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Numerical Verifications of Solutions for Nonlinear Hyperbolic Equations

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In this paper, we consider a numerical technique to enclose the solutions with guaranteed error bounds for nonlinear hyperbolic initial boundary value problems as well as to verify the existence of solutions. Using a finite element approximation and explicit error estimates for a certain simple linear hyperbolic problem, we construct, by computer, a set of functions which satisfies the condition of Schauder's fixed point theorem in some appropriate function space. In order to obtain such a numerical set, we use a kind of multivalued iterative procedure with efficient use of an initial approximate solution. A numerical example is provided.

Численная верификация решений нелинейных гиперболических уравнений

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Рассматривается численный метод гарантированной оценки решений нелинейных гиперболических краевых задач с начальными условиями и проверки существования решений. Используя приближение конечными элементами и явные оценки погрешностей для некоторой простой линейной гиперболической задачи, с помощью компьютера строится множество функций, удовлетворяющее условию теоремы о неподвижной точке Шаудера в некотором подходящем функциональном пространстве. Для получения такого численного множества мы используем вариант многозначной итеративной процедуры с эффективным использованием начального приближенного решения. Представлен численный пример.

1 Introduction

In the preceding papers [6–9, 11], we presented some of the computer assisted verification methods for the solutions of Dirichlet problems of second order, using the finite element approximation and Schauder's or Sadovskii's fixed point theorems. Also in [10], we extended the method to initial boundary value problems for some nonlinear parabolic equations. In this paper, we show that, under the setting of some appropriate function spaces, a similar verification principle to that in [10] can also be applied for hyperbolic problems of second order and that we can provide a computational verification procedure. Further, a prototype numerical example is presented.

In the following section, we formulate the nonlinear hyperbolic problem with homogeneous initial and boundary conditions as the fixed point equation of a compact operator. Also a fundamental theorem which is the base of our verification problem is proved. In Section 3, using the finite element approximation and its error estimates for a simple linear equation, the concepts of rounding and rounding error are introduced and a verification condition is presented. And we describe the concrete verification procedure in computer with a numerical example in Section 4.

2 Fixed point formulation

Consider the following nonlinear hyperbolic problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f(x,t,u), & (x,t) \in \Omega \times J, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times J, \\ u(x,0) = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial t}(x,0) = 0, & x \in \Omega \end{cases} \tag{1}$$

where Ω is a bounded and convex domain in $\mathbb{R}^n$ $(1 \le n \le 3)$ with piecewise smooth boundary $\partial\Omega$ and $J = (0,T)$ with $T > 0$. Set $Q = \Omega \times J$. For each nonnegative integer m, we denote by H^m and H_0^m the usual and homogeneous L^2-Sobolev spaces on Ω of order m with norm $\| \cdot \|_m$, respectively.

Particularly, $H^0 \equiv L^2$ and $(\cdot, \cdot)$ implies the L^2 inner product on Ω. Also note that, for $m = 1$, we use $(\nabla u, \nabla v)$ as the inner product on H_0^1 and thus $\|u\|_{H_0^1}^2 = (\nabla u, \nabla u)$.

Next, for nonnegative integers r and s, according to [3], let $H^r(J; H^s)$ denote the time-dependent type Sobolev space with norm:

$$\|u\|_{H^r(J;H^s)}^2 = \sum_{j=0}^{r} \int_0^T \left\| \frac{\partial^j u}{\partial t^j} \right\|_s^2 dt.$$

And $H^{r,s}(Q) \equiv H^{r,s}$ denotes the Hilbert space with the following norm:

$$\|u\|_{r,s}^2 = \int_0^T \left(\|u(t)\|_r^2 + \|u\|_{H^s(J;H^0)}^2 \right) dt.$$

Note that $H^r(J; H^s)$ and $H^{r,s}$ coincide with the closure of $C^\infty(\bar{Q})$ in the above norm ([3, 4]). Moreover, we define a Banach space $H \equiv H^2(Q) \cap L^\infty(Q)$ with norm: $\|u\|_H \equiv \|u\|_{H^2(Q)} + \|u\|_{L^\infty(Q)}$, where $H^2(Q)$ denotes the usual Sobolev space on Q. Also we denote $\|u\|_{L^2(Q)}$ by simply $\|u\|$.

We now suppose the following assumptions on the nonlinear map f in (1).
A1. $f(\cdot, u) \in H^1(J; L^2)$ for any $u \in H$.
A2. $f(\cdot, u)$ is bounded in $H^1(J; L^2)$ for any bounded subset in H.
A3. For each bounded subset U in H, f is the continuous map from U into $H^1(J; L^2)$ with $H^{1,1}$-norm.

For example, $f(x, t, u) = u^p$, where p is a nonnegative integer, satisfies these assumptions. Indeed, A1 and A2 follow easily from the imbedding theorem (e.g. [1]). Let U be a bounded subset of H and, for fixed $u \in U$, suppose that $v \to u$ in $H^{1,1}$ norm. Then observe that

$$\frac{\partial}{\partial t}(u^p - v^p) = pu^{p-1}(u_t - v_t) + p(u^{p-1} - v^{p-1})v_t.$$

It can be readily seen that the first term in the right hand side goes to zero in the L^2 norm as $v \to u$ in $H^{1,1}$ norm. Furthermore, there exists a constant K such that

$$\|(u^{p-1} - v^{p-1})v_t\|_{L^2(Q)} \leq K\|u - v\|_{L^2(Q)}\|v_t\|_{L^4(Q)}^2.$$

Noting that $v_t \in H^{1,1}$, by the imbedding $H^{1,1}(Q) \hookrightarrow L^4(Q)$, $\|v_t\|^2_{L^4(Q)}$ is bounded, and thus the second term also converges to zero. That is, A3 has been satisfied.

Next, it is known [4] that, for any $\psi \in H^1(J; L^2)$, there exists a unique solution $\phi \in H^{2,2} \cap H^1(J; H_0^1)$ to the following simple linear problem:

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial t^2} - \Delta\phi &= \psi, & (x,t) \in \Omega \times J, \\ \phi(x,t) &= 0, & (x,t) \in \partial\Omega \times J, \\ \phi(x,0) &= 0, & x \in \Omega, \\ \dfrac{\partial \phi}{\partial t}(x,0) &= 0, & x \in \Omega. \end{cases} \tag{2}$$

We denote the above correspondence by $\phi = A\psi$. Also (2) is equivalent to the following weak form: find ϕ such that

$$(\phi_{tt}, v) + (\nabla\phi, \nabla v) = (\psi, v), \qquad v \in H_0^1, \;\; t \in J. \tag{3}$$

So, we define a weak solution for (1) as an element $u \in L^2(J; H_0^1) \cap H^2(J; L^2)$ such that $f(\cdot, u) \in H^1(J; L^2)$ satisfying

$$(u_{tt}, v) + (\nabla u, \nabla v) = \big(f(\cdot, u), v\big), \qquad v \in H_0^1, \;\; t \in J. \tag{4}$$

Thus, using the nonlinear map $F \equiv Af$, we obtain the following fixed point formulation of the problem (1):

$$u = F(\cdot, u). \tag{5}$$

Now we have the following *a priori* estimates for the solution of (2).

Lemma 1. *Let ϕ be the unique solution for (2). Then*

$$\|\phi_{tt}\|^2 + \|\phi_t\|^2_{L^2(J;H_0^1)} + |\phi|^2_{L^2(J;H^2)} \le 2\|\psi\|^2 + 3T\big(\|\psi(0)\|^2 + \frac{1}{\epsilon}\|\psi_t\|^2\big)e^{\epsilon T} \tag{6}$$

where $|\phi|^2_{L^2(J;H^2)} = \int_0^T |\phi(t)|^2_{H^2(\Omega)} dt \equiv \sum_{i,j=1}^n \int_0^T \|\frac{\partial^2 u}{\partial x_i \partial x_j}\|_0^2 dt$ and ϵ means an arbitrary constant such that $0 < \epsilon < 1$.

Proof. The following arguments are along the same lines in [4]. First, differentiating (3) in t, we have

$$(\phi_{ttt}, v) + (\nabla\phi_t, \nabla v) = (\psi_t, v), \qquad v \in H_0^1, \quad t \in J.$$

Set $v = \phi_{tt}$ in the above and integrate it in time to get

$$\begin{aligned} \|\phi_{tt}(t)\|_0^2 + \|\phi_t(t)\|_1^2 &= \|\psi(0)\|_0^2 + 2\int_0^t (\psi_t, \phi_{tt})ds \\ &\leq \|\psi(0)\|_0^2 + \frac{1}{\epsilon}\int_0^t \|\psi_t\|_0^2 ds + \epsilon \int_0^t \|\phi_{tt}\|_0^2 ds \end{aligned}$$

where we have used the well known inequality: $ab \leq \frac{1}{2}(\frac{1}{\epsilon}a^2 + \epsilon b^2)$. Thus by the application of Gronwall's lemma, we obtain for each $t \in J$

$$\|\phi_{tt}(t)\|_0^2 + \|\phi_t(t)\|_1^2 \leq \Big(\|\psi(0)\|_0^2 + \frac{1}{\epsilon}\int_0^T \|\psi_t\|_0^2 dt\Big)e^{\epsilon T}. \tag{7}$$

Next, taking account of the relation $|\phi(t)|_{H^2(\Omega)}^2 = \|\Delta\phi(t)\|_0^2$, we have

$$|\phi|_{L^2(J;H^2)}^2 \leq 2\big(\|\phi_{tt}\|^2 + \|\psi\|^2\big). \tag{8}$$

Thus (7) and (8) yield the desired estimates. □

Lemma 2. *The map $A : H^{0,1} \to H^{1,1}$ defined above is compact.*

Proof. First, observe that for each $\psi \in H^{0,1}$

$$\psi(0) = \psi(t) - \int_0^t \psi_t(s)ds, \qquad t \in J.$$

Hence, it is seen that $\|\psi(0)\|_\Omega$ can be bounded by $\|\psi\|_{H^{0,1}}$. Therefore, by virtue of Lemma 1, it is sufficient to show that the inclusion $H^{2,2} \cap H^1(J;H^1) \hookrightarrow H^{1,1}$ is compact.

Let $\{u_n\}$ be a bounded sequence in $H^{2,2} \cap H^1(J;H^1)$. Then, by a well-known compactness theorem (e.g. [14], Chapter III, Theorem 2.1), $\{u_n\}$ is precompact in $L^2(J;H^1)$. Therefore, we can choose a convergent subsequence $\{u_{n'}\}$ of $\{u_n\}$. Since, $\{\frac{du_{n'}}{dt}\}$ is also precompact in $L^2(J;L^2)$, again there exists a subsequence $\{u_{n''}\}$ of $\{u_{n'}\}$ such that $\{\frac{du_{n''}}{dt}\}$ converges in $L^2(J;L^2)$. That is, $\{u_{n''}\}$ is a convergent sequence in $H^{1,1}$ and the proposition follows. □

We now prove a fundamental theorem which provides the principle of the verification. Set $\tilde{H} \equiv H \cap \{u(0) = 0\}$. Here, $u(0) = 0$ implies that $\lim_{t\to 0} u(t) = 0$ in $L^\infty(\Omega)$ sense.

Theorem 1. *Let U be a bounded, convex and nonempty subset of $\tilde{H}$ such that $FU \subset \overline{U}$. Then, there exists a solution $u \in \overline{U}$ for (1). Here, $\overline{U}$ means the closure of U with respect to the $H^2(Q)$ norm.*

Proof. First, notice that $\overline{U}$ is also closed in $H^{1,1}$. Since this fact follows by the quite similar arguments to that in [10] (Theorem 1), it is omitted here. Next, we show that $\overline{U}$ is a bounded subset in H. For arbitrary $u \in \overline{U}$, there exists a sequence $\{u_n\} \subset U$ such that $u_n \to u$ in $H^2(Q)$. Particularly, $u_n \to u$ in $L^2(Q)$. Thus, there is a subsequence $\{u_{n'}\}$ which converges to u in pointwise for almost everywhere in Q. Therefore, the boundedness of U in $L^\infty(Q)$ also assures the same property for $\overline{U}$.

Next, using the assumption A3 and the continuity of the map $F \equiv Af$, we have by the hypothesis

$$Af(\cdot, \overline{U}) \subset \overline{Af(\cdot, U)} \subset \overline{U}.$$

Therefore, by the assumption A3, Lemma 2 and application of Schauder's fixed point Theorem, we have the desired conclusion. □

3 Rounding and verification condition

Since the operator $F \equiv Af$ is infinite dimensional, it is impossible to compute FU for given $U \subset \tilde{H}$ directly with a computer. We thus introduce the rounding $R(FU)$ and the rounding error $RE(FU)$ as in [6, 10] etc.

Let S_h be a finite dimensional subspace of $L^2(J; H_0^1) \cap H^2(J; L^2)$ dependent on h $(0 < h < 1)$. Usually, S_h is taken to be a finite element subspace with mesh size h which satisfies the initial and boundary conditions.

Now define a projection $P_h : L^2(J; H_0^1) \cap H^2(J; L^2) \longrightarrow S_h$, associated with the solution to (2), by the following simultaneous discretization scheme in space and time:

$$\int_0^T \int_0^t \left\{ (\phi_{ss}^h, v_s)_\Omega + (\nabla \phi^h, \nabla v_s)_\Omega \right\} ds\, dt = \int_0^T \int_0^t (\psi, v_s)_\Omega \, ds\, dt, \quad v \in S_h \quad (9)$$

where $\phi^h \equiv P_h\phi$. Although (9) seems to be a somewhat peculiar scheme, it will be proved later to be more easy to obtain the constructive error estimates for (9) than in other existing approximation schemes.

Proposition 1. *(9) has a unique solution in S_h for each $\psi \in H^1(J; L^2)$.*

Indeed, when $f \equiv 0$, setting $v = \phi^h$ in (9) we have

$$\int_0^T \int_0^t \frac{d}{ds}\big[\|\phi_s^h\|^2_{L^2(\Omega)} + \|\nabla\phi^h\|^2_{L^2(\Omega)}\big]\,ds\,dt = 0.$$

Therefore,

$$\int_0^T \big(\|\phi_s^h\|^2_{L^2(\Omega)} + \|\nabla\phi^h\|^2_{L^2(\Omega)}\big)\,dt = 0$$

which implies $\phi^h \equiv 0$, thus, the proposition follows from the well known property of the solution for linear system of equations.

Next, we have the following error estimates.

Theorem 2. *Let ϕ and ϕ^h be the solutions for (2) and (9), respectively. If S_h is a subspace of $H^3(Q)$, which means in general a set of C^2 class piecewise polynomials both in space and time, then for $e = \phi - \phi^h$, we have*

$$\|e_t\|^2 + \|\nabla e\|^2 \le 2\big(\|\psi - \phi^h_{tt} + \Delta\phi^h\| + T\|\psi_t - \phi^h_{ttt} + \Delta\phi^h_t\|\big) \inf_{v\in S_h}\|\phi - v\|. \quad (10)$$

Proof. Observe that using (2) and (9), for arbitrary $v \in S_h$,

$$\begin{aligned}\frac{1}{2}\int_0^T \big(\|e_t\|^2 + \|\nabla e\|^2\big)\,dt &= \int_0^T\int_0^t \frac{1}{2}\frac{d}{ds}\big[(e_s, e_s) + (\nabla e, \nabla e)\big]\,ds\,dt \\ &= \int_0^T\int_0^t \big[(e_{ss}, e_s) + (\nabla e, \nabla e_s)\big]\,ds\,dt \\ &= \int_0^T\int_0^t \Big[\big(\psi - \phi^h_{ss}, (\phi - v)_s\big) \\ &\qquad - \big(\nabla\phi^h, \nabla(\phi - v)_s\big)\Big]\,ds\,dt.\end{aligned}$$

Thus, integrating by parts and using the Schwarz inequality, we have the desired estimates. □

We now suppose that S_h has the following approximation property.

For any $u \in H^{2,2} \cap H^1(J; H_0^1)$,

$$\inf_{v \in S_h} \|u - v\| \leq C_1 h^2 |u|_{H^2(Q)} \tag{11}$$

where $|u|^2_{H^2(Q)} \equiv \sum_{i,j=1}^{n+1} \|\frac{\partial^2 u}{\partial x_i \partial x_j}\|^2$ and $x_{n+1} \equiv t$. Also C_1 is supposed to be a positive constant, independent of h, which can be numerically estimated. This is a natural assumption for many finite element subspaces (cf. [10]).

We now give an $L^\infty(Q)$ estimate for the solution of (2).

Lemma 3. *Let ϕ be the solution for (2). Then there exists a positive constant $C = C(\Omega, \epsilon)$ such that*

$$\|\phi\|_{L^\infty(Q)} \leq C\big(\|\psi(\cdot, 0)\|_0 + \|\psi\| + \|\psi_t\|\big)$$

where ϵ is the same parameter as in Lemma 1.

Proof. For almost everywhere $t \in J$ and $x \in \Omega$, by the Sobolev imbedding theorem and well-known estimates (e.g. in [5]), there exists constants C_2 and C_3 such that

$$\begin{aligned} |\phi(x,t)| &\leq C_2 \|\phi(\cdot, t)\|_{H^2(\Omega)} \\ &\leq C_2 C_3 \|\Delta \phi(\cdot, t)\|_0 \\ &= C_2 C_3 \|\phi''(\cdot, t) - \psi(\cdot, t)\|_0. \end{aligned} \tag{12}$$

Here, for example, we can choose the above constant as $C_2 \leq 1.476$ and $C_3 \leq 3$, respectively, for the two dimensional unit square ([5]).

Further, using the relation $\psi(x,t) = \psi(x,0) + \int_0^t \psi_t(x,s) ds$, we get

$$\|\psi(\cdot, t)\|_0 \leq \|\psi(\cdot, 0)\|_0 + T\|\psi_t\|.$$

Therefore, by (12) and the estimates (7) in the proof of Lemma 1, we obtain the conclusion of the Lemma. □

Now, based upon the above arguments, for any $\psi \in H^1(J; L^2)$, we define the rounding $R(A\psi)$ and the rounding error $RE(A\psi)$ as follows:

$$R(A\psi) \equiv P_h(A\psi)$$

and

$$RE(A\psi) = \{\phi \in H \mid \|\phi\|_{H^{1,1}} \le h\sqrt{2C_1K_1K_2}\}$$

respectively. Here, C_1 is the constant in (11) and

$$K_1 \equiv K_1(\phi^h, \psi) = 2\big(\|\psi - \phi^h_{tt} + \Delta\phi^h\| + T\|\psi_t - \phi^h_{ttt} + \Delta\phi^h_t\|\big),$$

$$K_2 \equiv K_2(\psi, \epsilon) = 2\|\psi\|^2 + 3T\left(\|\psi(0)\|_0^2 + \frac{1}{\epsilon}\left\|\frac{d\psi}{dt}\right\|^2\right)e^{\epsilon T}$$

where $\phi^h = R(A\psi)$.

Moreover, the definitions of $R(AG)$ and $RE(AG)$ for the set of functions $G \subset H^1(J; L^2)$ are defined in the obvious manner (*cf.* [6]).

Now let $\{\phi_j\}_{j=1,\ldots,M}$ be a basis for S_h and let $\mathcal{S}_{I,h}$ denote the set of all linear combinations of $\{\phi_j\}$ with interval coefficients. And let $\mathbb{R}^+$ be the set of nonnegative real numbers. For any $\alpha \in \mathbb{R}^+$, set $[\alpha] \equiv \{\phi \in \tilde{H} \mid \|\phi\|_{H^{1,1}} \le \alpha\}$. Also for $U_h \in \mathcal{S}_{I,h}$ and $\alpha, \beta \in \mathbb{R}^+$, we define the ordered triple (U_h, α, β) as

$$(U_h, \alpha, \beta) \equiv \{\phi \in \tilde{H} \mid \phi \in U_h + [\alpha] \text{ and } \|\phi\|_{L^\infty(Q)} \le \beta\}.$$

Then, we have the following verification condition.

Theorem 3. *For $U_h \in \mathcal{S}_{I,h}$ and $\alpha, \beta \in \mathbb{R}^+$, set $G = f(\cdot, U)$ where $U = (U_h, \alpha, \beta)$.*

Suppose that

$$\begin{cases} R(AG) & \subset & U_h, \\ \|RE(AG)\|_{H^{1,1}} & \le & \alpha, \\ C\big(\|G(0)\|_0 + \|G\| + \|G_t\|\big) & \le & \beta \end{cases} \tag{13}$$

where C is the same constant in Lemma 3 and the norm for a set of functions means the supremum value for norms of all functions in it. Then, there exists a solution $u \in \overline{U}$ for (1), where $\overline{U}$ means the closure of U with $H^2(Q)$ norm.

Proof. First, note that by Theorem 2, (11), Lemma 1 and the definitions of rounding and rounding error, we have

$$FU \subset R(FU) + RE(FU). \tag{14}$$

Next, taking account of Lemma 3, the last condition in the proposition means that

$$\|FU\|_{L^\infty(Q)} \le \|U\|_{L^\infty(Q)}. \tag{15}$$

We now set $\gamma = 2\|G\|^2 + 3T\big(\|G(0)\|_0^2 + \frac{1}{\epsilon}\|\frac{dG}{dt}\|^2\big)e^{\epsilon T}$ and $U_{2\gamma} \equiv U \cap \{|u|^2_{H^2(Q)} \le 2\gamma\}$. Then, Lemma 1 implies that

$$|FU|^2_{H^2} \le 2\gamma. \tag{16}$$

On the other hand, by virtue of the continuity of f in t, we have

$$\|FU(t)\|_{H^2(\Omega)} \to 0, \qquad \text{as } \ t \to 0. \tag{17}$$

Then the imbedding theorem yields that $\|FU(t)\|_{L^\infty(\Omega)} \to 0$ which implies $FU_{2\gamma} \subset \tilde{H}$. Hence, from (14) – (16) we obtain $FU_{2\gamma} \subset U_{2\gamma}$ and thus, by Theorem 1 we have the desired conclusion. □

4 Verification procedures and a numerical example

In the present section, we describe an actual computing algorithm for generation of the set U which satisfies the verification conditions in Theorem 3, and also give a numerical example of verification.

We use an iterative procedure, which is similar to that in [6, 10] etc., except for the use of the smooth approximation space S_h in x and t, i.e. $S_h \subset H^2(Q)$ with homogeneous initial and boundary conditions. Let $\hat{u}$ be some smooth approximate solution of the problem (1) which may not be necessarily in S_h. By setting $w \equiv u - \hat{u}$, we rewrite (1) as follows:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} - \Delta w = d + g(x,t,w), & (x,t) \in \Omega \times J, \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times J, \\ w(x,0) = 0, & x \in \Omega, \\ \dfrac{\partial w}{\partial t}(x,0) = 0, & x \in \Omega \end{cases} \tag{18}$$

where $d = f(x,t,\hat{u}) - \hat{u}_{tt} + \Delta\hat{u}$ and $g(x,t,w) = f(x,t,w+\hat{u}) - f(x,t,\hat{u})$.

We now set $w_0^h = 0 \in S_h$, $\alpha_0 = \beta_0 = 0$, and $W_0 \equiv (w_0^h, \alpha_0, \beta_0)$. Let $\varepsilon_k > 0$, $1 \le k \le 3$, be given small numbers. For $i \ge 1$, when

$$w_{i-1}^h = \sum_{j=1}^{M} \big[\underline{A}_j^{(i-1)}, \overline{A}_j^{(i-1)}\big]\phi_j \in \mathcal{S}_{I,h}$$

and $\alpha_{i-1}, \beta_{i-1}$ are nonnegative real numbers, set

$$\begin{cases} \hat{w}_{i-1}^h \equiv \displaystyle\sum_{j=1}^{M} \big[\underline{A}_j^{(i-1)} - \varepsilon_1, \overline{A}_j^{(i-1)} + \varepsilon_1\big]\phi_j, \\ \hat{\alpha}_{i-1} \equiv \alpha_{i-1} + \varepsilon_2, \\ \hat{\beta}_{i-1} \equiv \beta_{i-1} + \varepsilon_3 \end{cases} \tag{19}$$

and also set $\hat{W}_{i-1} \equiv (\hat{w}_{i-1}^h, \hat{\alpha}_{i-1}, \hat{\beta}_{i-1})$, which are so-called ε_k-inflations (*cf.* [12]).

Then, we choose $w_i^h \in \mathcal{S}_{I,h}$ and $\alpha_i, \beta_i \in \mathbb{R}^+$ satisfying, for $G_{i-1} \equiv d + g(\cdot, \hat{W}_{i-1})$,

$$\begin{cases} \big((w_i^h)_{tt}, \phi_j) + (\nabla w_i^h, \nabla\phi_j\big) \supset (G_{i-1}, \phi_j), \qquad 1 \le j \le M, \\ \alpha_i = he\sqrt{2C_1K_1(w_i^h, G_{i-1})K_2(G_{i-1}, \epsilon)}, \\ \beta_i = C\big(\|G_{i-1}(0)\|_0 + \|G_{i-1}\| + \|(G_{i-1})_t\|\big) \end{cases} \tag{20}$$

respectively. Here, C_1, K_1, K_2, C, and ϵ are previously defined constants and parameter. Also the first formula in (20) means that u_i^h is determined by an interval vector solution for the system of linear equations with interval right hand side.

Then, from Theorem 3, we have the following actual verification conditions in computer.

Theorem 4. *If there exists some integer N such that*

$$w_N^h \subset \hat{w}_{N-1}^h, \quad \alpha_N \le \hat{\alpha}_{N-1}, \quad \text{and} \quad \beta_N \le \hat{\beta}_{N-1}.$$

Then there exists a solution $u \in \hat{u} + \overline{W}_N$ *for (1), where* $W_N = (w_N^h, \alpha_N, \beta_N)$ *and* $\overline{W}_N$ *means the closure of* W_N *in the* $H^2(Q)$ *norm. And* $w_N^h \subset \hat{w}_{N-1}^h$ *implies that each coefficient interval in* w_N^h *is included in the corresponding interval in* $\hat{w}_{N-1}^h$.

Now we will provide some numerical examples which were actually verified by computer with the procedure described above.

The model equation is as follows:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = Ku^2 + P\sin^2 \pi x(2 + t^2\pi^2 - KPt^4 \sin \pi x), & (x,t) \in \Omega \times J, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times J, \\ u(x,0) = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial t}(x,0) = 0, & x \in \Omega \end{cases} \tag{21}$$

where $\Omega = (0,1)$, $J = (0,1)$, and K, P are constants. The exact solution of this problem is $u = Pt^2 \sin \pi x$.

Now, in order to construct the approximation space, let $\delta_x : 0 = x_0 < x_1 < \cdots < x_L = 1$ be a uniform partition of the interval Ω and set $h = 1/L$. For simplicity, define the partition of J as $\delta_t = \delta_x$. We denote by $\mathcal{M}^3$ the set of C^2 class piecewise cubic functions, i.e. cubic splines, on Ω or J, let $S_x = \{v \in \mathcal{M}^3(\Omega) \mid v(0) = v(1) = 0\}$ and let $S_t = \{w \in \mathcal{M}^3(J) \mid w(0) = w_t(0) = 0\}$. And we adopt $S_{xt} = S_x \otimes S_t$ as the approximation space S_h. Then, we have $\dim S_{xt} = (L+1)^2$.

Next, we estimate the constant C_1 in the approximation property (11). Let P_x and P_t denote L^2 projections from $H^2(\Omega)$ and $H^2(J)$ into S_x and S_t, respectively. Then, it is easily seen (e.g. [13]) that, for any $w \in H^2(\Omega)$,

$$\|w - P_x w\|_{L^2(\Omega)} \le \inf_{v \in S_x} \|w - v\|_{L^2(\Omega)} \le \frac{2}{\pi^2} h^2 \|w_{xx}\|_{L^2(\Omega)}.$$

The same estimates hold for P_t. For any $u \in H^2(Q)$, notice that $\|u - P_x u\| \le \|u\|$, where $P_x u$ is a function of space and time in the sense of $P_x u(\cdot, t)$ for each $t \in J$. Then we have

$$\inf_{\chi \in S_h} \|u - \chi\| \le \|u - P_t P_x u\|$$

$$\begin{aligned} &\le \|u - P_x u\| + \|u - P_t u\| + \|u - P_x u - P_t(u - P_x u)\| \\ &\le \frac{2}{\pi^2} h^2 \big(\|u_{xx}\| + \|u_{tt}\| + \|(u - P_x u)_{tt}\| \big) \\ &\le \frac{2}{\pi^2} h^2 \big(\|u_{xx}\| + 2\|u_{tt}\| \big). \end{aligned}$$

Thus from the above inequality, we can take the constant in (11) as $C_1 = \frac{2}{\pi^2}\sqrt{6}$. And observe that

$$u(x,t) = \int_0^x u_x(\xi,t)d\xi = \int_0^x \int_0^t u_{xt}(\xi,\eta)d\eta\, d\xi \le \|u_{xt}\|.$$

Hence, using Lemma 1, we can estimate C in Lemma 3.

Thus we can implement the verification procedures (19), (20), using the above estimates and some of the calculation techniques which are similar to those in [7, 10].

We actually verified for several cases. For example, in the case that $K = 0.5$, $P = 0.1$, we completed the procedure with iteration numbers $N = 7$ and error $\alpha_N = 0.0702$ under the conditions of $\|d\| < 10^{-2}$ and mesh size $h = 0.1$ (i.e. $L = 10$).

Remark. From our experience, it is expected that we would also be able to verify for the problems with larger K and P than the present case provided that we can use smaller mesh size. However, owing to the various limitations of our computer facility, we could not use such a sufficiently small partition. Further, it was observed that better initial approximation yielded easier verification with the same mesh size. Anyway, as the main purpose of this report, we can show that a similar verification principle to that in [7, 10] can also be applied to hyperbolic problems of second order, but the development of the practical verification procedure is left as a future subject which may be dependent on the computer technology itself, e.g. such as the appearence of super parallel computers, etc.

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