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An Analytical Method of Constructing Hurwitz Interval Polynomials

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An analytical method of constructing Hurwitz interval polynomials which accompany a given Hurwitz polynomial based on a vector-matrix form of the Viète formulae is proposed.

Аналитический метод коструирования гурвицевых интервальных полиномов

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Предложен аналитический метод коструирования гурвицевых интервальных полиномов, сопровождающих заданный гурвицев многочлен, основанный на векторно-матричной записи формул Виета.

1 Introduction

When using a mathematical model of a control object, the natural question, "to what degree do the model properties correspond to those of the object described by this model?" arises. Since any mathematical model describes its object in an approximate way, only those model properties which hold under variations of its parameters within some limits can be of interest, because only in this case can we expect the object properties to coincide with those of the model. Varying the model parameters is necessary to take into account the features of the physical components of the control law applied, as well as to accomodate the changes in the object's physical parameters due to its aging.

If a property of the model motion is preserved under variations of the model parameters, then this property is called robust. It is known that asymptotic stability and instability properties are robust.

One of the most widespread mathematical models is the characteristic polynomial, which is called a Hurwitz polynomial if its roots are located in the left half of the plane of complex variable S. A Hurwitz polynomial is endowed with an asymptotic stability property that is robust.

The quantitative robustness analysis of the asymptotic stability property of the Hurwitz polynomial

$$d(S) = S^n + \sum_{i=1}^{n} \alpha_i * S^{n-i}, \qquad \alpha_i \in R^n \tag{1.1}$$

is connected with constructing the Hurwitz interval polynomial that accompanies the given polynomial (1.1)

$$D(S) = S^n + \sum_{i=1}^{n} [\underline{\alpha}_i, \overline{\alpha}_i] * S^{n-i} \tag{1.2}$$

i.e. the polynomial with coefficients that satisfy the inequality $\underline{\alpha}_i < \alpha_i < \overline{\alpha}_i$, $i = \overline{1, n}$.

The interval polynomial (1.2) provides an answer for the question: what are the variations of coefficients under which (1.1) is still a Hurwitz polynomial?

The history of this problem is discussed in detail in [1], where a method for its solution by Hermite-Biler polynomials was introduced. That method is based on an alternative formulation of V. L. Kharitonov's [2] necessary and sufficient conditions for the Hurwitzness of an interval polynomial (1.2) in the form of the Hurwitzness of the following four polynomials:

$$
\begin{aligned}
d_1(S) &= \overline{\alpha}_n + \overline{\alpha}_{n-1}S + \underline{\alpha}_{n-2}S^2 + \underline{\alpha}_{n-3}S^3 + \overline{\alpha}_{n-4}S^4 + \ldots, \qquad (1.1.3)\\
d_2(S) &= \overline{\alpha}_n + \underline{\alpha}_{n-1}S + \underline{\alpha}_{n-2}S^2 + \overline{\alpha}_{n-3}S^3 + \overline{\alpha}_{n-4}S^4 + \ldots,\\
d_3(S) &= \underline{\alpha}_n + \underline{\alpha}_{n-1}S + \overline{\alpha}_{n-2}S^2 + \overline{\alpha}_{n-3}S^3 + \underline{\alpha}_{n-4}S^4 + \ldots,\\
d_4(S) &= \underline{\alpha}_n + \overline{\alpha}_{n-1}S + \overline{\alpha}_{n-2}S^2 + \underline{\alpha}_{n-3}S^3 + \underline{\alpha}_{n-4}S^4 + \ldots
\end{aligned}
$$

To clarify the essence of the proposed method, we will present this formulation as in [1, 3].

Hermite-Biler polynomials are related to the polynomial (1.1) by the equality

$$
d(S) = G(\alpha_n, \alpha_{n-2}, \alpha_{n-4}, \ldots) + S * H(\alpha_{n-1}, \alpha_{n-3}, \alpha_{n-5}, \ldots) \qquad (1.4)
$$

where the following symbols are introduced ($z = S^2$):

$$
\begin{aligned}
G(\alpha_n, \alpha_{n-2}, \alpha_{n-4}, \ldots) &= \alpha_n + \alpha_{n-2}S^2 + \alpha_{n-4}S^4 + \cdots \qquad (1.1.5)\\
&= G(z) = \alpha_1^{n-2[\frac{n}{2}]} * \varphi_G(z), \quad m = \left[\frac{n}{2}\right],
\end{aligned}
$$

$$
\varphi_G(z) = \prod_{i=1}^{m}(z - \mu_i) = z^m + \beta_1^G z^{m-1} + \beta_2^G z^{m-2} + \beta_3^G z^{m-3} + \beta_4^G z^{m-4} + \cdots,
$$

$$
\begin{aligned}
H(\alpha_{n-1}, \alpha_{n-3}, \alpha_{n-5}, \ldots) &= \alpha_{n-1} + \alpha_{n-3}S^2 + \alpha_{n-5}S^4 + \cdots\\
&= H(z) = \alpha_1^{n-1-2\left[\frac{n-1}{2}\right]} * \varphi_H(z), \quad l = \left[\frac{n-1}{2}\right],
\end{aligned}
$$

$$
\varphi_H(z) = \prod_{i=1}^{l}(z - \eta_i) = z^l + \beta_1^H z^{l-1} + \beta_2^H z^{l-2} + \beta_3^H z^{l-3} + \beta_4^H z^{l-4} + \cdots
$$

where [...] denotes the operation of extracting the integer part of a real number, and φ_G, φ_H are the reduced polynomials that accompany the Hermite-Biler polynomials G and H.

In terms of Hermite-Biler polynomials, Hurwitzness of the polynomial $d(S)$ with $\alpha_1 > 0$ means that the roots of the corresponding reduced Hermite-Biler polynomials φ_G, φ_H are negative, real, and intermittent, i.e. they satisfy the inequalities

$$0 > \mu_1 > \eta_1 > \mu_2 > \eta_2 > \cdots > \mu_i > \eta_i > \cdots \tag{1.6}$$

The reduced Hermite-Biler polynomials that satisfy the inequalities (1.6) are said to form a positive pair. In this case a positive pair is also formed by the initial Hermite-Biler polynomials G and H.

The technique of Hermite-Biler polynomials allows computation of the roots of the corresponding positive pair of the reduced Hermite-Biler polynomials φ_G and φ_H from the characteristic Hurwitz polynomial (1.1), such that the roots satisfy the inequality (1.6). The negative semiaxis of the real axis is then split into non-intersecting intervals of possible variations of these roots according to the inequality:

$$0 > \overline{\mu}_1 > \mu_1 > \underline{\mu}_1 > \overline{\eta}_1 > \eta_1 > \underline{\eta}_1 > \cdots > \overline{\mu}_i > \mu_i > \underline{\mu}_i > \overline{\eta}_i > \eta_i > \underline{\eta}_i > \cdots \tag{1.7}$$

where $\overline{\mu}_i$, $\underline{\mu}_i$, $\overline{\eta}_i$, $\underline{\eta}_i$ are negative real numbers, to be determined, that represent the boundaries of the possible root variations of the positive pair φ_G, φ_H. These boundaries are the roots of the Hurwitz polynomials (1.3) which correspond to Hermite-Biler polynomials, and can be described by introducing the corresponding symbols:

$$\begin{aligned}
d_1(S) &= G_1(\overline{\alpha}_n, \underline{\alpha}_{n-2}, \overline{\alpha}_{n-4}, \ldots) + S * H_1(\overline{\alpha}_{n-1}, \underline{\alpha}_{n-3}, \overline{\alpha}_{n-5}, \ldots), \quad (1.1.8)\\
d_2(S) &= G_1(\overline{\alpha}_n, \underline{\alpha}_{n-2}, \overline{\alpha}_{n-4}, \ldots) + S * H_2(\underline{\alpha}_{n-1}, \overline{\alpha}_{n-3}, \underline{\alpha}_{n-5}, \ldots),\\
d_3(S) &= G_2(\underline{\alpha}_n, \overline{\alpha}_{n-2}, \underline{\alpha}_{n-4}, \ldots) + S * H_2(\underline{\alpha}_{n-1}, \overline{\alpha}_{n-3}, \underline{\alpha}_{n-5}, \ldots),\\
d_4(S) &= G_2(\underline{\alpha}_n, \overline{\alpha}_{n-2}, \underline{\alpha}_{n-4}, \ldots) + S * H_1(\overline{\alpha}_{n-1}, \underline{\alpha}_{n-3}, \overline{\alpha}_{n-5}, \ldots).
\end{aligned}$$

If we denote the reduced polynomials corresponding to the polynomials G_i, H_j $(i, j = 1, 2)$ by

$$\begin{aligned}
\Phi_{G1}(z) &= z^m + \overline{\beta}_1^G z^{m-1} + \underline{\beta}_2^G z^{m-2} + \overline{\beta}_3^G z^{m-3} + \underline{\beta}_4^G z^{m-4} + \cdots \quad (1.1.9)\\
&= (z - \underline{\mu}_m)(z - \overline{\mu}_{m-1})(z - \underline{\mu}_{m-2})(z - \overline{\mu}_{m-3}) \ldots,\\
\Phi_{G2}(z) &= z^m + \underline{\beta}_1^G z^{m-1} + \overline{\beta}_2^G z^{m-2} + \underline{\beta}_3^G z^{m-3} + \overline{\beta}_4^G z^{m-4} + \cdots\\
&= (z - \overline{\mu}_m)(z - \underline{\mu}_{m-1})(z - \overline{\mu}_{m-2})(z - \underline{\mu}_{m-3}) \ldots,
\end{aligned}$$

$$\begin{aligned}
\Phi_{H1}(z) &= z^l + \overline{\beta}_1^H z^{l-1} + \underline{\beta}_2^H z^{l-2} + \overline{\beta}_3^H z^{l-3} + \underline{\beta}_4^H z^{l-4} + \cdots \\
&= (z - \underline{\eta}_l)(z - \overline{\eta}_{l-1})(z - \underline{\eta}_{l-2})(z - \overline{\eta}_{l-3}) \ldots, \\
\Phi_{H2}(z) &= z^l + \underline{\beta}_1^H z^{l-1} + \overline{\beta}_2^H z^{l-2} + \underline{\beta}_3^H z^{l-3} + \overline{\beta}_4^H z^{l-4} + \cdots \\
&= (z - \overline{\eta}_l)(z - \underline{\eta}_{l-1})(z - \overline{\eta}_{l-2})(z - \underline{\eta}_{l-3}) \ldots
\end{aligned}$$

The necessary and sufficient conditions for Hurwitzness of the polynomial (1.2) (equivalent to V. L. Kharitonov's conditions) can be formulated as a requirement that the polynomials G_i, H_j $(i, j = 1, 2)$ defined by the expressions

$$\begin{array}{llll}
 & \text{for odd } n & & \\
G_1(z) = \underline{\alpha}_1 * \Phi_{G1}(z), & G_2(z) = \overline{\alpha}_1 * \Phi_{G2}(z), \\
H_1(z) = \Phi_{H1}(z), & H_2(z) = \Phi_{H2}(z), \\
 & \text{for even } n & & \\
G_1(z) = \Phi_{G1}(z), & G_2(z) = \Phi_{G2}(z), \\
H_1(z) = \underline{\alpha}_1 * \Phi_{H1}(z), & H_2(z) = \overline{\alpha}_1 * \Phi_{H2}(z)
\end{array} \tag{1.10}$$

constitute positive pairs.

The accompanying polynomial pair of the polynomial $\varphi_G(\varphi_H)$, that is Φ_{G1}, Φ_{G2} (Φ_{H1}, Φ_{H2}), whose coefficients, according to (1.9), are intermittent and satisfy the inequality

$$\underline{\beta}_i^G < \beta_i^G < \overline{\beta}_i^G \quad (i = \overline{1, m}), \qquad \underline{\beta}_j^H < \beta_j^H < \overline{\beta}_j^H \quad (j = \overline{1, l}) \tag{1.11}$$

was called a Lobachevsky pair in [1].

Thus, Hermite-Biler polynomials allow construction of the following algorithm to solve the problem being discussed:

1) Form the Hermite-Biler polynomials G and H associated with the given characteristic Hurwitz polynomial $d(S)$;

2) Form the reduced Hermite-Biler polynomials φ_G, φ_H associated with the polynomials G and H;

3) Find the roots of the reduced Hermite-Biler polynomials φ_G, φ_H;

4) Split the negative semiaxis of the real axis into non-intersecting intervals of possible variations of the roots computed in Step 3 according to the inequalities:

for odd n $(l = m)$

$$\begin{aligned} 0 &> \overline{\mu}_1 > \mu_1 > \underline{\mu}_1 > \overline{\eta}_1 > \eta_1 > \underline{\eta}_1 > \cdots \\ &> \overline{\mu}_i > \mu_i > \underline{\mu}_i > \overline{\eta}_i > \eta_i > \underline{\eta}_i > \cdots \\ &> \overline{\mu}_m > \mu_m > \underline{\mu}_m > \overline{\eta}_m > \eta_m > \underline{\eta}_m, \end{aligned} \tag{1.1.12}$$

for even n $(l = m - 1)$

$$\begin{aligned} 0 &> \overline{\mu}_1 > \mu_1 > \underline{\mu}_1 > \overline{\eta}_1 > \eta_1 > \underline{\eta}_1 > \cdots \\ &> \overline{\mu}_i > \mu_i > \underline{\mu}_i > \overline{\eta}_i > \eta_i > \underline{\eta}_i > \cdots \\ &> \overline{\mu}_{i+1} > \mu_{i+1} > \underline{\mu}_{i+1} > \cdots \\ &> \overline{\eta}_{m-1} > \eta_{m-1} > \underline{\eta}_{m-1} > \overline{\mu}_m > \mu_m > \underline{\mu}_m \end{aligned} \tag{1.1.13}$$

under the condition that the reduced Hermite-Biler polynomials Φ_{G1}, Φ_{G2}, Φ_{H1}, Φ_{H2} associated with the polynomials φ_G, φ_H constitute Lobachevsky pairs satisfying the inequalities (1.11);

5) Choose upper and lower bounds of the coefficient α_1 to try to satisfy the conditions following from (1.10):

for odd n

$$\begin{aligned} \underline{\alpha}_1 &< \min[\alpha_1, \alpha_5/\underline{\beta}_2^G, \alpha_9/\underline{\beta}_4^G, \alpha_{13}/\underline{\beta}_6^G, \dots], \\ \underline{\alpha}_1 &> \max[\alpha_3/\overline{\beta}_1^G, \alpha_7/\overline{\beta}_3^G, \alpha_{11}/\overline{\beta}_5^G, \dots], \\ \overline{\alpha}_1 &> \max[\alpha_1, \alpha_5/\overline{\beta}_2^G, \alpha_9/\overline{\beta}_4^G, \alpha_{13}/\overline{\beta}_6^G, \dots], \\ \overline{\alpha}_1 &< \min[\alpha_3/\underline{\beta}_1^G, \alpha_7/\underline{\beta}_3^G, \alpha_{11}/\underline{\beta}_5^G, \dots], \end{aligned} \tag{1.1.14}$$

for even n

$$\begin{aligned} \underline{\alpha}_1 &< \min[\alpha_1, \alpha_5/\underline{\beta}_2^H, \alpha_9/\underline{\beta}_4^H, \alpha_{13}/\underline{\beta}_6^H, \dots], \\ \underline{\alpha}_1 &> \max[\alpha_3/\overline{\beta}_1^H, \alpha_7/\overline{\beta}_3^H, \alpha_{11}/\overline{\beta}_5^H, \dots], \\ \overline{\alpha}_1 &> \max[\alpha_1, \alpha_5/\overline{\beta}_2^H, \alpha_9/\overline{\beta}_4^H, \alpha_{13}/\overline{\beta}_6^H, \dots], \\ \overline{\alpha}_1 &< \min[\alpha_3/\underline{\beta}_1^H, \alpha_7/\underline{\beta}_3^H, \alpha_{11}/\underline{\beta}_5^H, \dots] \end{aligned} \tag{1.1.15}$$

6) Form the Hurwitz interval polynomial (1.2) according to (1.10).

Step 4 of the above algorithm (i.e. splitting the negative semiaxis into non-intersecting intervals of possible variations of the reduced Hermite-Biler polynomial roots and constructing the Lobachevsky pairs associated with these polynomials from the bounds so obtained) is not formalized. Unfortunately, the recursive procedure proposed for this task in [3] is very time-consuming; its nature is similar to mere exhaustive search.

Thus, formalization of the procedure of constructing Lobachevsky pairs by arbitrary splitting of the negative semiaxis into intervals of possible root variations is the main purpose of this paper.

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2 The vector-matrix form of the Viète formulae and the main result

We propose an analytical method of constructing Hurwitz interval polynomials based on the following special representation of the vector-matrix form of the Viète formulae.

It is well known that for the polynomial

$$f(s) = s^m + a_1 s^{m-1} + a_2 s^{m-2} + \cdots + a_{m-1} s + a_m = \prod_{i=1}^{m} (s - x_i) \qquad (2.1)$$

these formulae which relate its roots to its coefficients can be written as:

$$\begin{aligned} a_1 &= -x_1 - x_2 - x_3 - x_4 - \cdots - x_{m-1} - x_m = -\sum_{i=1}^{m} x_i, \qquad (2.2.2) \\ a_2 &= x_1(x_2 + x_3 + x_4 + \cdots + x_{m-1} + x_m) + x_2(x_3 + x_4 + \cdots \\ &\quad + x_{m-1} + x_m) + \cdots + x_{m-2}(x_{m-1} + x_m) + x_{m-1}x_m \\ &= \sum_{\substack{i_1=1 \\ i_2=1 \\ i_1<i_2}}^{m} x_{i_1} x_{i_2}, \\ a_3 &= -x_1x_2(x_3 + x_4 + \cdots + x_{m-1} + x_m) - x_1x_3(x_4 + x_5 + \cdots \\ &\quad + x_{m-1} + x_m) - \cdots - x_1x_{m-1}x_m - x_2x_3(x_4 + x_5 + \cdots \end{aligned}$$

$$
\begin{aligned}
& +x_{m-1}+x_m) - \cdots - x_2x_{m-1}x_m - \cdots - x_{m-2}x_{m-1}x_m \\
&= -\sum_{\substack{i_1=1\\ i_2=2\\ i_3=3\\ i_1<i_2<i_3}}^{m} x_{i_1}x_{i_2}x_{i_3},
\end{aligned}
$$

$\ldots$

$$
\begin{aligned}
a_{m-1} &= (-1)^{m-1}[x_1(x_2x_3\ldots x_{m-2}x_{m-1} + x_2x_3\cdots x_{m-2}x_m) \\
&\quad +x_2(x_3x_4\ldots x_m)] \\
&= (-1)^{m-1} \sum_{\substack{i_1=1\\ \cdots \\ i_{m-1}=m-1\\ i_1<\cdots<i_{m-1}}}^{m} x_{i_1}x_{i_2}\ldots x_{i_{m-1}}, \\
a_m &= (-1)^m x_1[x_2x_3\ldots x_{m-1}x_m].
\end{aligned}
$$

Hence, introducing the symbols

$$a = \text{colon}[a_1, a_2, \ldots, a_m], \tag{2.3}$$

$$x = \text{colon}[x_1, x_2, \ldots, x_m],$$

$$
W = \begin{vmatrix}
-1 & -1 & -1 & -1 & \cdots & -1 & -1 \\
w_{21} & w_{22} & w_{23} & w_{24} & \cdots & w_{2,m-1} & 0 \\
w_{31} & w_{32} & w_{33} & w_{34} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
w_{m-i,1} & w_{m-i,2} & 0 & 0 & \cdots & 0 & 0 \\
w_{m,1} & 0 & 0 & 0 & \cdots & 0 & 0
\end{vmatrix},
$$

$$w_{ij} = (-1)^i \sum_{\substack{i_1=j+1\\ \cdots \\ i_{i-1}=i\\ i_1<\cdots<i_{i-1}}}^{m} x_{i_1}\ldots x_{i_{i-1}}, \qquad (i,j=\overline{2,m-1}),$$

$$\ldots$$

$$w_{m,1} = (-1)^m x_2\ldots x_m$$

we get

$$a = W(x_2,\ldots,x_m)x. \tag{2.4}$$

We can see from (2.3) that the matrix W does not depend on the root x_1, and provided that there are no zero roots it is always invertible, i.e.,

$$x = W^{-1}a. \tag{2.5}$$

Let us demonstrate use of the formula obtained in constructing the Lobachevsky pair associated with the polymonial φ_G. As shown in [1], the relationship between the coefficients and the roots of this pair is expressed by the inequalities:

$$\begin{aligned} \underline{\beta}_i^G(\overline{\mu}_m, \underline{\mu}_{m-1}, \overline{\mu}_{m-2}, \underline{\mu}_{m-3}, \ldots) &< \beta_i^G \\ &< \overline{\beta}_i^G(\underline{\mu}_m, \overline{\mu}_{m-1}, \underline{\mu}_{m-2}, \overline{\mu}_{m-3}, \ldots) \\ &(i = 1, 3, 5, \ldots), \\ -\overline{\beta}_j^G(\overline{\mu}_m, \underline{\mu}_{m-1}, \overline{\mu}_{m-2}, \underline{\mu}_{m-3}, \ldots) &< -\beta_j^G \\ &< -\underline{\beta}_j^G(\underline{\mu}_m, \overline{\mu}_{m-1}, \underline{\mu}_{m-2}, \overline{\mu}_{m-3}, \ldots) \\ &(j = 2, 4, 6, \ldots). \end{aligned} \tag{2.2.6}$$

Using (2.4), we can write the inequalities (2.6) that define the Lobachevsky pair in a vector-matrix form. For this purpose we introduce the following symbols:

$$\begin{aligned} B_G &= \text{colon}[\beta_1^G, -\beta_2^G, \beta_3^G, -\beta_4^G, \ldots], \\ \overline{B}_G^* &= \text{colon}[\overline{\beta}_1^G, -\underline{\beta}_2^G, \overline{\beta}_3^G, -\underline{\beta}_4^G, \ldots], \\ \underline{B}_G^* &= \text{colon}[\underline{\beta}_1^G, -\overline{\beta}_2^G, \underline{\beta}_3^G, -\overline{\beta}_4^G, \ldots], \\ X &= \text{colon}[\ldots, \mu_{m-3}, \mu_{m-2}, \mu_{m-1}, \mu_m], \\ \overline{X} &= \text{colon}[\ldots, \underline{\mu}_{m-3}, \overline{\mu}_{m-2}, \underline{\mu}_{m-1}, \overline{\mu}_m], \\ \underline{X} &= \text{colon}[\ldots, \overline{\mu}_{m-3}, \underline{\mu}_{m-2}, \overline{\mu}_{m-1}, \underline{\mu}_m], \\ W^* &= \begin{vmatrix} -1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ -w_{21} & -w_{22} & -w_{23} & -w_{24} & \cdots & -w_{2,m-1} & 0 \\ w_{31} & w_{32} & w_{33} & w_{34} & \cdots & 0 & 0 \\ -w_{41} & -w_{42} & -w_{43} & -w_{44} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{vmatrix}. \end{aligned} \tag{2.2.7}$$

We can then write:

$$\underline{B}_G^* = W^*(\overline{X})\overline{X} < B_G = W^*(X)X < \overline{B}_G^* = W^*(\underline{X})\underline{X}. \tag{2.8}$$

The inequality (2.8) allows finding, from arbitrary bounds of possible root variations μ_i $(i = \overline{2, m})$, the bounds of possible variations of the root μ_1, i.e. $\underline{\mu}_1$, $\overline{\mu}_1$, within which the polynomials Φ_{G1}, Φ_{G2} form a Lobachevsky pair. Moreover, to find these bounds, only the lower line of the scalar form (2.8) is needed, which is the main result of the paper.

Provided the initial characteristic polynomial (1.1) has an even degree n, we can assume the bounds of possible root variations η_i $(i = \overline{1, l = m-1})$ to be arbitrarily close (but not equal) to the respective root bounds μ_i $(i = \overline{1, m})$, and then construct the second Lobachevsky pair from (2.8).

3 The algorithm for an odd power of the polynomial (1.1)

If the initial characteristic polynomial (1.1) has an odd degree n, the construction of the Lobachevsky pair Φ_{G1}, Φ_{G2} is performed as described in Section 2 of this paper. It is obvious from the inequality (1.12) that if the bounds of possible root variations η_i $(i = \overline{1, m})$ are arbitrarily close to the corresponding bounds of possible root variations $\mu_i (i = \overline{1, m})$, we have to choose only the root η_m.

The analytical choice of this root requires that vector-matrix form of Viète formulae that allows the matrix (2.4) to be independent of the root x_m. This form is attained by introducing the following symbols:

$$a = \text{colon}[a_1, a_2, \dots, a_m], \tag{3.1}$$

$$x = \text{colon}[x_1, x_2, \dots, x_m],$$

$$F = \begin{vmatrix} -1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ 0 & f_{22} & f_{23} & f_{24} & \cdots & f_{2,m-1} & f_{2,m} \\ 0 & 0 & f_{33} & f_{34} & \cdots & f_{3,m-1} & f_{3,m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & f_{m-1,m-1} & f_{m-1,m} \\ 0 & 0 & 0 & 0 & \cdots & 0 & f_{m,m} \end{vmatrix},$$

$$f_{ij} = (-1)^i \sum_{\substack{i_1=1 \\ \cdots \\ i_{i-1}=i-1 \\ i_1<\cdots<i_{i-1}}}^{j-1} x_{i_1} \dots x_{i_{i-1}}, \qquad (i, j = \overline{2, m-1}),$$

$$\cdots$$

$$f_{m,m} = (-1)^m x_1 \dots x_{m-1}.$$

Thus we can write:

$$a = F(x_1, \ldots, x_{m-1})x. \tag{3.2}$$

Analogously to the previous section of the paper, we introduce the symbols

$$\begin{aligned}
B_H &= \text{colon}[\beta_1^H, -\beta_2^H, \beta_3^H, -\beta_4^H, \ldots], \qquad (3.3.3)\\
\overline{B}_H^* &= \text{colon}[\overline{\beta}_1^H, -\underline{\beta}_2^H, \overline{\beta}_3^H, -\underline{\beta}_4^H, \ldots],\\
\underline{B}_H^* &= \text{colon}[\underline{\beta}_1^H, -\overline{\beta}_2^H, \underline{\beta}_3^H, -\overline{\beta}_4^H, \ldots],\\
X &= \text{colon}[\ldots, \eta_{m-3}, \eta_{m-2}, \eta_{m-1}, \eta_m],\\
\overline{X} &= \text{colon}[\ldots, \underline{\eta}_{m-3}, \overline{\eta}_{m-2}, \underline{\eta}_{m-1}, \overline{\eta}_m],\\
\underline{X} &= \text{colon}[\ldots, \overline{\eta}_{m-3}, \underline{\eta}_{m-2}, \overline{\eta}_{m-1}, \underline{\eta}_m],\\
F^* &= \begin{vmatrix} -1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ 0 & -f_{22} & -f_{23} & -f_{24} & \cdots & -f_{2,m-1} & -f_{2,m} \\ 0 & 0 & f_{33} & f_{34} & \cdots & f_{3,m-1} & f_{3,m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{vmatrix}.
\end{aligned}$$

for the Lobachevsky pair Φ_{H1}, Φ_{H2} from (3.2). Thus we get an inequality for determining the root to be found and the coefficients of the Lobachevsky pair being constructed:

$$\underline{B}_H^* = F^*(\overline{X})\overline{X} < B_H = F^*(X)X < \overline{B}_H^* = F^*(\underline{X})\underline{X}. \tag{3.4}$$

As in the previous section of the paper, we use the lower line of the scalar form to find the bounds of this root.

4 Example

Let us demonstrate the efficiency of the method presented by the following example:

$$d(S) = S^5 + 2S^4 + 10S^3 + 12S^2 + 16S + 10.$$

From (1.4) we obtain

$$d(S) = G(S) + SH(S)$$

where $G(S) = 2S^4 + 12S^2 + 10$ and $H(S) = S^4 + 10S^2 + 16$ are the Hermite-Biler polynomials. Substituting $S^2 = Z$, we get

$$G(Z) = 2Z^2 + 12Z + 10, \quad H(Z) = Z^2 + 10Z + 16$$

and the accompanying reduced polynomials are

$$\varphi_G(Z) = Z^2 + 6Z + 5, \quad \varphi_H(Z) = H(Z) = Z^2 + 10Z + 16.$$

The polynomial roots are as follows: for $\varphi_G(Z)$, the roots are $\mu_1 = -1$, $\mu_2 = -5$; for $\varphi_H(Z)$, the roots are $\eta_1 = -2$, $\eta_2 = -8$.

Thus, the roots satisfy the inequality (1.7).

Now we can determine the upper and lower root bounds for μ_1 and μ_2, i.e. $\overline{\mu}_1$, $\underline{\mu}_1$, $\overline{\mu}_2$, $\underline{\mu}_2$.

Assuming that $\underline{\mu}_2 = -6.5$, $\overline{\mu}_2 = -3.1$, we construct the inequality (2.8). In our case it looks like this:

$$\begin{aligned} \left|\begin{matrix} \underline{\beta}_1^G \\ -\overline{\beta}_2^G \end{matrix}\right| &= \left|\begin{matrix} -1 & -1 \\ -w_{21}(\overline{\mu}_2) & 0 \end{matrix}\right| \left|\begin{matrix} \underline{\mu}_1 \\ \overline{\mu}_2 \end{matrix}\right| = \left|\begin{matrix} -1 & -1 \\ 3.1 & 0 \end{matrix}\right| \left|\begin{matrix} \underline{\mu}_1 \\ -3.1 \end{matrix}\right| \\ &< \left|\begin{matrix} \beta_1^G \\ -\beta_2^G \end{matrix}\right| = \left|\begin{matrix} -1 & -1 \\ -w_{21}(\mu_2) & 0 \end{matrix}\right| \left|\begin{matrix} \mu_1 \\ \mu_2 \end{matrix}\right| = \left|\begin{matrix} -1 & -1 \\ 5 & 0 \end{matrix}\right| \left|\begin{matrix} -1 \\ -5 \end{matrix}\right| = \left|\begin{matrix} 6 \\ -5 \end{matrix}\right| \\ &< \left|\begin{matrix} \overline{\beta}_1^G \\ -\underline{\beta}_2^G \end{matrix}\right| = \left|\begin{matrix} -1 & -1 \\ -w_{21}(\underline{\mu}_2) & 0 \end{matrix}\right| \left|\begin{matrix} \overline{\mu}_1 \\ \underline{\mu}_2 \end{matrix}\right| = \left|\begin{matrix} -1 & -1 \\ 6.5 & 0 \end{matrix}\right| \left|\begin{matrix} \overline{\mu}_1 \\ -6.5 \end{matrix}\right|. \end{aligned} \tag{4.4.1}$$

To find the desired bounds, let us write out the lower line of the inequality:

$$3.1\underline{\mu}_1 < -5 < 6.5\overline{\mu}_1$$

and hence

$$\overline{\mu}_1 > -5/6.5 = -0.7692307, \quad \underline{\mu}_1 < -5/3.1 = -1.6129032.$$

Thus, we can assume $\overline{\mu}_1 = -0.1$, $\underline{\mu}_1 = -1.7$, which on the basis of (4.1) allows determination of the Lobachevsky pair coefficients Φ_{G1}, Φ_{G2} by the equalities

$$\overline{\beta}_1^G = 6.6; \quad \underline{\beta}_1^G = 4.8; \quad \overline{\beta}_2^G = 5.27; \quad \underline{\beta}_2^G = 0.65.$$

Since the power of the characteristic polynomial $d(S)$ $n = 5$ is odd, we will use the results of Section 3 to construct the Lobachevsky pair.

Let us assume $\overline{\eta}_1 = -1.701$, $\underline{\eta}_1 = -3.099$, $\overline{\eta}_2 = -6.501$, i.e. close to the corresponding roots μ_i with an accuracy of 0.001 (which according to [1] is the "unimprovability" interval of the polynomial constructed). Let us write down the inequality (3.4) for our case:

$$\begin{aligned}\begin{vmatrix}\underline{\beta}_1^H\\ -\overline{\beta}_2^H\end{vmatrix} &= \begin{vmatrix}-1 & -1\\ 0 & -f_{22}(\underline{\eta}_1)\end{vmatrix}\begin{vmatrix}\underline{\eta}_1\\ \overline{\eta}_2\end{vmatrix} = \begin{vmatrix}-1 & 1\\ 0 & 3.099\end{vmatrix}\begin{vmatrix}-3.099\\ \overline{\eta}_2\end{vmatrix}\\ &< \begin{vmatrix}\beta_1^H\\ -\beta_2^H\end{vmatrix} = \begin{vmatrix}-1 & -1\\ 0 & -f_{22}(\eta_1)\end{vmatrix}\begin{vmatrix}\eta_1\\ \eta_2\end{vmatrix} = \begin{vmatrix}-1 & -1\\ 0 & 2\end{vmatrix}\begin{vmatrix}-2\\ -8\end{vmatrix} = \begin{vmatrix}10\\ -16\end{vmatrix}\\ &< \begin{vmatrix}\overline{\beta}_1^H\\ -\underline{\beta}_2^H\end{vmatrix} = \begin{vmatrix}-1 & -1\\ 0 & -f_{22}(\overline{\eta}_1)\end{vmatrix}\begin{vmatrix}\overline{\eta}_1\\ \underline{\eta}_2\end{vmatrix} = \begin{vmatrix}-1 & -1\\ 0 & 1.751\end{vmatrix}\begin{vmatrix}-1.701\\ \underline{\eta}_2\end{vmatrix}.\end{aligned} \tag{4.4.2}$$

To find the desired bounds, we write down the lower line of the inequality

$$3.099\overline{\eta}_2 < -16 < 1.701\underline{\eta}_2$$

and hence

$$\overline{\eta}_2 < -16/3.099 = -5.162956, \quad \underline{\eta}_2 > -16/1.701 = -9.4062$$

i.e., the chosen value $\overline{\eta}_2 = -6.501 < -5.162956$. Let us assume that $\underline{\eta}_2 = -9.4 > -9.4062$. Then, from (4.2), we get

$$\underline{\beta}_1^H = 9.6; \quad \overline{\beta}_1^H = 11.101; \quad \underline{\beta}_2^H = 15.9894; \quad \overline{\beta}_2^H = 20.1466.$$

We then choose the bounds of the coefficient α_1 according to (1.14):

$$\begin{aligned}\underline{\alpha}_1 < \min[\alpha_1, \alpha_5/\underline{\beta}_2^G] &= \min[2; 10/0.65 = 15.384615] = 2,\\ \underline{\alpha}_1 > \alpha_3/\overline{\beta}_1^G &= 12/6.6 = 1.8181818,\\ \overline{\alpha}_1 > \max[\alpha_1, \alpha_5/\overline{\beta}_2^G] &= \max[2; 10/5.27 = 1.8975332] = 2,\\ \overline{\alpha}_1 < \alpha_3/\underline{\beta}_1^G &= 12/4.8 = 2.5\end{aligned}$$

assuming $\underline{\alpha}_1 = 1.82$; $\overline{\alpha}_1 = 2.49$.

The final interval polynomial will be as follows:

$$\begin{aligned}D(S) = {} & S^5 + [1.82; 2.49]S^4 + [9.6; 11.101]S^3 + [11.952; 12.0012]S^2\\ & + [15.9894; 20,1466]S + [5.915; 13.1223].\end{aligned}$$

5 Discussion of the results

Summarizing the above, we can conclude that the analytical method proposed allows construction, for any given Hurwitz characteristic polynomial (1.1), the associated Hurwitz interval polynomial (1.2). In the process of solving this problem, we *cannot* arbitrarily choose the following root bounds of Hermite-Biler polynomials:

- $\underline{\mu}_1$, $\overline{\mu}_1$ for an even power of (1.1);
- $\underline{\mu}_1$, $\overline{\mu}_1$, $\underline{\eta}_m$ for an odd power of (1.1);
- α_1 in both above cases.

It should be noted that the visual geometrical interpretion of the arbitrary rule for choosing the bounds of possible variations of the other Hermite-Biler polynomial roots is an important feature of this method. Besides, as emphasized in [1], the bounds of possible variations of the coefficients of the Hurwitz polynomial (1.1) that are found from the bounds of possible root variations are "unimprovable" with respect to the distances between the bounds of possible variations of neighbour roots, chosen to satisfy the condition that roots should be intermittent:

$$\epsilon_{1i} = |\overline{\eta}_i - \underline{\mu}_i| \quad (i = \overline{1, m}), \quad \epsilon_{2i} = |\overline{\mu}_{i+1} - \underline{\eta}_i| \quad (i = \overline{1, m-1}) \qquad (5.1)$$

and thus can be chosen infinitesimal.

A large number of works concerning the subject of this paper (see, e.g., [5–7]) are dedicated to estimating the stability radius in the space of coefficients of the polynomial (1.1) for different norms, assuming that in addition to the coefficients of (1.1) the scales of possible errors for each coefficient are given. It is easy to see that the "greatest" interval polynomial which is obtained in this way may prove "improvable" with respect to the distances (5.1) because of the arbitrarily chosen scales of possible errors for each coefficient of (1.1).

In our method, the distances (5.1) may be infinitesimal, which is an advantage in comparison with other methods. However, our method does not reject the above approach since it allows applying reasonable scales for the given errors, but in this case the concept of a radius of robust stability loses its sense.

A problem which is closer to the subject of this paper was considered in [8], namely the problem of finding the greatest deviation for each given coefficient of the polynomial (1.1) that does not break Hurwitzness of the accompanying interval polynomial (1.2). However, in our opinion, expressing this problem in terms of Hermite-Biler polynomial roots instead of the given Hurwitz characteristic polynomial coefficients has an advantage of visual geometrical interpretion provided by the inequalities (1.12), (1.13) and equalities (5.1).

In conclusion it should be noted that the algorithm described here is easily implemented on personal computers.

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