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A new characterization of the set of all intervals, based on the necessity to check consistency easily

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The purpose of this paper is to present a new characterization of the set of all intervals. This characterization is based on several natural properties useful in mathematical modeling; the main of these properties is the necessity to easily check consistency of incomplete knowledge This characterization is obtained both for one-dimensional and for multidimensional cases

Новая характеризация множества всех интервалов, основанная на требовании простой проверки непротиворечивости

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Представлена новая характеризация множества всех интервалов. Эта характеризация основана на нескольких очевидных свойствах, применяющихся в математическом моделировании, главное из которых — требование простой проверки непротиворечивости неполного знания. Данная характеризация применима как к случаю одной размерности, так и к многомерным случаям

1. Formulation of the problem: How to describe incomplete knowledge?

Measurements always lead to approximate values of the measured physical quantities. So, if we measure a physical quantity x, and the result of this measurement is $\tilde{x}$, then, because of the possible error, several different values of x are possible. Let's denote the set of all possible values (that are consistent with our measurement result) by X.

The manufacturers of a measuring device usually provide us with an upper bound Δ for possible errors. Since the error $\Delta x = \tilde{x} - x$ is limited by Δ, we can guarantee that possible values of the measured physical quantity belong to the interval $[\tilde{x} - \Delta, \tilde{x} + \Delta]$. But are all values from this interval possible? In other words, is this interval the desired set X?

Usually, it is not. The reason for that is as follows: The manufacturer's bound is often an overestimate, because it is difficult to estimate Δ precisely, and so the manufacturer, because of his desire to *guarantee* the accuracy, prefers to give an upper estimate for this bound. Because of that, the actual set X is usually smaller than the interval $[\tilde{x} - \Delta, \tilde{x} + \Delta]$.

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Assume now that we know the exact upper bound Δ_e for the error This means that all possible values of x belong to an interval $[\tilde{x} - \Delta_e, \tilde{x} + \Delta_e]$, or, that the set X is a subset of this interval. Then our question is: is X equal to this interval? I.e., *is X an interval?*

In some cases, it may not be an interval. Then, we must somehow approximate it. What family of sets should we use for this approximation? To find that out, let us take into consideration the fact that measurements can go wrong, and therefore, several results of measuring one and the same physical quantity can be inconsistent.

2. The necessity to check consistency easily

2.1. Main idea

Assume that we fix a physical quantity x, and we measure it several times. Each measurement provides us with an additional knowledge about x.

Measuring instruments can go wrong. As a result, we may get inconsistent data. For example, assume that we have measured the same current I twice, both times with an accuracy 0.1, and the results are 1.0 and 2.0. According to the first measurement, the actual value I of the current must belong to an interval $[0.9, 1.1]$; according to the second measurement, this value must belong to an interval $[1.9, 2.1]$. Therefore, I must belong to the intersection of these two intervals. But this intersection is empty, which means that during (at least) one of the measurements, something went wrong with the measuring instrument.

In view of this possibility, before we start processing measurement results, it would be nice to check the existing knowledge for possible inconsistencies.

Therefore, it is desirable to choose the family of approximating sets in such a way that this check will not be computationally very complicated.

Comments.

1. The idea that checking consistency can detect crude errors, and thus improve the accuracy of the resulting estimates, is definitely not new. What is new in this paper is our proof that:

   - for this idea to be (easily) applicable, i.e., in order to be able to easily check consistency, we must represent all incomplete knowledge in terms of intervals, and that

   - therefore, if we use sets from any other class (ellipsoids, etc) to describe incomplete knowledge, then checking consistency will not be as easy as for intervals (the precise formulation of what "easy" means will be given later).

2. Some of this paper's results first appeared in the Technical Report [13] and in the abstract [12].

2.2. Motivations for the (following) precise definition

Let's formulate this necessity in mathematical terms. Assume that we have several pieces of knowledge about x, that come from several measurements. Each piece of knowledge can be formulated as a set of all the values x that are consistent with this particular knowledge.[1] So, instead of saying that we have several pieces of knowledge, we can say that we have several sets $X, Y, \ldots, Z \subseteq R$. Consistency means that it is possible that a certain value $x \in R$ is consistent with all these measurements, i.e., x belongs to all these sets $X, Y, \ldots, Z$ (in other words, that $X \cap Y \cap \cdots \cap Z \neq \varphi$).

How can we actually check consistency? Knowledge usually comes piece after piece, so a typical situation is as follows:

- We already have a consistent "knowledge base"[2]. In other words, we have the pieces of knowledge represented by sets $X^{(1)}, \ldots, X^{(s)}$, and these pieces of knowledge are consistent.
- Then, a new piece of knowledge arrives, described by a set X.

In the case when we are sure that all measuring instruments function properly, and therefore, all the sets $X^{(i)}$ do contain the actual value of the measured quantity, there is no need to store all these sets in the knowledge base: in this case, the actual value belongs to the intersection of these sets, so, we can as well keep the intersection only. However, the motivation for this paper is to consider the case when measuring instruments can go wrong. Therefore, if it turns out that the new measurement is inconsistent with the results of the previous measurements, we want to be able to analyze the records of the related measurements, find the measurement that could be wrong, and discard its result. If we store the sets that describe the results of all previous measurements, then we discard one of them by simply deleting the correspondent set from the collection of sets $X^{(1)}, \ldots, X^{(s)}$. If we only store the intersection (so that the sets themselves have been deleted), then there is no easy way to discard just the result of one measurement. So, in our situation, we *have to* keep the sets that describe the results of all the measurements.

Another possibility could be to keep *both* the sets $X^{(i)}$ that describe the results of the measurements, *and* the intersection of these sets. This is a reasonable alternative, but it has one serious drawback: it is not always applicable. Modern knowledge bases are distributed, i.e., some data are stored in one place, and some are stored in other places (see, e.g., [15]; this is how, e.g., some knowledge is stored in the World Wide Web and Mosaic). One of the major advantages of such storage is that we can do operations like search and update in *parallel* on different parts of this knowledge base (and therefore, much *faster* than if all the knowledge was located in one place only). If, however, we were required to also store the intersection, then we would need to actually re-compute this intersection every time we add a new piece of knowledge. This necessity will make update a sequential operation and thus, one of the main advantages of distributed databases will be gone.

Therefore, in this paper, we will consider the situation in which we store the sets $X^{(i)}$ corresponding to different pieces of knowledge but *not* their intersection.

In this case, if we have a new set X (representing the new piece of knowledge) added to the sets $X^{(i)}$, a natural way to check consistency is to check whether X is consistent with

[1] This is a very specific type of knowledge, called *incomplete knowledge*. There exist other types of knowledge that describe different types of uncertainty. For a brief description of different types of uncertainty, see, e.g., [14].

[2] The notion of a knowledge base is a general notion, that is used to describe different types of knowledge. In this paper, we will be using only one specific type of knowledge bases: namely, collections of sets representing different pieces of incomplete knowledge.

each of the existing sets $X^{(i)}$. The consistency between X and each of $X^{(i)}$ is, of course, necessary for the new knowledge base $\{X^{(1)}, \ldots, X^{(s)}, X\}$ to be consistent. It is easy to see, however, that this comparison is not always sufficient. For example, if $s = 2$, $X^{(1)} = \{0, 1\}$, $X^{(2)} = \{1, 2\}$, and $X = \{0, 2\}$, then the original knowledge base $\{X^{(1)}, X^{(2)}\}$ is consistent, and X is consistent with each of the sets $X^{(i)}$, but there is no number common to all three sets.

So, since we want to express the fact that the above procedure does lead to checking consistency easily, we arrive at the following definition:

2.3. Formal definition

Definition 1. *We say that a family $\mathcal{S}$ of sets allows checking consistency easily if the following is true: For every s, and for every tuple of sets $X^{(1)} \in \mathcal{S}, \ldots, X^{(s)} \in \mathcal{S}, X \in \mathcal{S}$, for which:*

- *sets $X^{(1)}, \ldots, X^{(s)}$ are consistent (i.e., $X^{(1)} \cap \cdots \cap X^{(s)} \neq \varphi$) and*
- *X is consistent with all $X^{(i)}$ (i.e., $X \cap X^{(i)} \neq \varphi$ for all $i = 1, \ldots, s$),*

all $s + 1$ sets $X^{(1)}, \ldots, X^{(s)}, X$ are consistent (i.e., $X^{(1)} \cap \cdots \cap X^{(s)} \cap X \neq \varphi$).

3. Translation and dilation invariance: Additional demands on the desired family of sets

We are interested in measurements, so it is natural to assume that if X is a reasonable approximation to sets that describe the incomplete knowledge, then a set $X + a$ (that is obtained from X by a translation) is also a reasonable approximation. For example, assume that we are measuring time, and as a result we get 35 ± 5 (i.e., a set $X = [30, 40]$). If we now change the starting point for measuring time, e.g., take -5 as the new starting point, then the same result will be expressed as $X + 5 = 40 \pm 5 = [35, 45]$. This new set $X + 5$ must also be a reasonable approximation. If instead of changing the starting point, we change the measuring unit (i.e., consider minutes instead of seconds), then in the new units, we get λX instead of X. So, it is also natural to assume that if X is a reasonable approximation, and $\lambda > 0$, then λX is a reasonable approximation as well.

Comment. In terms of measurement theory (see, e.g., [16]), this means that we are considering measuring scales that are determined modulo arbitrary linear transformations. Such scales are also called *interval scales*. To avoid confusion, we must mention that the very fact that we are considering an *interval* scale (e.g., time) does not necessarily mean that the resulting incomplete knowledge is represented by an interval (see, e.g., [10]). For example, suppose that we are measuring time by using an electromagnetic clock in the close vicinity of a computer memory element. In this situation, the external field caused by this element is the main source of error. This element can be in two possible states (depending on whether it represents bit "1" or bit "0"), so we have two possible values of an error. Crudely speaking, for this situation, the set of possible values of measurement error consists of only two points $\{-\varepsilon, \varepsilon\}$, and does not contain any internal values at all. So, if the measured time is $\tilde{t}$, then the set $X^{(i)}$ of possible values of time is $\{\tilde{t} - \varepsilon, \tilde{t} + \varepsilon\}$. This set is *not* an interval, but measuring time is definitely an *interval scale*.

If in addition to this main source of error, we take into consideration other possible sources of error, then the resulting set $X^{(i)}$ of possible values of time becomes a union of two small intervals: one close to $\tilde{t} - \varepsilon$, and a one close to $\tilde{t} + \varepsilon$. Still, it is not an interval.

Definition 2. *We say that a family S of sets is translation-invariant if for every $X \in S$, and for every $a \in R$, the set $X + a$ also belongs to S (this set $X + a$ is called a translate of X).*

Definition 3. *We say that a family S of sets is dilation-invariant if for every $X \in S$, and for every $\lambda > 0$, the set λX also belongs to S.*

Since we have already argued that each set X belongs to an interval (namely, to $[\tilde{x} - \Delta, \tilde{x} + \Delta]$), each set X is bounded. It is also natural to assume that each set X is closed. Now, we are ready to formulate our main result.

4. First result: one-dimensional case

Proposition 1. *For an arbitrary family of sets S, the following conditions are equivalent to each other:*

i) S is a non-empty translation- and dilation-invariant family of bounded closed sets that allows checking consistency easily;

ii) S is either:

- *the family of all intervals, or*
- *the family of all non-degenerate intervals $[a, b]$, $a < b$.*

(For reader's convenience, all the proofs are moved to the last section).

Comment. This result *does not* mean that in all the cases, incomplete knowledge must be described by intervals. It could as well be that our incomplete knowledge can be described by the 2-element set $\{-1, 1\}$. This Proposition proves that if we use non-interval sets to describe incomplete knowledge, then checking consistency of the resulting knowledge will not be so straightforward. So, if the actual knowledge corresponds to non-interval sets (domains representing incomplete knowledge), then we have two options:

- We can replace these domains by intervals (e.g., use $[-1, 1]$ instead of $\{-1, 1\}$).
  - The main *advantage* of this replacement is that we will now be able to easily check consistency.
  - The main *disadvantage* is that we have increased the domain, and therefore, the resulting estimates may be *overestimates*. Let's describe an example of this overestimation. Suppose that we know that $x \in X = \{-1, 1\}$. In this case, we use $[-1, 1]$ to describe this incomplete knowledge. Suppose that we are now interested in the set Y of possible values of $y = x^2$. If when estimating Y, we can only use the fact that $X \subsetneq [-1, 1]$, then, as a result, we can only conclude that $Y \subsetneq \{x^2 \mid x \in [-1, 1]\} = [0, 1]$, which is an overestimate for the desired set $Y = \left\{x^2 \mid x \in \{-1, 1\}\right\} = \{1\}$.
- We can also use the original (non-interval) sets to describe knowledge. Then, the estimates will be correct, but (due to Proposition 1) checking consistency will be difficult.

5. Multi-dimensional case

5.1. Motivation and the formulation of the main result

Motivations. If we have several physical quantities $x_1, \ldots, x_n$, it is natural to ask the following question: what is the set of possible tuples $\vec{x} = (x_1, \ldots, x_n) \in R^n$? Definition 1 does not depend on the dimension. Definitions 2 and 3 can be easily adjusted to $n-$dimensional case, if we take into consideration that in case of n quantities, we can independently change the value of each of them:

Definition 2′. *We say that a family S of sets is translation-invariant if for every $X \in S$, and for every $\vec{a} \in R^n$, the set $X + \vec{a}$ also belongs to S (this set $X + \vec{a}$ is called a translate of X).*

Definition 3′. *We say that a family S of sets is dilation-invariant if for every $X \in S$, and for every vector $\vec{\lambda}$ with positive components $\lambda_i > 0$, the set $\vec{\lambda}X$ also belongs to S, where $\vec{\lambda}X = \{\vec{\lambda}\vec{x} \mid \vec{x} \in X\}$ and $\vec{\lambda}\vec{x} = (\lambda_1 x_1, \ldots, \lambda_n x_n)$.*

For n quantities, one of the natural situations is when we have some knowledge about each quantities, and these pieces of knowledge are independent. In such a situation, whether x_i is a possible value of i-th quantity or not does not depend on what the values of other quantities are. So, in this case, a set of values $\vec{x} = (x_1, \ldots, x_n)$ is possible iff each of x_i is possible. In mathematical terms, it means that the set S of possible values of $\vec{x}$ can be represented as a Cartesian product $S_1 \times S_2 \times \cdots \times S_n$, where S_i is a set of possible values of i-th quantity x_i. Hence, we arrive at the following definition:

Definition 4. *We say that a set S describes independent knowledge of x_i is $S = S_1 \times \cdots \times S_n$ for some sets $S_i \subseteq R$.*

If one of these sets S_i consists of only one point $\{s_i\}$, this means that we actually know the value of i-th quantity exactly. For this x_i, our knowledge about x_i is complete. So, we arrive at the following definition:

Definition 5. *We say that a set $S = S_1 \times \cdots \times S_n$ describes independent incomplete knowledge of x_i if each of the sets S_i contains at least two different points.*

Proposition 2. *($R^n, n > 1$) For an arbitrary family of sets S, the following conditions are equivalent to each other:*

i) S is a non-empty translation- and dilation-invariant family of bounded closed sets that allows checking consistency easily, and that contains a set S describing independent incomplete knowledge of x_i.

ii) There exists n values $n_i \in \{0, 1\}$ such that S coincides with the family of all n-dimensional intervals $X = [a_1, b_1] \times \cdots \times [a_n, b_n]$ for which all component intervals $[a_i, b_i]$ with $n_i = 1$ are non-degenerate.

Comments.

1. In particular, if all $n_i = 1$, S coincides with the set of all non-degenerate n-dimensional intervals, and if all $n_i = 0$, then S coincides with the set of all n-dimensional intervals.

2. Similarly to Proposition 1, this result *does not* mean that in all the cases, incomplete knowledge must be described by intervals. It could as well be described by, e.g., an ellipsoid. This proposition proves that if we use non-interval sets (e.g., ellipsoids) to describe incomplete knowledge, then checking consistency of the resulting knowledge

will not be so straightforward. So, if we want to be able to check consistency fast, we must enclose ellipsoids and other sets in n-dimensional intervals, and use these enclosures instead of the original domains.

5.2. The auxiliary result

A similar result can be proven if instead of dilation-invariance, we assume that all sets from X are *convex*.

Proposition 3. *If S is a non-empty translation-invariant family of bounded closed convex sets that allows checking consistency easily, then all sets from S are parallelepipeds.*

Comment. This result *explains why parallelepipeds are often used to describe incomplete knowledge.*

A possible meaning of convexity of domains representing incomplete knowledge. Let's consider the case when all the knowledge about the quantities $x_1, \dots, x_n$ comes from measurements, and when the errors of these measurements are so small that terms quadratic in these errors can be neglected. This means that after processing the results of all the measurements, we get the (approximate) values $\tilde{x}_1, \dots, \tilde{x}_n$ of the quantities x_i, and we know that the errors $\Delta_i = \tilde{x}_i - x_i$ of these measurements are small.

How does the domain X representing incomplete knowledge, i.e., the set of all possible values of $\vec{x}$, look in this case? Since we assumed that knowledge comes only from measurements, X is the set of all vectors $\vec{x}$ that are consistent with all the measurements. How to describe this consistency? Suppose that we measure a quantity y that is related to x_i, i.e., that can be expressed as a function of x_i: $y = f(x_1, \dots, x_n)$. The measured quantity y can coincide with one of the x_i, or it could be some combination of x_i. Assume that the result of measurement is $\tilde{y}$, and that we know the bound Δ for the measurement error guaranteed by the manufacturer of the used measuring instrument. This means that the actual value $f(x_1, \dots, x_n)$ of y must belong to an interval $[y^-, y^+]$, where $y^\pm = \tilde{y} \pm \Delta$. Since we assumed that the errors are small, and their errors are negligible, we can expand the function $f(x_1, \dots, x_n)$ in Taylor series, and retain only linear terms in this expansion: $f(x_1, \dots, x_n) \approx f_{\text{lin}}(x_1, \dots, x_n)$, where we denoted $f_{\text{lin}}(x_1, \dots, x_n) = f(\tilde{x}_1, \dots, \tilde{x}_n) + f_{,1}(\tilde{x}_1, \dots, \tilde{x}_n) \cdot (x_1 - \Delta x_1) + \dots + f_{,n}(\tilde{x}_1, \dots, \tilde{x}_n) \cdot (x_n - \Delta x_n)$. So, instead of the inequalities $y^- \le f(x_1, \dots, x_n)$ and $f(x_1, \dots, x_n) \le y^+$, we consider the inequalities $y^- \le f_{\text{lin}}(x_1, \dots, x_n)$ and $f_{\text{lin}}(x_1, \dots, x_n) \le y^+$. For a linear function f_{lin}, each of these inequalities describes a semi-space.

Elements from X must satisfy all inequalities that stem from all the measurements. Therefore, X is an intersection of semi-spaces that correspond to all these inequalities. A semi-space is convex, so, this intersection is also convex.

5.3. A hypothesis

Propositions 2 and 3 naturally lead to the following Hypothesis:

Hypothesis ($R^n, n > 1$). *If S is a non-empty translation- and dilation-invariant family of bounded closed sets that allows checking consistency easily, then all the sets from S are parallelepipeds.*

6. Proofs

Proof of Proposition 1. $ii) \to i)$ is evident. So, let us prove that $i) \to ii)$. This proof will also be relatively simple. Assume that $\mathcal{S}$ is a non-empty translation- and dilation-invariant family of bounded closed sets that allows checking consistency easily.

First, we will prove that all sets from $\mathcal{S}$ are intervals. Indeed, assume that $X \in \mathcal{S}$. Since X is closed and bounded, it is compact. Therefore, the set X contains its sup and inf: $x^- = \inf X \in X$ and $x^+ = \sup X \in X$. Hence,

$$\{x^-, x^+\} \subseteq X \subseteq [x^-, x^+]. \tag{1}$$

Let us prove that $X = [x^-, x^+]$. Indeed, take $x \in (x^-, x^+)$, and let us prove that $x \in X$.

Due to translation- and dilation-invariance, the sets

$$X^{(1)} = x^- + \frac{x - x^-}{x^+ - x^-}(X - x^-)$$

and

$$X^{(2)} = x + \frac{x^+ - x}{x^+ - x^-}(X - x^-)$$

belong to $\mathcal{S}$. From (1) and from the definitions of $X^{(1)}$ and $X^{(2)}$, we conclude that

$$\begin{array}{ccccc} \{x^-, x\} & \subseteq & X^{(1)} & \subseteq & [x^-, x], \\ \{x, x^+\} & \subseteq & X^{(2)} & \subseteq & [x, x^+]. \end{array}$$

Let us now apply to the sets $X^{(1)}, X^{(2)}$, and X the assumption that $\mathcal{S}$ allows checking consistency easily. All conditions of Definition 1 are satisfied; indeed:

- the sets $X^{(1)}$ and $X^{(2)}$ have a common element x, so $X^{(1)}$ and $X^{(2)}$ are consistent;
- the sets X and $X^{(1)}$ have a common element x^-, so X is consistent with $X^{(1)}$;
- the sets X and $X^{(2)}$ have a common element x^+, so X is consistent with $X^{(2)}$.

As a result, we conclude that all three sets X, $X^{(1)}$, and $X^{(2)}$ are consistent, i.e., that they have a common point y. But all elements of $X^{(1)}$ are $\leq x$, all elements of $X^{(2)}$ are $\geq x$, so the only common point of $X^{(1)}$ and $X^{(2)}$ is x. Therefore, $y = x$ and $y \in X$, i.e., $x \in X$.

So, we have proved two statements about X:

- First, we have proved that $\{x^-, x^+\} \subseteq X \subseteq [x^-, x^+]$.
- Second, we have proved that an arbitrary real number x from (x^-, x^+) belongs to X.

Therefore, $X = [x^-, x^+]$.

Hence, all sets from $\mathcal{S}$ are intervals. To complete our proof, we must show that every interval belongs to $\mathcal{S}$. Indeed, $\mathcal{S}$ is non-empty, and we already know that all its sets are intervals. Let's take one of these sets $X = [x^-, x^+] \in \mathcal{S}$, and let us prove that any other interval $[a, b]$ also belongs to $\mathcal{S}$. Indeed,

$$[a, b] = a + \frac{b - a}{x^+ - x^-}(X - x^-)$$

so the fact that $[a, b] \in \mathcal{S}$ follows from the assumption that the family $\mathcal{S}$ is dilation- and translation-invariant.

Similarly, it is easy to prove that if $\mathcal{S}$ contains at least one degenerate interval $[a, a]$, then $\mathcal{S}$ contains all possible degenerate intervals $[a, a]$. □

Proof of Proposition 2. $ii) \to i)$ is evident. So, let us prove that $i) \to ii)$. Assume that $\mathcal{S}$ is a non-empty translation- and dilation-invariant family of bounded closed sets that allows checking consistency easily, and that contains a set S describing independent incomplete knowledge of x_i.

1. Let us first prove that the given set S (that describes independent knowledge) is an n-dimensional interval.

Indeed, according to Definitions 4 and 5, $S = S_1 \times S_2 \times \cdots \times S_n$, where each of the sets S_i contains at least two different elements. Since $\mathcal{S}$ is translation- and dilation-invariant, we can conclude that for every $a \in R$, and for every $\lambda > 0$, the sets $(S_1 + a) \times S_2 \times \cdots \times S_n$ and $\big(\lambda(S_1 + a)\big) \times S_2 \times \cdots \times S_n$ also belong to $\mathcal{S}$.

All resulting sets have the same 2-nd, …, n-th components $S_2, \ldots, S_n$. For such sets,

$$(S_1 \times S_2 \times \cdots \times S_n) \cap (S_1' \times S_2 \times \cdots \times S_n) = (S_1 \cap S_1') \times S_2 \times \cdots \times S_n$$

so $(S_1 \times S_2 \times \cdots \times S_n) \cap (S_1' \times S_2 \times \cdots \times S_n) = \varphi$ iff $S_1 \cap S_1' = \varphi$. Therefore, from the fact that the family $\mathcal{S}$ allows checking consistency easily, we can conclude that the first components of the resulting sets also allow checking consistency easily. So, these first components satisfy the conditions of part $i)$ of Proposition 1. So, these first components are intervals. In particular, S_1 is an interval. Since S_1 contains at least two different elements, it is a non-degenerate interval.

Similarly, we can conclude that each of the components S_i is a non-degenerate interval and therefore, that $S = S_1 \times \cdots \times S_n$ is an n-dimensional interval.

2. We have just proved that $\mathcal{S}$ contains an n-dimensional interval. By applying appropriate translations and dilations in each dimension, we can conclude (similarly to Proposition 1) that every non-degenerate n-dimensional interval I belongs to $\mathcal{S}$.

3. Let us now prove that every set $X \in \mathcal{S}$ is convex.

By definition of convexity, it means that if $\vec{a} \in X$, $\vec{b} \in X$, and $\alpha \in (0, 1)$ is a real number, then $\vec{c} = \alpha\vec{a} + (1 - \alpha)\vec{b} \in X$. To prove this inclusion, let us construct the following intervals $X_i^{(1)}$ and $X_i^{(2)}$:

- If $a_i < b_i$, then $a_i < c_i < b_i$. In this case, we take $X_i^{(1)} = [a_i, c_i]$ and $X_i^{(2)} = [c_i, b_i]$.
- If $a_i > b_i$, then $a_i > c_i > b_i$. In this case, we take $X_i^{(1)} = [c_i, a_i]$ and $X_i^{(2)} = [b_i, c_i]$.
- If $a_i = b_i$, then $a_i = c_i = b_i$. In this case, we take $X_i^{(1)} = [c_i - 1, c_i]$ and $X_i^{(2)} = [c_i, c_i + 1]$.

In all three cases, $X_i^{(1)}$ and $X_i^{(2)}$ have the following properties:

- $a_i \in X_i^{(1)}$ and $b_i \in X_i^{(2)}$;
- $X_i^{(1)}$ and $X_i^{(2)}$ are non-degenerate intervals;
- $X_i^{(1)} \cap X_i^{(2)} = \{c_i\}$.

Let us now consider n-dimensional intervals

$$X^{(1)} = X_1^{(1)} \times \cdots \times X_n^{(1)} \tag{2}$$

and

$$X^{(2)} = X_1^{(2)} \times \cdots \times X_n^{(2)}. \tag{3}$$

According to Part 2 of this proof, $X^{(1)} \in \mathcal{S}$ and $X^{(2)} \in \mathcal{S}$. Because of the properties of $X_i^{(1)}$ and $X_i^{(2)}$, we have $\vec{a} = (a_1, \ldots, a_n) \in X^{(1)}$, $\vec{b} \in X^{(2)}$, and

$$\begin{aligned} X^{(1)} \cap X^{(2)} &= (X_1^{(1)} \times \cdots \times X_n^{(1)}) \cap (X_1^{(2)} \times \cdots \times X_n^{(2)}) \\ &= (X_1^{(1)} \cap X_1^{(2)}) \times \cdots \times (X_n^{(1)} \cap X_n^{(2)}) = \{c_1\} \times \cdots \times \{c_n\} = \{\vec{c}\}. \end{aligned}$$

Let us now apply to the sets $X^{(1)}$, $X^{(2)}$, and X the assumption that $\mathcal{S}$ allows checking consistency easily. All conditions of Definition 1 are satisfied; indeed:

- the sets $X^{(1)}$ and $X^{(2)}$ have a common element $\vec{c}$, so $X^{(1)}$ and $X^{(2)}$ are consistent;
- the sets X and $X^{(1)}$ have a common element $\vec{a}$, so X is consistent with $X^{(1)}$;
- the sets X and $X^{(2)}$ have a common element $\vec{b}$, so X is consistent with $X^{(2)}$.

As a result, we conclude that all three sets X, $X^{(1)}$, and $X^{(2)}$ are consistent, i.e., that they have a common point. But the only common point of $X^{(1)}$ and $X^{(2)}$ is $\vec{c}$. Therefore, $\vec{c} \in X$. So, X is convex.

4. Let us now prove that every set $X \in \mathcal{S}$ satisfies the following *recombination property*: for every i from 1 to n, if $\vec{a} = (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \in X$, and $\vec{b} = (b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_n) \in X$, then $\vec{c} = (a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n) \in X$.

Comment. In other words, instead of each component a_i of $\vec{a} \in X$, we can substitute the corresponding component of any other vector $\vec{b} \in X$, and still get an element of X. This operation is a particular case of what is happening when two DNA's from two parents are transformed into a single DNA of a child. This procedure has been formally described under the name of *recombination* in the first algorithms that simulated biological evolution [2, 3], and it is actively used in *genetic algorithms* that use a simulated evolution to solve optimization problems [4–8, 11].

To prove the recombination property, let us consider the following intervals $X_i^{(1)}$ and $X_i^{(2)}$:

- If $a_i < b_i$, then $X_i^{(1)} = [a_i, b_i]$ and $X_i^{(2)} = [b_i, b_i + 1]$;
- If $a_i > b_i$, then $X_i^{(1)} = [b_i, a_i]$ and $X_i^{(2)} = [b_i - 1, b_i]$;
- If $a_i = b_i$, then $\vec{c} = \vec{a}$, so the desired conclusion $\vec{c} \in X$ immediately follows from $\vec{a} \in X$. In this case, there is no need to construct the intervals $X_i^{(k)}$.

For $j \neq i$, we will consider the following intervals $X_j^{(1)}$ and $X_j^{(2)}$:

- If $a_j < b_j$, then $X_j^{(1)} = [a_j - 1, a_j]$ and $X_j^{(2)} = [a_j, b_j]$;
- If $a_j > b_j$, then $X_j^{(1)} = [a_j, a_j + 1]$ and $X_j^{(2)} = [b_j, a_j]$;

- If $a_j = b_j$, then $X_j^{(1)} = [a_j, a_j + 1]$ and $X_j^{(2)} = [b_j, a_j]$.

  In all three cases, these intervals $X_j^{(1)}$ and $X_j^{(2)}$, $j = 1, 2, \ldots, i, \ldots, n$, satisfy the following properties:

- $a_j \in X_j^{(1)}$ and $b_j \in X_j^{(2)}$;

- $X_j^{(1)}$ and $X_j^{(2)}$ are non-degenerate intervals;

- $X_j^{(1)} \cap X_j^{(2)} = \{c_j\}$ for all j (including $j = i$).

So, if we use formulas (2) and (3) to define n-dimensional intervals $X^{(1)}$ and $X^{(2)}$, we will have $\vec{a} \in X^{(1)}$, $\vec{b} \in X^{(2)}$, and $X^{(1)} \cap X^{(2)} = \{\vec{c}\}$. Applying the condition that the family $\mathcal{S}$ allows checking consistency easily, we conclude that $\vec{c} \in X$.

5. Let us now prove that an arbitrary set X from the given class $\mathcal{S}$ is an n-dimensional interval.

Indeed, since X is bounded and closed, it is compact, and so, for each i, the set X contains the points in which the i-th coordinate attains its supremum and infimum. Let's denote these supremum and infimum by x_i^+ and x_i^- correspondingly. Then, there is a point $\vec{x}^{(+i)} \in X$ with coordinates $(\ldots, x_i^+, \ldots)$ (i.e., for which $(x^{(+i)})_i = x^+S_i$), and there is another point $\vec{x}^{(-i)} \in X$, with coordinates $(\ldots, x_i^-, \ldots)$.

Now, we can apply recombination property to the points $\vec{x}^{(+1)} \in X$ and $\vec{x}^{(+2)} \in X$, and get points from X with coordinates $(x_1^+, x_2^+, \ldots)$. Similarly, by combining different pairs of vectors $\vec{x}^{(\pm 1)} \in X$ and $\vec{x}^{(\pm 2)} \in X$, we get 3 more points, with coordinates correspondingly $(x_1^+, x_2^-, \ldots)$, $(x_1^-, x_2^+, \ldots)$, and $(x_1^-, x_2^-, \ldots)$. We can recombine each of these 4 elements with one of the elements $\vec{x}^{(\pm 3)} \in X$ and get 8 vectors with coordinates $(x_1^\pm, x_2^\pm, x_3^\pm, \ldots)$ that all belong to our set X. Repeating this procedure n times, we conclude that X contains all 2^n points with coordinates $(x_1^\pm, x_2^\pm, \ldots, x_n^\pm)$. In other words, X contains all vertices of an n-dimensional interval $I = [x_1^-, x_1^+] \times \cdots \times [x_n^-, x_n^+]$.

We have already proven (in Part 3 of this proof) that X is convex. Therefore, X contains the convex combination of these vertices, i.e., X contains the entire n-dimensional interval I: $I \subseteq X$. On the other hand, since x_i^- and x_i^+ have been chosen as the infimum and supremum of x_i, we have $x_i \in [x_i^-, x_i^+]$ for all i and for all $\vec{x} \in X$. Therefore, if $\vec{x} \in X$, then $\vec{x} \in I$, i.e., $X \subseteq I$. Since we already know that $I \subseteq X$, we conclude that $X = I$, and X is an n-dimensional interval. □

Proof of Proposition 3. This result is a simple corollary of a known result from combinatorial geometry.

1. Let us first prove that our Definition 1 can be proved to be equivalent to the so-called 2–*Helly* property (see, e.g., [1, 9]): *if every two sets from $\mathcal{S}$ have a common point, then any finite number of sets from $\mathcal{S}$ also have a common point.*

Comment. The name of this property came from *Helly theorem* from combinatorial geometry.

Proof of Statement 1. Assume that $\mathcal{S}$ allows checking consistency easily, and the sets $X^{(1)}, \ldots, X^{(s)}$ from $\mathcal{S}$ have pairwise non-empty intersections. To prove that

$$X^{(1)} \cap \cdots \cap X^{(s)} \neq \varphi$$

let's start with a consistent class $\{X^{(1)}, X^{(2)}\}$. Since $\mathcal{S}$ allows checking consistency easily, and $X^{(3)}$ is consistent with both sets $X^{(1)}$ and $X^{(2)}$, we can conclude that $X^{(1)} \cap X^{(2)} \cap X^{(3)} \neq \varphi$.

Now, likewise, we can add $X^{(4)}$, and conclude that $X^{(1)} \cap X^{(2)} \cap X^{(3)} \cap X^{(4)} \neq \varphi$. Adding the sets $X^{(i)}$ one by one, we finally conclude that $X^{(1)} \cap \cdots \cap X^{(s)} \neq \varphi$.

Vice versa, assume that 2−Helly property is true, $X^{(1)} \cap \cdots \cap X^{(s)} \neq \varphi$, and $X^{(i)} \cap X \neq \varphi$ for all i from 1 to s. Then, $X^{(i)} \cap X^{(j)} \neq \varphi$ for all i, j and therefore, due to 2−Helly property, $X^{(1)} \cap \cdots \cap X^{(s)} \cap X \neq \varphi$. The equivalence is proven.

2. It is known ([1], p. 237; [17]) that the *translates of a compact convex set X satisfy a 2−Helly property if and only if X is a parallelepiped.* So, a *convex* compact set X can be a reasonable representation of incomplete knowledge iff X is a parallelepiped. □

7. Conclusions

In this paper, we give a characterization of the set of all intervals and (in multi-dimensional case) the set of all parallelepipeds as natural and useful settings in mathematical modeling. Namely, we show that these two sets can be uniquely characterized by the condition of invariance with respect to shifts and dilations (that correspond to change the unit measure and the starting point) plus the possibility to check easily the consistency of incomplete knowledge.

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