Interval Computations
No 1, 1994

Interval Spanning Trees Problem: Solvability and Computational Complexity

Galina L. Kozina and Vitaly A. Perepelitsa

The optimization Spanning Trees Problem on graphs with interval weights is presented. The interval function is defined as the sum of interval weights of feasible spanning tree edges. The relation order introduced into set of feasible solutions generates the Pareto set which is considered as the solution of the interval problem. The questions of solvability and computational complexity are investigated by applying the multicriterial approach.

Интервальная задача о остовных деревьях: разрешимость и вычислительная сложность

Г. Л. Козина, В. А. Перепелица

Рассматривается оптимизационная задача о **остовных деревьях на графах с интервальными весами. Целевая функция представляет соой сумму интервальных весов реер, входящих в допустимое остовное дерево. Введенный на множестве допустимых решений порядок порождает паретовское множество, которое является решением интервальной задачи. Исследуются вопросы разрешимости и вычислительной сложности с помощью аппарата многокритериальности.**

For achieving adequacy of more real situation in mathematical modeling it is expedient in some cases to use the interval descriptions. By applying intervals we may account approximate nature of initial data [1–3].

For extremal problems on graphs (for example Spanning Trees Problem) the interval weights of edges may denote the resource (time, materials) consumption for network objects or technical parameters (noise immunity, electromagnetic interaction) in Computer-Aided Design, or yield capacity of crops in land tenure. The Optimization Spanning Trees Problem on graph with interval parameters was considered by A. I. Demchenko in [4].

1 Interval definition of Spanning Trees Problem

In general, let $G = (V, E)$, $|V| = n$ be a graph with set of vertices V and set of edges E. Every edge $e \in E$ is weighted by interval weight $w(e) = [w_1(e), w_2(e)]$.

Let us denote the set of all spanning trees of graph G by

$$X = \{x \mid x = (V, E_x)\}.$$

The weight of the tree x is

$$w(x) = \sum_{e \in E_x} w(e). \tag{1}$$

Obviously, $w(x) = [w_1(x), w_2(x)]$, where

$$w_i(x) = \sum_{e \in E_x} w_i(e), \qquad i = 1, 2. \tag{2}$$

The problem is to find the spanning tree of "minimum" weight [3, 4]

$$\min_{x \in X} w(x). \tag{3}$$

In a classical noninterval case the described problem is successfully solved, for example, by algorithms of R. C. Prim or J. B. Jr. Kruskal [5] and has a unique solution. In an interval case the solution of the problem generally is not unique, so we must introduce the order on the set of intervals for definition of the solutions set.

Definition 1. *The solution x is "better" than the solution y, $x \prec y$, if $w_i(x) \leq w_i(y), i = 1, 2$ and at least one of these inequalities is strict; otherwise x and y are incomparable.*

Such order generates the Pareto Set (PS) $\tilde{X} \subseteq X, \tilde{X} = \{\tilde{x}\}$, including all the Pareto optimums.

Definition 2. *The solution $\tilde{x} \in X$ is called the Pareto optimum for the problem (1), (3) if there does not exist such $x \in X$ that $x \prec \tilde{x}$.*

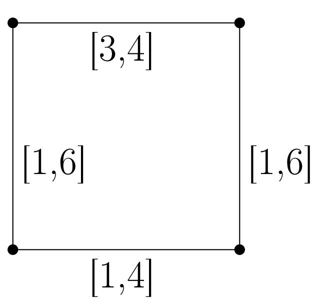
PS consists of incomparable solutions.

Two solutions x, y belong to one class of equivalence (CE) if $w(x) = w(y)$. The Complete Set of Alternatives (CSA) $\hat{X}$ is a subset of PS which contains by ones representatives of each CE.

The solution of the problem (1), (3) may be PS or CSA.

Example 1. Consider graph $G = (V, E)$ with interval weights (Fig. 1). Set of feasible solutions X (all spanning trees of graph G) consists of four trees.

According to the weights obtained we have only 3 trees in Pareto set $\tilde{X} = \{x, y, z\}$. The complete set of alternatives $\hat{X}$ is not uniquely defined. We may choose, for example, x and y as CSA.



Graph G

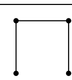
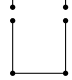
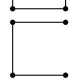
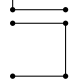
X	SFS		PS $\tilde{X}$	CSA $\hat{X}$
t		$w(t) = [5, 16]$		
x		$w(x) = [3, 16]^V$	x	x
y		$w(y) = [5, 14]^V$	y	y
z		$w(x) = [5, 14]^V$	z	

Figure 1: Set of feasible solutions, Pareto set, complete set of alternatives for graph G (example 1)

2 Reduction of interval Spanning Trees Problem to multiobjective optimization problem

Problem (1), (3) is reduced to multiobjective optimization problem with Vector Objective Function (VOF) [3, 4]

$$
\begin{aligned}
F(x) &= \{w_1(x), w_2(x)\} && (4)\\
w_i(x) &\rightarrow \min_{x \in X}, \qquad i = 1, 2, && (5)\\
w_1(x) &\leq w_2(x). && (6)
\end{aligned}
$$

The solution of the problem (4), (5), (6) is a Pareto set (or complete set of alternatives).

Definition 3. *The solution $\tilde{x} \in X$ is called Pareto optimum for problem (4), (5), (6) if there does not exist any $x \in X$ such that $F(x) \leq F(\tilde{x})$ and at least one inequality is strict.*

According to definitions 2 and 3, PS-s of the interval problem (1), (3) and the multiobjective optimization problem (4), (5), (6) coincide. (Complete set of alternatives for the problem (4), (5), (6) is defined as for the problem (1), (3).)

The condition (6) is proved to be unessential for the given problem. It directly follows from the two lemmas.

Lemma 1. *Let $X = \{x\}$ be a set of feasible solutions for a problem with the vector objective function*

$$F(x) = \{F_1(x), F_2(x)\} \tag{7}$$

and $\tilde{X}$ is a Pareto set generated by VOF (7). Then for any constant C the Pareto sets $\tilde{X}$ and $\tilde{X}_C$, generated by VOF

$$F_C(x) = \{F_1(x), F_2(x) + C\}$$

coincide.

The proof follows directly from definition 3.

Lemma 2. *Pareto sets of problems with criteria (4), (5) and (4), (5), (6) coincide.*

Proof. The problem (4), (5), (6) is a partial case of the problem (4), (5). Let's show that we may transform any problem (4), (5) into the problem (4), (5), (6). Really, let every edge $e \in E$ of the graph $G = (V, E)$ be weighed by the two weights $w_1(e)$ and $w_2(e)$. Add a constant k to the second weight $w_2(e)$ of every edge $e \in E$ such that $w_1(e) \leq w_2(e) + k$ for all $e \in E$.

The transformed problem (4), (5) has the form (4), (5), (6). Let us denote it as (4), (5), $(6)_k$. It is obvious that Pareto sets of both problems coincide. Clearly, the set of feasible solutions X will not change after the operation. At the same time for every $x \in X$ the value of the first criterion (2) $w_1^k(x)$ will not change, $w_1^k(x) = w_1(x)$, but the value of the second criterion (2) $w_2^k(x)$ will be equal to

$$w_2^k(x) = w_2(x) + (n-1)k$$

(every spanning tree $x = (V, E_x), |V| = n$, has exactly $n - 1$ edges).

Denote $C = (n-1)k$. Then using Lemma 1 we have that the Pareto sets of problems (4), (5) and (4), (5), $(6)_k$ coincide.

Hence, we may apply for the problem (1), (3) all the assertions concerning the Pareto set of the problem (4), (5).

3 Main results

3.1 Solvability by linear convolution algorithms

The problem (4), (5) is investigated by multiobjective optimization methods [6].

Substitute vector objective function (4), (5) for linear convolution of criteria

$$F^{\lambda}(x) = \lambda_1 \cdot w_1(x) + \lambda_2 \cdot w_2(x), \tag{8}$$

where

$$\lambda_1, \lambda_2 > 0, \tag{9}$$

$$\lambda_1 + \lambda_2 = 1. \tag{10}$$

Any exact algorithm optimizing linear convolution objective function in form (8) is called linear convolution algorithm (LCA) [6, 7]. It is well known that the solution $x^* \in X$ optimizing linear convolution objective function is Pareto optimum.

Thus, the problem (4), (5) is called solvable by LCA if for any $\tilde{x} \in \tilde{X}$ there exists vector λ satisfying (9), (10) such that

$$F^\lambda(\tilde{x}) = \min_{x \in X} F^\lambda(x).$$

Otherwise, the problem (4), (5) is unsolvable by LCA if there exists such instance problem with PS $\tilde{X}$ that

$$\exists \tilde{x} \in \tilde{X} \qquad F^\lambda(\tilde{x}) > \min_{x \in X} F^\lambda(x) \tag{11}$$

for all λ, satisfying (9), (10).

It is easy to see that condition (10) is not essential.

The following result is obtained in [6].

Theorem 1. *For all $n \geq 3$ the problem (4), (5) is unsolvable by LCA.*

As a proof we consider

Example 2. Let $G = (V, E)$ be graph with interval weights (Fig. 2). The set of feasible solutions X consists of three trees and Pareto set $\tilde{X}$ and complete set of alternatives $\hat{X}$ consist of 3 trees. So we have $X = \tilde{X} = \hat{X}$.

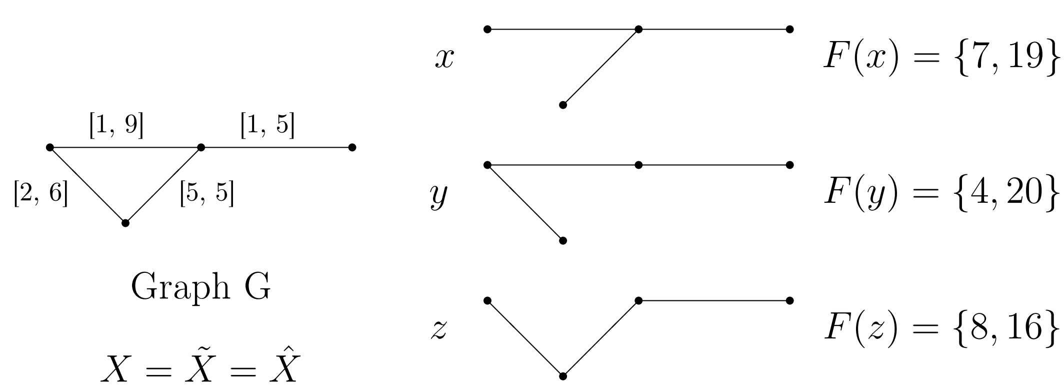


Figure 2: Set of feasible solutions, Pareto set, complete set of alternatives for graph G (example 2).

But for the solution x there does not exist any vector λ satisfying (8), (9) that

$$F^\lambda(x) = \min\{F^\lambda(x), F^\lambda(y), F^\lambda(z)\}.$$

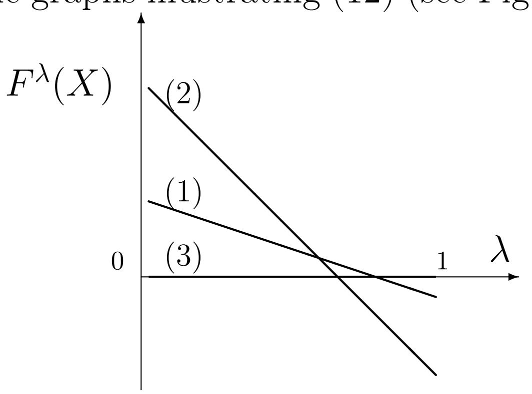
Really, unsolvability condition is written as follows (see (8), (11))

$$\begin{aligned} &\forall\lambda \in (0,1) \\ \lambda \cdot w_1(x) \ &+ \ (1-\lambda)\cdot w_2(x) \\ &> \ \min\{\lambda w_1(x) + (1-\lambda)w_2(x), \lambda w_1(y) \\ &\qquad +(1-\lambda)w_2(y), \lambda w_1(z) + (1-\lambda)w_2(z)\} \end{aligned}$$

or after substitution and transformation

$$\forall\lambda \in (0,1) \qquad 3-4\lambda > \min\{3-4\lambda, 4-8\lambda, 0\}. \tag{12}$$

The graphs illustrating (12) (see Fig. 3) show that condition (12) is fulfilled.



(1) $F^\lambda(x) \leftrightarrow 3-4\lambda$

(2) $F^\lambda(y) \leftrightarrow 4-8\lambda$

(3) $F^\lambda(z) \leftrightarrow 0$

Figure 3: Graphs for $F^\lambda(X)$

3.2 Computational complexity

We may consider the cardinality of CSA as a lower bound of Computational Complexity for its finding. If the maximal cardinality of CSA has exponential estimate then problem is intractable [8]. Investigation of complexity of problem (4), (5) has given the following results. In terms of multiobjective optimization the problem (4), (5) is complete, i.e. for any graph $G = (V, E)$ there exists such weighting by $w_i(e)$, $e \in E$, $i = 1, 2$, that $X = \tilde{X} = \hat{X}$.

Since the complete graph $G = (V, E)$ with $|V| = n$ has n^{n-2} spanning trees, computational complexity is exponential in worse. Hence we have

Theorem 2. *Interval Spanning Trees Problem (1), (3) and Multiobjective Spanning Trees Problem (4), (5) are intractable.*

3.3 Polynomial solvability

However, if the problem has such description where the interval weight of tree is not important but lower bound of weight and maximal width of the interval weights of spanning tree edges are taken into consideration, then we have more successful result.

Let's introduce the new criterion

$$d(x) = \max_{e \in E} d(e), \qquad d(e) = w_2(e) - w_1(e).$$

On the set X we consider the new VOF

$$\begin{array}{rcl} F(x) & = & \{w_1(x), d(x)\}, \\ w_1(x) & \to & \min\limits_{x \in X}, \\ d(x) & \to & \min\limits_{x \in X}. \end{array} \tag{13}$$

Solution of problem (13) (see definition 3) is complete set of alternatives.

Theorem 3 [9]. *The problem of finding CSA of Spanning Trees Problem with criteria (13) is solvable with polynomial complexity $O(n^4)$.*

The proof is constructive. In [9] the algorithm including two stages is described. The algorithm has not more than n^2 iterations and it is an algorithm of linear convolution. Every iteration yields one element of CSA.

4 Conclusion

Application of multiobjective optimization methods in interval modeling has allowed to formulate and to substantiate methodologically difficult assertions:

- principal unsufficiency of classical optimization methods used for linear convolution algorithms is assertained;
- computational complexity of interval spanning trees problem is estimated in worse and its intractability is assertained;
- the sufficient conditions of polynomial solvability is found.

References

[1] Alefeld, G. and Herzberger, J. *Introduction to interval computations.* Academic Press, New York, 1983.

[2] Kalmykov, S. A., Shokin, Yu. I., and Yuldashev Z. Kh. *Methods of interval analysis.* Nauka, Novosibirsk, 1986 (in Russian).

[3] Voshchinin, A. P. and Sotirov, G. R. *Optimization under uncertainty conditions.* Moscow Power engineering Institute, Publishing House Technika (Bulgaria), Moscow-Sofia, 1989 (in Russian).

[4] Demchenko, A. I. *Transport networks synthesis under incertainty conditions of initial information.* In: "Proceedings of Seminar on Interval Mathematics, Saratov, 1990", Saratov, 1990, pp. 10–16 (in Russian).

[5] Christofides, N. *Graph theory. An algorithmic approach.* Academic Press, New York, London, San Francisco, 1975.

[6] Emelichev, V. A. and Perepelitsa, V. A. *Complexity of vector optimization problems on graphs.* Optimization **22** (6) (1991), pp. 903–918.

[7] Bruker, P. *Discrete parameter optimization problem and essential efficient points.* Operat. Res. **16** (5) (1972), pp. 189–197.

[8] Garey, M. R. and Johnson, D. S. *Computers and intractability. A guide to the theory of NP-completeness.* Freeman and Co., San Francisco, 1979.

[9] Sergienko, I. V. and Perepelitsa, V. A. *On the problem of finding sets of alternatives in discrete multiobjective problems.* Kibernetika **5** (1987), pp. 85–93 (in Russian).

Received: July 30, 1993
Revised version: December 10, 1993

Zaporozhye State University
Zhukovsky Street, 66
330600 Zaporozhye, GSP–41
Ukraine
e-mail: `root@zgu.zaporizhzhe.ua`