Interval Computations
No 1, 1994

On Different Definitions of Interval Extension: Problems of Teaching

Grigory G. Menshikov

О различных определениях интервального расширения: проблемы преподавания

Г. Г. Меньшиков

Several definitions for the *interval extension* (IE) $F(X)$ of a function $f(x)$ are known. To facilitate constructing and analysing interval algorithms, it is desirable that the three general requirements of IE be fulfilled:

1. Reliability in computation.
2. Compositional property (CP): if mappings $F_i(x)$ are IE's of functions $f_i(x)$ then the composition of F_i is IE for analogous composition of f_i (here x, $f(x) \in R$; X, $F(X)$ are closed intervals).
3. Capability to describe as many as possible of the known interval processes.

We consider four models of IE. Firstly, the classical definition was proposed by R. E. Moore in 1966 [1]. It supposes two conditions:

$$f(x) \subseteq F(x), \quad \text{where } f(X) = \{f(x) : x \in X\}, \tag{1}$$
$$f(x) = F(x) = F\big([x, x]\big). \tag{2}$$

Secondly, the Shokin's definition [2] supposes only (1).

This condition is very important: when it is off, interval computations do not yield guaranteed results. Thus, the condition (1) might be named "basic inclusion of interval analysis."

Thirdly, the definition from [3] supposes only (2). The basic inclusion (1) follows from (2) when mapping $F(X)$ is *inclusion monotonic* (IM).

Fourthly, the definition from [4] supposes besides (1) two following conditions:

(3) mapping $F(X)$ is IM;

(4) for each $\varepsilon > 0$ such a number $\delta > 0$ exists, that $w(X) < \delta \Rightarrow w\bigl(F(X)\bigr) < \varepsilon$, where $w(X)$ is the width of interval X.

The IE for functions of several variables are defined analogously.

We can see that condition (1) is a part of all definitions. It is the only condition of Shokin's IE, which thus can be easily implemented on computer, and its CP evidently holds.

Further, the equality (2) is not guaranteed in practical computations, since real values of f are rarely represented by computer numbers. Therefore, Moore's IE cannot be implemented on computer at all.

Furthermore, IE definition from [3] is not applicable to interval processes with non-proved inclusion monotonicity.

And finally, definition from [4] causes difficulties in investigations of computations. Indeed, δ cannot be less than minimal distance between adjacent computer numbers. Thus, finding proper δ is sometimes impossible.

Thus, the IE as defined in [1, 3, 4] is convenient only for description of purely theoretical, "paper" interval processes.

That is why I use Shokin's IE from [2] as the main concept of interval computations in my lectures. One might introduce a special name for such an extension, for example *inclusion function* [5], *weak interval extension* and *interval enclosure* [6], *interval expansion* [7], or use just *interval extension.*

For the IE in this sense, the two theorems on the compositions are true:

Theorem 1. *The above-mentioned* CP *is valid.*

Theorem 2. *If mappings* $X \to F_i(X)$ *are* IM*, then their composition is also* IM*.*

These statements are very convenient in lectures on interval computations as basic theorems.

References

[1] Moore, R. E. *Interval analysis.* Prentice-Hall, Englewood Cliffs, N. J., 1966.

[2] Shokin, Yu. I. *Interval analysis.* Nauka, Novosibirsk, 1981 (in Russian).

[3] Kalmykov, S. A., Shokin, Yu. I., and Yuldashev, Z. Kh. *Methods of interval analysis.* Nauka, Novosibirsk, 1986 (in Russian).

[4] Bauch, H., Jahn, K. U., Oelschlagel, D., Susse, H., and Wiebigke, V. *Intervallmathematik. Theorie und Anwendungen.* BSB B.G Teubner Verlaggesellschaft, Leipzig, 1987.

[5] Ratschek, H. and Rokne, J. *New computer methods for global optimization.* Ellis Norwood, Chichester, 1988.

[6] Neumaier, A. *Interval methods for systems of equations.* Cambridge University Press, 1990.

[7] Men'shikov, G. G. *Problems of interval computations teaching: the two basic theorems on compositions.* In: "International Congress on Computer Systems and Applied Mathematics CSAM'93. St. Petersburg, July 19–23, 1993. Abstracts", pp. 97–98.

Received: September 15, 1993
Revised version: June 30, 1994

St. Petersburg State University
Faculty of Applied Mathematics
and Control Processes
Bibliotechnaya sq. 2
St. Petersburg 198904
Russia