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Enclosures for Solutions of Parameter-Dependent Nonlinear Elliptic Boundary Value Problems: Theory and Implementation on a Parallel Computer

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We consider a method for proving the existence and computing enclosures for solutions of parameter-dependent nonlinear elliptic boundary value problems, which is applicable also in (neighborhoods of) simple turning points. This goal is achieved by a combination of earlier existence and enclosure results with the technique of change of parameters. Significant parts of our numerical algorithm possess a high degree of parallelism and have been implemented on a T 800 Transputer System.

Оболочки для решений параметрически зависимых нелинейных эллиптических краевых задач: теория и реализация на параллельном компьютере

М. Плум

Рассматривается метод доказательства существования и вычисления включений для решений зависящих от параметров нелинейных эллиптических краевых задач, который также применим в окрестности простых экстремальных точек. Эта цель достигается за счет комбинации ранее известных результатов,касающихся существования и включения, с техникой изменения параметров. Значительная часть нашего численного алгоритма имеет высокую степень параллелизма и реализована на транспьютерной системе Т 800.

1 Introduction

Consider the parameter-dependent nonlinear boundary value problem

$$-\Delta U + F(x, U, \lambda) = 0 \text{ on } \Omega, \quad U = 0 \text{ on } \partial\Omega \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ (with $n \in \{2, 3\}$) is a bounded domain with sufficiently regular boundary $\partial\Omega$, and $F : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, together with its derivatives $\partial F/\partial U$ and $\partial F/\partial\lambda$.

In previous articles (e. g. [17–19]) the author derived *existence* and *enclosure* results for solutions of problem (1) with fixed parameter λ, provided that an approximate solution

$$\omega \in H_{2,0}(\Omega) := \text{closure}_{H_2(\Omega)} \left\{ u \in C_2(\bar{\Omega}) : u\big|_{\partial\Omega} \equiv 0 \right\}$$

with sufficiently (L_2-)small defect $-\Delta\omega + F(\cdot, \omega, \lambda)$ can be computed such that the inverse of the operator $L : H_{2,0}(\Omega) \to L_2(\Omega)$ given by

$$L[u] := -\Delta u + c \cdot u, \qquad c(x) := \big(\partial F/\partial U\big)\big(x, \omega(x), \lambda\big) \tag{2}$$

can be bounded suitably.

Considering problem (1) as a bifurcation problem with parameter λ, we therefore find that this existence and enclosure method is not applicable in (neighborhoods of) turning or bifurcation points (ω, λ), since the operator L is not invertible in such points.

To overcome this difficulty for simple turning points, we *change the parametrization* of the problem: we choose some suitable C_1-smooth function $\Phi : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and adjoin the scalar equation

$$\int_\Omega \Phi\Big(x, U(x), \lambda\Big)\, dx = \mu \tag{3}$$

to problem (1). The augmented problem (1), (3), with $\mu \in \mathbb{R}$ as new (input-)parameter and with λ now being a part of the solution, is (locally) turning-point-free, if Φ has been chosen appropriately.

This technique of parameter change is well known to be a powerful tool in the treatment of turning point problems, both from the theoretical and the numerical point of view (e. g., [1, 9, 10, 21, 22, 24]). Here, we will combine this technique with *existence* and *enclosure* methods for nonlinear boundary value problems, which we believe is new. In [20], we worked out the corresponding theory for boundary value problems with *ordinary* differential equations. Most of the results and proofs can be carried over to elliptic problems without any significant changes. Therefore, we omit some proofs here and refer the reader to [20] for more theoretical details. That paper also contains references to other relevant enclosure methods for boundary value problems (e. g., [5, 8, 12–15, 23]).

More differences between ordinary and elliptic problems are present in the *numerical* procedures needed for our method. In [20], we proposed a collocation method with polynomial basis functions for ordinary differential equations, while for elliptic problems a finite-element method appears to be more appropriate for several reasons, one of them being the possibility of *parallelization*: many time-consuming parts of our algorithm are carried out independently on each element, which allows a highly parallel implementation. We used a T 800 Transputer System with 32 transputers to test the parallel aspects of our existence and enclosure method.

2 Existence and enclosure for the augmented problem

Throughout this paper, we will assume that the domain Ω is regular in the sense that $\partial\Omega$ is Lipschitz, and the Laplacian maps the space $H_{2,0}(\Omega)$ *onto* $L_2(\Omega)$. This condition is satisfied, for instance, for $C_{1,1}$-smooth boundaries $\partial\Omega$, for convex polygonal domains $\Omega \subset \mathbb{R}^2$ and their $C_{1,1}$-diffeomorphic images, and for many more cases (see [17, Section 5]); it fails for domains with reentrant corners (such as L-shaped domains).

We wish to derive an existence and enclosure result for problem (1) which, in particular, is applicable in some neighborhood of some (conjectured) simple *turning point,* where the linearized operator L defined in (2) is not invertible, so that a direct application of our former enclosure method is impossible.

For this purpose, we choose some C_1-smooth function $\Phi : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that the expression

$$\int_\Omega \Phi\Big(x, U_\lambda(x), \lambda\Big)\, dx$$

is (expected to be) strictly *monotonic* along the conjectured solution branch (U_λ) containing the turning point. Then, the augmented problem (1), (3) is locally turning-point-free, since the branch (U_λ) is obtained by *monotone* variation of the new parameter μ.

We will restrict ourselves to choices where $\partial\Phi/\partial U \equiv \partial F/\partial\lambda$, which is sufficient for many practical cases and facilitates our theory and numerics.

Suppose that, for some given μ, an *approximate* solution

$$(\omega, \tilde{\lambda}) \in H_{2,0}(\Omega) \times \mathbb{R}$$

of problem (1), (3) has been computed, as well as bounds δ_1, δ_2 for its *defects:*

$$\big\| -\Delta\omega + F(\cdot, \omega, \tilde{\lambda}) \big\|_2 \le \delta_1, \qquad \left| \int_\Omega \Phi\Big(x, \omega(x), \tilde{\lambda}\Big)\, dx - \mu \right| \le \delta_2. \tag{4}$$

Furthermore, let constants K and K_0 be known such that

$$\begin{array}{rcl} \|u\|_\infty & \le & K \big\| \mathcal{L}[(u,\sigma)] \big\| \\ |\sigma| & \le & K_0 \big\| \mathcal{L}[(u,\sigma)] \big\| \end{array} \qquad \forall (u,\sigma) \in H_{2,0}(\Omega) \times \mathbb{R} \tag{5}$$

where $\mathcal{L}$ is the linearization of the augmented problem (1), (3) at $(\omega, \tilde{\lambda})$, i. e.,

$$\mathcal{L}[(u,\sigma)] := \left(L[u] + \psi \cdot \sigma, \int_\Omega \psi(x) u(x)\, dx + \tau \cdot \sigma \right) \in L_2(\Omega) \times \mathbb{R}$$

with L given by (2) (with $\tilde{\lambda}$ in place of λ), and

$$\begin{aligned} \psi(x) &:= \frac{\partial F}{\partial \lambda}\Big(x, \omega(x), \tilde{\lambda}\Big) = \frac{\partial \Phi}{\partial U}\Big(x, \omega(x), \tilde{\lambda}\Big), \\ \tau &:= \int_\Omega \hat{\tau}(x)\, dx, \quad \hat{\tau}(x) := \frac{\partial \Phi}{\partial \lambda}\Big(x, \omega(x), \tilde{\lambda}\Big) \end{aligned}$$

and with $\|(v,\varrho)\| := \left[\|v\|_2^2 + \varrho^2\right]^{1/2}$. The constants K and K_0 required in (5) constitute bounds for the inverse operator $\mathcal{L}^{-1}$, which replace the bounds for L^{-1} in our former "direct" approach, and which may be expected to be moderate, due to our considerations made above.

Finally, let functions $G_1, G_2 : [0,\infty) \times [0,\infty) \to [0,\infty)$ be chosen which are monotonically nondecreasing with respect to both arguments and satisfy

$$\begin{aligned} \sqrt{\operatorname{meas}(\Omega)}\left|F(x,\omega(x)+y,\tilde{\lambda}+z) - F(x,\omega(x),\tilde{\lambda}) - c(x)y - \psi(x)z\right| \\ \leq G_1(|y|,|z|), \\ \operatorname{meas}(\Omega)\left|\Phi(x,\omega(x)+y,\tilde{\lambda}+z) - \Phi(x,\omega(x),\tilde{\lambda}) - \psi(x)y - \hat{\tau}(x)z\right| \\ \leq G_2(|y|,|z|) \end{aligned} \tag{6}$$

for all $x \in \bar{\Omega}$, $y, z \in \mathbb{R}$, and

$$G_i(t,s) = o(t+s) \qquad (t,s \to 0,\ i = 1,2) \tag{7}$$

which is possible due to the smoothness assumptions on F and Φ. Usually, such functions G_i can easily be computed directly if constant upper and lower bounds for ω are at hand.

Theorem. *Suppose that some $\alpha \geq 0$ exists such that*

$$\alpha^2 \geq \left[\delta_1 + G_1(K\alpha, K_0\alpha)\right]^2 + \left[\delta_2 + G_2(K\alpha, K_0\alpha)\right]^2. \tag{8}$$

Then, there exists a solution $(U,\lambda) \in H_{2,0}(\Omega) \times \mathbb{R}$ of problem (1), (3) (for the given μ) such that

$$\|U - \omega\|_\infty \leq K\alpha, \qquad |\lambda - \tilde{\lambda}| \leq K_0\alpha.$$

Due to the growth property (7), our crucial assumption (8) is satisfied if the defect bounds δ_1 and δ_2 are sufficiently small, i. e., if the approximate solution $(\omega,\tilde{\lambda})$ has been computed with *sufficient accuracy.*

The *proof* of the Theorem is based on Schauder's Fixed-Point-Theorem and can be carried over almost word by word from the ordinary differential case (see [20]). The only part which changes significantly is the proof of the following

Lemma 1. *$\mathcal{L} : H_{2,0}(\Omega) \times \mathbb{R} \to L_2(\Omega) \times \mathbb{R}$ is one-to-one and onto.*

Proof. The first assertion follows immediately from (5). Due to our general regularity assumption on Ω, Fredholm's alternative holds for the boundary value problem $u \in H_{2,0}(\Omega)$, $L[u] = r$ on Ω, with given $r \in L_2(\Omega)$.

Now let $r \in L_2(\Omega)$ and $\varrho \in \mathbb{R}$ be given. In order to prove that the problem

$$L[u] + \psi \cdot \sigma = r \text{ on } \Omega, \qquad \int_\Omega \psi(x)u(x)\,dx + \tau \cdot \sigma = \varrho \tag{9a, b}$$

has a solution $(u, \sigma) \in H_{2,0}(\Omega) \times \mathbb{R}$, suppose first that $L : H_{2,0}(\Omega) \to L_2(\Omega)$ is one-to-one (and thus, due to Fredholm's alternative, also onto). Then, (9a) is solved by $u := L^{-1}[r] - \sigma L^{-1}[\psi]$, for arbitrary $\sigma \in \mathbb{R}$. Inserting u into (9b), we are left with the equation

$$\left(\tau - \int_\Omega \psi \cdot L^{-1}[\psi]\,dx\right) \cdot \sigma = \varrho - \int_\Omega \psi \cdot L^{-1}[r]\,dx \tag{10}$$

for σ. The second estimate in (5) shows that $\mathcal{L}\big[(L^{-1}[\psi], -1)\big] \not\equiv 0$ and thus, that the term in parantheses in (10) is not zero. Consequently, (10) is (uniquely) solvable for σ.

Now suppose that $L : H_{2,0}(\Omega) \to L_2(\Omega)$ is not one-to-one, so that linearly independent functions $g_1, \dots, g_s \in H_{2,0}(\Omega)$ exist which span the nullspace of L. For each $(\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s \setminus \{(0, \dots, 0)\}$, $\sum_{i=1}^s \alpha_i g_i$ does not vanish identically, so that the first estimate in (5) implies

$$\mathcal{L}\left[\left(\sum_{i=1}^s \alpha_i g_i, 0\right)\right] \not\equiv 0$$

and thus,

$$\sum_{i=1}^s \alpha_i \int_\Omega \psi g_i\,dx \neq 0.$$

Since this is true for each nontrivial $(\alpha_1, \dots, \alpha_s)$, it follows that $s = 1$ and

$$\int_\Omega \psi \cdot g\,dx \neq 0 \tag{11}$$

for $g := g_1$. Due to Fredholm's alternative, equation (9a) is solvable if and only if $r - \sigma\psi$ is orthogonal to g in $L_2(\Omega)$, which provides a unique value for σ, according to (11). The general solution of (9a) then is $u = w + \kappa \cdot g$ ($\kappa \in \mathbb{R}$), with $w \in H_{2,0}(\Omega)$ denoting some particular solution. Inserting into (9b) and using (11), we obtain a unique value for κ. □

3 Computation of K and K_0

In this section, we show how constants K and K_0 bounding $\mathcal{L}^{-1}$ via (5) can be computed. Since $\mathcal{L}$ is symmetric on $H_{2,0}(\Omega) \times \mathbb{R}$ and $\mathcal{L}^{-1} : L_2(\Omega) \times \mathbb{R} \to H_{2,0}(\Omega) \times \mathbb{R} \hookrightarrow L_2(\Omega) \times \mathbb{R}$ is compact, $\mathcal{L}$ possesses a complete orthonormal system of eigenelements, and one easily finds by series expansion that

$$\|(u,\sigma)\| \leq K_0 \|\mathcal{L}[(u,\sigma)]\| \quad \text{for} \quad (u,\sigma) \in H_{2,0}(\Omega) \times \mathbb{R} \tag{12}$$

with

$$K_0 := \left[\min\{|\lambda| : \lambda \text{ eigenvalue of } \mathcal{L} \text{ on } H_{2,0}(\Omega) \times \mathbb{R}\}\right]^{-1}. \tag{13}$$

Obviously, (12) implies the second estimate in (5). The calculation of K_0 via (13) requires the computation of bounds for the *eigenvalues* of $\mathcal{L}$ neighboring 0. Numerical methods concerned with eigenvalue bounds may be found, for instance, in [2, 6, 11, 16]. In [16], a homotopy method is presented for eigenvalue problems with purely differential operators, which however can easily be transferred to augmented problems of the type occurring here; see [20] for more details.

To calculate a constant K satisfying the first estimate in (5), we compute constants K_1 and K_2 such that, for $(u,\sigma) \in H_{2,0}(\Omega) \times \mathbb{R}$,

$$\|u_x\|_2 \leq K_1 \|\mathcal{L}[(u,\sigma)]\|, \qquad \|u_{xx}\|_2 \leq K_2 \|\mathcal{L}[(u,\sigma)]\| \tag{14}$$

(where $\|u_x\|_2^2 = \sum_{i=1}^n (\partial u/\partial x_i)^2$, $\|u_{xx}\|_2^2 = \sum_{i,j=1}^n (\partial^2 u/\partial x_i \partial x_j)^2$) and combine (12), (14) with an explicit version of Sobolev's embedding $H_2(\Omega) \hookrightarrow C(\bar{\Omega})$ derived in [17], which reads

$$\|u\|_\infty \leq C_0 \|u\|_2 + C_1 \|u_x\|_2 + C_2 \|u_{xx}\|_2 \quad (u \in H_2(\Omega)) \tag{15}$$

with explicitly known constants C_0, C_1, C_2. From (12), (14), (15), we obviously obtain the first estimate in (5) with $K := C_0 K_0 + C_1 K_1 + C_2 K_2$.

The following lemma shows how constants K_1 and K_2 satisfying (14) can be calculated if the domain Ω is convex. In other cases, Section 4 in [17] shows how to generalize the results.

Lemma 2. *Let (12) hold for some constant K_0, and let $\underline{c}$ and $\bar{c}$ denote constant lower and upper bounds for the coefficient function c defined in (2) (with $\tilde{\lambda}$ in place of λ). Moreover, let*

$$c_0 := \frac{1}{2}(\underline{c}+\tau) - \sqrt{\frac{1}{4}(\underline{c}-\tau)^2 + \|\psi\|_2^2}.$$

Then (14) holds with

$$K_1 := \begin{cases} \left[K_0(1-c_0K_0)\right]^{1/2} & \text{if } c_0K_0 \le \frac{1}{2} \\ \dfrac{1}{2\sqrt{c_0}} & \text{otherwise} \end{cases}$$

and, if Ω is convex, with

$$K_2 := 1 + \left[\left(\max\left\{\frac{1}{2}(\bar{c}-\underline{c}), -\underline{c}\right\}\right)^2 + \|\psi\|_2^2\right]^{1/2} \cdot K_0.$$

Proof. By Schwarz's inequality and partial integration we obtain, for $(u,\sigma) \in H_{2,0}(\Omega) \times \mathbb{R}$,

$$\begin{aligned} &\|(u,\sigma)\| \cdot \left\|\mathcal{L}[(u,\sigma)]\right\| \\ &\quad \ge \int_\Omega u \cdot (-\Delta u + cu + \psi \cdot \sigma)\, dx + \sigma\left(\int_\Omega \psi u \, dx + \tau\sigma\right) \\ &\quad \ge \|u_x\|_2^2 + \underline{c}\|u\|_2^2 - 2|\sigma|\,\|\psi\|_2\,\|u\|_2 + \tau\sigma^2 \\ &\quad \ge \|u_x\|_2^2 + c_0\|(u,\sigma)\|^2. \end{aligned}$$

Consequently, $\|u_x\|_2^2 \le \|(u,\sigma)\| \cdot \left[\|\mathcal{L}[(u,\sigma)]\| - c_0\|(u,\sigma)\|\right]$, and combination with (12) provides our first assertion.

If Ω is convex, the inequality $\|u_{xx}\|_2 \le \|\Delta u\|_2$ holds for all $u \in H_{2,0}(\Omega)$ (see [17], Section 4). Moreover, $\|\Delta u\|_2 \le \|-\Delta u + \kappa u\|_2$ for $\kappa \ge 0$. Choosing $\kappa := \max\left\{\frac{1}{2}(\underline{c}+\bar{c}), 0\right\}$ we therefore obtain, for $(u,\sigma) \in H_{2,0}(\Omega) \times \mathbb{R}$,

$$\begin{aligned} \|u_{xx}\|_2 &\le \|-\Delta u + \kappa u\|_2 \\ &\le \|\Delta u + cu + \psi\sigma\|_2 + \|c-\kappa\|_\infty \|u\|_2 + \|\psi\|_2\,|\sigma| \\ &\le \left\|\mathcal{L}[(u,\sigma)]\right\| + \max\left\{\frac{1}{2}(\bar{c}-\underline{c}), -\underline{c}\right\} \|u\|_2 + \|\psi\|_2\,|\sigma| \end{aligned}$$

so that Schwarz's inequality (in $\mathbb{R}^2$) and (12) provide our second assertion. □

4 Computation of $(\omega, \tilde{\lambda})$ and δ_1, δ_2 for rectangular domains in $\mathbb{R}^2$

In our examples, we use a *Newton-iteration* to calculate an aproximate solution $(\omega, \tilde{\lambda}) \in H_{2,0}(\Omega) \times \mathbb{R}$ of the augmented problem (1), (3): Starting with some rough approximation $(\omega_0, \tilde{\lambda}_0) \in H_{2,0}(\Omega) \times \mathbb{R}$ (which we obtain by a homotopy method), we compute iteratively approximate solutions $(u_{k+1}, \sigma_{k+1}) \in H_{2,0}(\Omega) \times \mathbb{R}$ of the linear problems

$$\mathcal{L}_k[(u, \sigma)] = -(d_{1k}, d_{2k}) \tag{16}$$

with $\mathcal{L}_k$ denoting the linearization of problem (1), (3) at $(\omega_k, \tilde{\lambda}_k)$, and $(d_{1k}, d_{2k}) \in L_2(\Omega) \times \mathbb{R}$ the defect pair of $(\omega_k, \tilde{\lambda}_k)$. The iteration step is completed by the update $\omega_{k+1} := \omega_k + u_{k+1}, \tilde{\lambda}_{k+1} := \tilde{\lambda}_k + \sigma_{k+1}$.

To determine $(u, \sigma) := (u_{k+1}, \sigma_{k+1})$ we use a *finite element* procedure with rectangular elements; here, we suppose that Ω is itself a rectangle. On each element, u is put up as a biquintic polynomial. The local basis functions are chosen such that the 36 coefficients determining u coincide with the values of u, $\partial u/\partial x_1$, $\partial u/\partial x_2$, and $\partial^2 u/\partial x_1 \partial x_2$ in 9 knots of the element, namely the corners, the midpoints of the sides, and the midpoint of the element. Simple results from Hermite interpolation theory show that the corresponding *global* basis functions φ_i $(i = 1, \ldots, N)$ are C_1-functions, so that

$$u = \sum_{i=1}^{N} a_i \varphi_i \tag{17}$$

belongs to $H_2(\Omega)$. To ensure that $u \in H_{2,0}(\Omega)$ (i. e., that u satisfies the required boundary conditions), several of the coefficients in (17) have to be set to zero. The remaining coefficients are determined, together with σ, by the usual Ritz-procedure for the differential equation contained in (16), and the additional real equation in (16).

All occurring integrals are approximated by the composite trapezoidal quadrature (product-) formula, applied on each element.

The matrix of the resulting linear algebraic system is symmetric and has band structure, except that its last row and its last column are dense. (Here, we assume that equations and variables are ordered such that the additional equation in (16) constitutes the last equation, and σ is the last variable.) This system is solved approximately by a band-Gauss algorithm.

The Newton-iteration it terminated when, for some $k \in \mathbb{N}$, the coefficients a_i of $u = u_k$ in (17), and the value σ_k, are (in modulus) below some tolerance. Then, we choose $(\omega, \tilde{\lambda}) := (\omega_k, \tilde{\lambda}_k)$. ω is therefore available in the form (17), provided that the starting approximation ω_0 has that form.

To compute the defect bounds δ_1 and δ_2 required in (4) we have to enclose two *integrals*. For this purpose, we apply the composite trapezoidal product formula

$$S[f] := \frac{h_1 h_2}{4} \sum_{j=0}^{m_1} \sum_{k=0}^{m_2} w_j^{(1)} w_k^{(2)} f\left(x_1^{(j)}, x_2^{(k)}\right) \tag{18}$$

separately on each finite element Ω_k (to the integrands f occurring in (4)), and bound the quadrature error according to

$$\left| \int_{\Omega_k} f(x)\,dx - S[f] \right| \leq \frac{\text{meas}\,(\Omega_k)}{12} \cdot \left[h_1^2 \left\| \frac{\partial^2 f}{\partial x_1^2} \right\|_{\infty,\Omega_k} + h_2^2 \left\| \frac{\partial^2 f}{\partial x_2^2} \right\|_{\infty,\Omega_k} \right]. \tag{19}$$

(Here, we have to assume that the nonlinearities F and Φ are sufficiently smooth.) To compute the required bounds for $\int_{\Omega_k} f(x)\,dx$ we must therefore

i) enclose $S[f]$,

ii) bound $\partial^2 f / \partial x_i^2$ $(i = 1, 2)$ *roughly*.

(Observe that the right-hand side of (19) can be made arbitrarily small by the choice of sufficiently small quadrature stepsizes h_1 and h_2.)

To enclose $S[f]$ (for the integrands f occuring in (4)) we use the representation (17) for $u = \omega$ (and the polynomial form of the basis functions φ_i on Ω_k) to compute enclosures for ω and $\Delta\omega$ at each quadrature point $\left(x_1^{(j)}, x_2^{(k)}\right)$. Supposing that interval-evaluators for F and Φ are available, we can therefore enclose $f\left(x_1^{(j)}, x_2^{(k)}\right)$ and, via (18), $S[f]$.

To bound $\partial^2 f / \partial x_i^2$ (roughly), we first compute bounds for the x_i-derivatives (up to the second order) of ω and $\Delta\omega$, using a two-dimensional version of a theorem in [4] which reduces the calculation of bounds for a polynomial (on a compact set) to its evaluation at finitely many points. Next, we calculate all derivatives of $F(x_1, x_2, y, \lambda)$ and $\Phi(x_1, x_2, y, \lambda)$ up to the second order by hand (here, automatic differentiation techniques will certainly facilitate the algorithm) and compute rough bounds for them on $\bar{\Omega}_k \times [\underline{\omega}, \bar{\omega}] \times \{\tilde{\lambda}\}$, with $[\underline{\omega}, \bar{\omega}]$ denoting the enclosure for ω on $\bar{\Omega}_k$ obtained before. Now it is a simple process to compute rough bounds for $\partial^2 f/\partial x_i^2$.

All the computations needed to enclose $S[f]$ and to bound $\|\partial^2 f/\partial x_i^2\|_{\infty,\Omega_k}$ (and some more very simple operations needed to obtain the defect bounds δ_1 and δ_2) have to be carried out in *interval-arithmetic,* so that rounding errors are taken into account. In our examples, we used ACRITH-subroutines [7] resp. their analogues implemented on a T 800 Transputer System.

5 Parallel implementation

The numerical procedure for the computation of $(\omega, \tilde{\lambda})$ and δ_1, δ_2 described in the previous section contains essentially the following time-consuming parts.

1. (Approximate) computation of the integrals determining the matrix elements and the right-hand side (in each Newton-step).

2. (Approximate) solution of the linear algebraic system (in each Newton-step).

3. (Interval) computation of the integrals occurring on the left-hand sides of (4).

The most efficient way of treating Parts 1 and 3 is to compute the integrals separately on each finite element and then to sum up the results. This provides a high degree of "natural" parallelism, since the computations on the single elements may be carried out by separate processors. They all work on identical tasks, and there is no need for data processing during the computation of the integrals. Thus, the efficiency of the procedure is 1 in this part, if the number of the elements is a multiple of the number of processors available.

Furthermore, this parallelization is really relevant: on a *serial* IBM 4381 machine, Parts 1 and 2 needed approximately the same CPU time; Part 3 was even much more time-consuming since we had to choose many quadrature points here in order to reduce the bound for the quadrature error in (19).

We implemented the parallel algorithm on the T 800 Transputer System of the Mathematics Institute of the Cologne University. This system provides 32 transputers which we used in a ring topology. One transputer serves as a host which distributes and collects date before and after the computation of the integrals.

Since in Part 3 we need *interval-arithmetic*, we use the package T 800 BAR which was developed in joint work of the Universities of Karlsruhe and Cologne, and which provides interval-arithmetical routines with the same syntax as in ACRITH [7].

The approximative solution of the linear algebraic systems (Part 2) is carried out, in our procedure, in the "classical" serial way (on *one* transputer) even if parallelization would be possible. The same is true for the computation of the eigenvalue bounds needed to calculate the constant K_0 via (13). Thus, further parallelization remains to be done.

6 A numerical example

To test our method we treated the example

$$-\Delta U = \lambda e^U \text{ on } \Omega := (0,1)^2, \qquad U = 0 \text{ on } \partial\Omega. \tag{20}$$

It is easy to see that, for $\lambda \neq 0$, exact solutions of this problem are not C_2-smooth in the corners of $\bar{\Omega}$. Since these corner singularities cannot be represented by a finite element approximation, we *transformed* (20) into a problem with smoother solutions, to which we applied our method. See [18] for the details of this transformation.

The following figure shows a bifurcation diagram for the *original* problem (20) which we obtained from several approximate solutions along the branch, and interpolation in between. This figure indicates the presence of a turning point at $\lambda^* \approx 6.808$, in a neighborhood of which our former "direct" existence and enclosure method cannot be applied (see [18]). Moreover, this figure indicates that the expression $\int_\Omega \exp(U_\lambda)\,dx$ may be expected to be monotonic along the conjectured solution branch (U_λ), so that

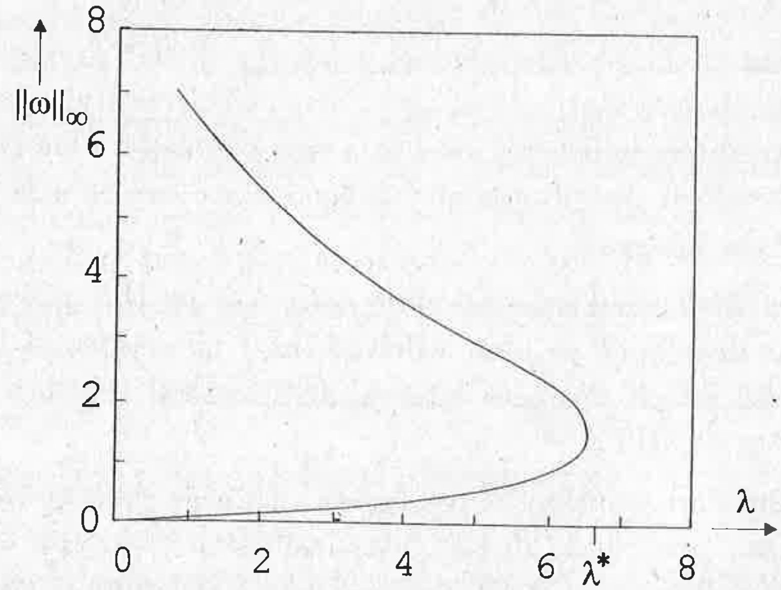

Figure 1: Bifurcation diagram for problem (20)

we hope to be successful in removing the turning point with the additional equation (compare (3))

$$-\int_{\Omega} e^{U(x)}\,dx = \mu. \tag{21}$$

It should be added that the augmented problem (20), (21) has to be *scaled*, in order to equilibrate the "main" part L and the "crossover" part ψ of the linear operator $\mathcal{L}$ defined in Section 2. Details of this scaling process can be found in [20].

We applied our existence and enclosure method with 8×8 finite elements to (the transformed and scaled version of) the augmented problem (20), (21). All practical work connected with the transputer system was done by H. Becker in the context of his diploma thesis [3].

For several selected values of the independent parameter μ, the following table shows the (rescaled and retransformed) computed values for $\omega\left(\frac{1}{2},\frac{1}{2}\right) = \|\omega\|_\infty$ and $\tilde{\lambda}$, the defect bound δ_1 (see (4); δ_2 has been omitted for reasons of space-saving), the constants K_0 and K satisfying (5), and the error bounds E_1 for $\|U-\omega\|_\infty$ and E_2 for $|\lambda-\tilde{\lambda}|$ provided by our Theorem. Obviously, there is no particular behaviour of the results close to the turning point of the original problem (20); the singularity has disappeared.

μ	$\|\omega\|_\infty$	$\tilde{\lambda}$	δ_1	K_0	K	E_1	E_2
−0.375	0.850	6.155	0.261E−02	0.0137	0.359	0.936E−03	0.427E−02
−0.400	0.985	6.465	0.312E−02	0.0132	0.366	0.114E−02	0.494E−02
−0.425	1.112	6.656	0.356E−02	0.0130	0.377	0.135E−02	0.554E−02
−0.450	1.231	6.761	0.395E−02	0.0129	0.390	0.155E−02	0.609E−02
−0.475	1.344	6.804	0.430E−02	0.0128	0.406	0.175E−02	0.660E−02
−0.500	1.451	6.802	0.459E−02	0.0129	0.423	0.194E−02	0.706E−02
−0.600	1.831	6.546	0.538E−02	0.0132	0.505	0.272E−02	0.852E−02
−0.700	2.151	6.135	0.581E−02	0.0137	0.599	0.349E−02	0.955E−02
−0.800	2.427	5.705	0.603E−02	0.0143	0.703	0.426E−02	0.104E−01
−0.900	2.670	5.299	0.616E−02	0.0148	0.814	0.504E−02	0.110E−01
−1.000	2.888	4.932	0.623E−02	0.0154	0.933	0.585E−02	0.116E−01
−1.100	3.084	4.603	0.627E−02	0.0159	1.058	0.670E−02	0.121E−01
−1.200	3.262	4.309	0.631E−02	0.0164	1.189	0.760E−02	0.126E−01

Table 1: Results for problem (20), (21)

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