &\le |\mathbf{f}'_i - \mathbf{a}'_i\mathbf{y}_{i-1}|\, d\left(\frac{1}{\mathbf{g}_i}\right) + d(\mathbf{f}'_i - \mathbf{a}'_i\mathbf{y}_{i-1})\left|\frac{1}{\mathbf{g}_i}\right|\\
&\le \frac{|\mathbf{f}'_i - \mathbf{a}'_i\mathbf{y}_{i-1}|}{\langle\mathbf{g}_i\rangle}\cdot\frac{d(\mathbf{g}_i)}{\langle\mathbf{g}_i\rangle} + \frac{\lambda+\lambda|\mathbf{y}_{i-1}| + |\mathbf{a}'_i| d(\mathbf{y}_{i-1})}{\langle\mathbf{g}_i\rangle}\\
&\le |\mathbf{y}_i|\frac{\lambda+\lambda|\mathbf{x}_{i-1}| + |\mathbf{a}'_i| d(\mathbf{x}_{i-1})}{\langle\mathbf{g}_i\rangle} + \frac{\lambda+\lambda|\mathbf{y}_{i-1}|}{\langle\mathbf{g}_i\rangle} + \frac{|\mathbf{a}'_i|}{\langle\mathbf{g}_i\rangle} d(\mathbf{y}_{i-1}), \quad i = 1,2,\ldots,n. \qquad (34)
\end{aligned}$$

Finally, from (12) we have

$$\begin{aligned}
|\mathbf{u}_n| &= |\mathbf{y}_n|, & |\mathbf{u}_i| &\le |\mathbf{y}_i| + |\mathbf{x}_i||\mathbf{u}_{i+1}|, & & (34')\\
d(\mathbf{u}_n) &= d(\mathbf{y}_n), & d(\mathbf{u}_i) &\le d(\mathbf{y}_i) + d(\mathbf{x}_i)|\mathbf{u}_{i+1}| + |\mathbf{x}_i| d(\mathbf{u}_{i+1}), \quad i = 1,2,\ldots,n. & & (34'')
\end{aligned}$$

From (29) and (8), it follows that

$$\begin{aligned}
\frac{|\mathbf{a}'_i|}{\langle\mathbf{g}_i\rangle} &\le \frac{a_\delta}{1-a_\delta(1-\delta)} = 1-\delta, \qquad (35)\\
\langle\mathbf{g}_i\rangle &\ge 1 - a_\delta(1-\delta) = 1/[1+(1-\delta)^2] > 1/2, \quad i = 1,2,\ldots,n. \qquad (36)
\end{aligned}$$

Expanding (32) and taking (35), (36) into account, we obtain

$$d(\mathbf{x}_i) \le \frac{6-6\delta+2\delta^2}{2-\delta}\cdot\frac{\lambda}{\delta}.$$

For $\delta \in (0,1)$, this implies

$$d(\mathbf{x}_i) \le 4\lambda/\delta, \quad i = 1,2,\ldots,n-1. \qquad (37)$$

This estimate is of independent importance, since it presents an estimate of the width of the continued fraction (16). Analogously we obtain

$$\begin{aligned} |\mathbf{y}_i| &\leq 2f + (1-\delta)|\mathbf{y}_{i-1}| \leq \cdots \leq 2f/[1-(1-\delta)] = 2f/\delta, \qquad (38)\\ d(\mathbf{y}_i) &\leq \frac{4f\lambda}{\delta}\left(2-\delta+\frac{3}{\delta}\right) + 2\lambda\left(1+\frac{2f}{\delta}\right) + (1-\delta)\,d(\mathbf{y}_{i-1}) \\ &\leq \left(\frac{12f}{\delta^2}+\frac{12f}{\delta}+2\right)\lambda + (1-\delta)\,d(\mathbf{y}_{i-1}) \\ \cdots &\leq \left(\frac{12f}{\delta^3}+\frac{12f}{\delta^2}+\frac{2}{\delta}\right)\lambda =: L\lambda, \quad i=1,2,\ldots,n \qquad (39) \end{aligned}$$

from (33), (34), (35), and (36). Finally, from (34′), (34″), (38), and (39), we obtain

$$\begin{aligned} |\mathbf{u}_i| &\leq \frac{2f}{\delta} + (1-\delta)|\mathbf{u}_{i+1}| \leq \cdots \leq \frac{2f}{\delta^2}, \\ d(\mathbf{u}_i) &\leq L\lambda + \frac{4\lambda}{\delta}\cdot\frac{2f}{\delta^2} + (1-\delta)d(\mathbf{u}_{i+1}) \\ &\leq \left(\frac{20f}{\delta^4}+\frac{12f}{\delta^3}+\frac{2}{\delta^2}\right)\lambda, \quad i=n,n-1,\ldots,1. \end{aligned}$$

□

Theorem 6. *Let the following conditions be satisfied for the interval coefficients* $\mathbf{a}_i \not\ni 0$, $\mathbf{b}_i \not\ni 0$, $i=1,2,\ldots,n$, $\mathbf{c}_i \neq 0$, $i=1,2,\ldots,n-1$ *of the system (4) (see (7)):*

$$\begin{aligned} \mathbf{c}_i' &\subseteq [0,\,1-\delta], \qquad (40)\\ (1-\delta)^2\overline{a}_i' + \overline{c}_i' &\leq 1-\delta, \qquad (41)\\ \mathbf{a}_i' &\subseteq \left[-1+\delta,\,\frac{1-\delta}{1+(1-\delta)^2}\right], \quad i=1,2,\ldots,n. \qquad (42) \end{aligned}$$

Then the estimates (30) and (31), which are independent of n, hold for the components of the interval vector $\mathbf{u}=(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n)$ *that is found from (10)–(12).*

Proof. By virtue of Theorem 2,

$$\frac{|\mathbf{a}_i'|}{\langle \mathbf{g}_i\rangle} \leq \begin{cases} |\underline{a}_i'|, & \text{for } \overline{a}_i' < 0, \\ \overline{a}_i'/[1-\overline{a}_i'(1-\delta)], & \text{for } \underline{a}_i' > 0. \end{cases}$$

This together with (42) implies (35) and (36). Then, if we repeat the proof of the previous theorem, merely changing the method of obtaining the estimates (35) and (36), we obtain the estimates (30) and (31). □

Analogously we can prove

Theorem 7. *Let the following conditions be satisfied for the interval coefficients* $\mathbf{a}_i \not\ni 0$, $\mathbf{b}_i \not\ni 0$, $i=1,2,\ldots,n$, $\mathbf{c}_i \neq 0$, $i=1,2,\ldots,n-1$ *of the system (4) (see (7)):*

$$\begin{aligned} \mathbf{c}_i' &\subseteq [-1+\delta,\ 0], \\ -(1-\delta)^2\underline{a}_i' - \underline{c}_i' &\leq 1-\delta, \qquad (43)\\ \mathbf{a}_i' &\subseteq \left[\frac{-(1-\delta)}{1+(1-\delta)^2},\ 1-\delta\right]. \end{aligned}$$

Then the estimates (30) and (31), which are independent of n, *hold for the components of the interval vector* $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)^\top$ *that is found from (10)–(12).*

Comment 4. The hypotheses of any of Theorems 6, 7 include the estimate (37).

Comment 5. As in [5], we do not consider λ small and suppose that λ and δ are independent from each other. Under the condition of strict interval diagonal predominance (9) for $\lambda < \delta/6$, the estimates of the kind (30), (31) can be found in [2].

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