Interval Computations
No 1, 1994

Verified Inclusion for Eigenvalues of Hill's Equation

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We present a method depending on matrix continued fractions and Sturm's comparison theorem to obtain verified inclusions for all eigenvalues of *Hill's* equation $u'' + (\lambda + \tilde{g})u = 0$, with an arbitrary periodic function $\tilde{g}$. As an example we treat the first-order phase locked loop equation.

Верифицированное включение собственных значений для уравнения Хилла

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Предлагается метод получения верифицированных включений всех собственных значений для уравнения Хилла: $u'' + (\lambda + \tilde{g})u = 0$, где $\tilde{g}$ — произвольная периодическая функция. Метод основан на применении матричных цепных дробей и теоремы сравнения Штурма. В качестве примера рассматривается уравнение фазово замкнутой петли первого порядка.

1 Introduction

The task of finding all eigenvalues and eigenfunctions of a boundary value problem has always been of mathematical interest. One special problem is the calculation of eigenvalues for *Hill's* equation $u'' + \big(\lambda + \tilde{g}(\varphi)\big)u = 0$, with an arbitrary periodic function $\tilde{g}(\varphi)$. This equation occurs in different problems of practical interest such as in investigation of cycle slipping of phase locked loops (PLL).

The phase locked loop (PLL) is an electrical control system which synchronizes an oscillator and a given signal in frequence and phase. If the phase error in this regulation leaves the interval $[0, 2\pi]$ the PLL slips a cycle. In many applications, for example in information transmission in satellite technique, the cycle slipping is very detrimental. To avoid unreliable and costly design of loop components an accurate calculation of the cycle slip rate is crucial.

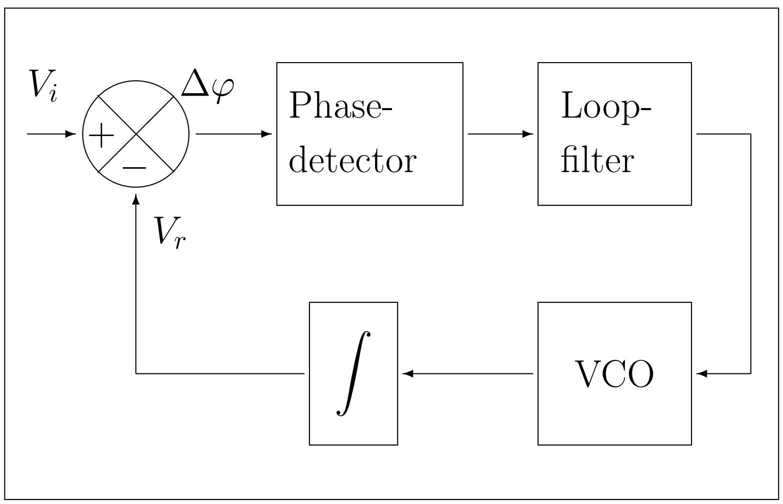

Figure 1: Basic block diagramm PLL MC14046

Studying the cycle slip rate leads to a stochastic differential equation for the phase error process $\varphi(t)$ of a first order phase locked loop (filter omitted) [2, 9, 15] with phase detector characteristic $g(\varphi)$, i.e. $\dot{\varphi}(t) = \Omega - K[Ag\big(\varphi(t)\big) + n(t)]$, where Ω is the initial frequence detuning, A, K characteristic loop constants and $n(t)$ a stationary white *Gaussian* noise process with zero mean and one-sided band limited spectral density N_0w/cps.

The corresponding *Markov* process is described by the transition probability density function $P(\varphi,\, t \,|\, \varphi_0,\, t_0)$, which satisfies a *Fokker-Planck* equation [4]. Applying the separation method to this equation we are led to the self-adjoint eigenvalue problem

$$Ly = \left\{ p\frac{\partial^2}{\partial\varphi^2} + \frac{\partial}{\partial\varphi} g(\varphi) \right\} y = -\lambda y \tag{1}$$

with a normed separation constant λ. Here $1/p := 4A/(N_0K) = A^2/(N_0B_L)$ is the signal-to-noise ratio SNR in the loop bandwith B_L.

Together with absorbing boundary conditions $P(\varphi_0 - 2\pi, t) = P(\varphi_0 + 2\pi, t) = 0$ the statistical description of cycle slipping indicates, as shown by *Meyr* and *Ascheid* [10], that the mean lifetime between cycle slips is related to the first non-zero eigenvalue $\lambda_0^{(2)}$ of the $2M\pi$-periodic problem with $M = 2$ as $E(T_s) = 1/\lambda_0^{(2)}$. Instead of absorbing the particles at the boundaries $\varphi = \varphi_0 \pm 2\pi$ one can measure the phase error modulo $2M\pi$. Thus the boundary conditions are of $2M\pi$-periodic type and one gets *Ryter's* multistable cyclic model with M attractors as in [10]. For moderate noise the transition cycle slipping rates in this model are determined by the $M-1$ smallest eigenvalues of the *Fokker-Planck* operator L. In both cases a precise determination of the eigenvalues and eigenfunctions is necessary.

The substitution $u(\varphi) := y(\varphi)\exp\big(\int_0^\varphi g(\theta)d\theta/(2p)\big)$ transforms our equation (1) to *Hill's* equation $u'' + \Big(\lambda/p + g'/(2p) - \big(g/(2p)\big)^2\Big)u = 0$. For big SNR's the eigenvalue problem is ill-conditioned, the error increases exponentially in $1/p$.

In former works we have studied the PLL equation with sinusoidal or finite trigonometric polynomials as phase detector characteristic [9]. Now we turn our attention to the general case

$$g(\varphi) = \sum_{m=1}^{\infty} (-1)^{m-1}\beta_m \sin m\varphi, \quad \beta_m \geq 0$$

which includes triangular, sawtooth and tanlock [1] characteristics.

The existence problem is treated in [8, 9]. For the 2π-periodic problem (1) $\big($and a function $g(\varphi)$ with only one extremal value in $(0,\pi)\big)$ we have a simple eigenvalue $\lambda_0^{(1)} = 0 \sim \lambda_0^{(2)}$ and in the case of $g(\varphi) = g(\pi - \varphi)$ double real eigenvalues $\lambda_\nu^{(1)}$ with $\lambda_{2\nu-1}^{(2)}, \lambda_{2\nu}^{(2)} \sim \lambda_\nu^{(1)} \sim g'(0)\nu$, ν small, $p \to 0$ and for large ν it holds $\lambda_\nu^{(1)} \sim p\,\nu^2$. Otherwise we can localize eigenvalues near

$g'(0)\mu$ and $-g'(\pi)\mu$, $\mu = 1, 2, 3, \ldots$, $\lambda_\nu^{(2)} \sim \lambda_\nu^{(1)}$, ν small, $p \to 0$ and for large ν it holds $\lambda_{2\nu-1}^{(1)}$, $\lambda_{2\nu}^{(1)} \sim p\,\nu^2$. This enables us to decide whether an interval inclusion contains one or more eigenvalues. For a more detailed discussion see [3, 6, 8].

2 The eigenvalue problem

In [9] we utilized complex continued fractions for the treatment of *Hill's* equation in the case of $g(\varphi) = \sin\varphi - \gamma$ (cf. also [12, 14]). By a generalized approach based on matrix continued fractions ([11]) we are also able to treat problem (1) with finite *Fourier* expansions

$$g(\varphi) = \sum_{m=1}^{k-1} (-1)^{m-1} \beta_m \sin m\varphi.$$

We start with the eigenvalue-problem

$$\begin{aligned} Ly &= \left\{ p\frac{\partial^2}{\partial\varphi^2} + \frac{\partial}{\partial\varphi} g(\varphi) \right\} y = -\lambda y, \qquad (2) \\ y(-M\pi) &= y(M\pi), \quad y'(-M\pi) = y'(M\pi), \quad \text{for one } M \in \mathbb{N} \end{aligned}$$

and insert the "ansatz" $y(\varphi) = \sum_{m\geq 0} A_m \cos m\varphi/M + B_m \sin m\varphi/M$. We find

$$\begin{aligned} 0 &= \sum_{m=0}^{\infty} \left(\lambda - p\frac{m^2}{M^2}\right) \left(A_m \cos\frac{m\varphi}{M} + B_m \sin\frac{m\varphi}{M}\right) \\ &+ \sum_{\nu=1}^{k-1} (-1)^{\nu-1} \beta_\nu \sum_{m=0}^{\infty} \frac{A_m}{2} \left(\frac{m+\nu M}{M} \cos\frac{m+\nu M}{M}\varphi - \frac{m-\nu M}{M} \cos\frac{m-\nu M}{M}\varphi\right) \\ &+ \sum_{\nu=1}^{k-1} (-1)^{\nu-1} \beta_\nu \sum_{m=0}^{\infty} \frac{B_m}{2} \left(\frac{m+\nu M}{M} \sin\frac{m+\nu M}{M}\varphi - \frac{m-\nu M}{M} \sin\frac{m-\nu M}{M}\varphi\right). \end{aligned}$$

After an index transformation we can determine the expansion coefficients of the *Fourier* series using the homogeneous recurrence relations ($m = 0, 1, 2, \ldots$)

$$\begin{aligned}\left(\lambda - p\frac{m^2}{M^2}\right) A_m \quad &- \quad \frac{m}{2M}\sum_{\nu=1}^{k-1}(-1)^{\nu-1}\beta_\nu(A_{m+\nu M} - A_{|m-\nu M|}) = 0,\\ \left(\lambda - p\frac{m^2}{M^2}\right) B_m \quad &- \quad \frac{m}{2M}\sum_{\nu=1}^{k-1}(-1)^{\nu-1}\beta_\nu(B_{m+\nu M} - \operatorname{sgn}(m-\nu M)B_{|m-\nu M|}) = 0.\end{aligned} \tag{3}$$

If λ vanishes, a direct integration is straightforward. So we are only interested in eigenvalues $\lambda \neq 0$ and immediately find $A_0 = B_0 = 0$. In the case $M > 1$ we substitute

$$\begin{array}{lll} x_\nu := (-1)^\nu A_{M\nu+r}, & y_\nu := (-1)^\nu B_{M\nu+r}, & r = 1,\dots,M-1,\\ x'_\nu := (-1)^\nu A_{M(\nu+1)-r}, & y'_\nu := (-1)^\nu B_{M(\nu+1)-r}, & \nu = 0,1,2,\dots, \end{array}$$

choose the other A_m, B_m to be zero and obtain independent recurrence relations of maximum order $2k-1$ ($r = 1,\dots,M$, $\nu = 0,1,2,\dots$) for x_ν, y_ν and replacing r by $M-r$ also for x'_ν, y'_ν. Thus, it holds for $i = 0,1,2,\dots$:

$$\begin{aligned} 0 \;=\; & \left(\lambda - p\frac{(iM+r)^2}{M^2}\right)\begin{Bmatrix} x_i \\ y_i \end{Bmatrix}\\ & + \frac{iM+r}{2M}\left(\sum_{\nu=1}^{k-1}\beta_\nu\begin{Bmatrix} x_{\nu+i} \\ y_{\nu+i} \end{Bmatrix} - \sum_{\nu=1}^{\min(i,k-1)}\beta_\nu\begin{Bmatrix} x_{i-\nu} \\ y_{i-\nu} \end{Bmatrix} \pm \sum_{\nu=i+1}^{k-1}\beta_\nu\begin{Bmatrix} x'_{\nu-i-1} \\ y'_{\nu-i-1} \end{Bmatrix}\right). \end{aligned}$$

Note that only the first k relations of both systems are coupled.

For a greatest common divisor $\gcd(r,M) > 1$ we are led to smaller M and $r = M$ is equivalent to $M = 1$. The problem is now to calculate the parameter λ (the eigenvalues) in (3) in such a way that the sequences $(A_m)_{m\geq 0}$ and $(B_m)_{m\geq 0}$ tend to zero for $m \to \infty$.

3 The verification of real eigenvalues

The inclusions for real eigenvalues are calculated with an algorithm based on matrix continued fractions. Both recurrences in (3) can be formulated as a recurrence of type $\vec{x}_m = b_m(\lambda)\vec{x}_{m+1} + a_{m+1}(\lambda)\vec{x}_{m+2}$ with the initial conditions $\vec{x}_0 = f_0(\lambda)\vec{x}_1$, $(k-1)\times(k-1)$-matrix coefficients b_m, a_m, f_0 and $(k-1)$-vectors $\vec{x}_m$ built from the above defined x_i and x'_i. With these matrices a continued fraction

$$C(\lambda) \;=\; b_0 + a_1\Big(b_1 + a_2\big(b_2 + a_3(b_3 + \cdots)^{-1}\big)^{-1}\Big)^{-1}$$

$$= \; b_0 + a_1\Bigg(b_1 + a_2\Big(b_2 + \cdots + a_{r-1}\big(b_{r-1} + C_r(\lambda)\big)^{-1}\cdots\Big)^{-1}\Bigg)^{-1} \quad (4)$$

is defined as in *Risken* [12]. In [11] the following theorem is proved:

Theorem 1. *Assume that* $\|a_m b_m^{-1}\| \le q_1 < 1/2$ *and* $\|b_m^{-1}\| \le q_2 < 1/2$ *for* $m \ge m_0$. *Then the continued fraction converges uniformly and for an eigenvalue, i.e.* $\|\vec{x}_m\| \to 0$, *follows:*

$$\lambda^* \quad \textit{is eigenvalue iff} \quad \vec{x}_0 = C(\lambda^*)\vec{x}_1.$$

With the initial conditions $\vec{x}_0 = f_0(\lambda)\vec{x}_1$ this leads to the homogeneous linear system $\big(E - f_0(\lambda)C^{-1}(\lambda)\big)\vec{x}_0 = 0$ with identity matrix E. For a nontrivial solution of this system we get $F(\lambda) := \det\big(E - f_0(\lambda)C^{-1}(\lambda)\big) = 0$, so that we have to determine inclusions of the zeros of $F(\lambda)$. To do this, verified inclusions of the continued fraction $C(\lambda)$ have to be calculated.

As shown in [11], the continued fraction remainder $C_r(\lambda)$ is enclosed by an interval matrix $R := (r_{ij})$ with $r_{ij} = [-\alpha(q_1, q_2),\, \alpha(q_1, q_2)]$. This matrix R is inserted into the continued fraction for the remainder $C_r(\lambda)$. If we now evaluate the continued fraction, using a backward recurrence algorithm with matrix interval arithmetic, we get verified inclusions of $C(\lambda)$. If the above assumptions are valid over a closed interval I the following algorithm to verify the zeros of $F(\lambda)$ can be used:

Algorithm 1.

1. *Real approximation:*
   Calculate an approximation $\tilde{\lambda} \in I$ *of the function* $F(\lambda) = \det(E - f_0(\lambda)C^{-1}(\lambda)) = 0$.

2. ϵ*-inflation:*
   Create an interval $I_0 = [\lambda_1,\ \lambda_2] = \tilde{\lambda} \cdot [1 - \epsilon,\ 1 + \epsilon]$ *through* ϵ*-inflation with* $\tilde{\lambda}$ *from Step 1. In a floating-point system with base* b *and mantissa-length* l *we choose* $\epsilon = b^{-l+1}$. *(In the examples below we choose* $b = 2$ *and* $l = 53$.*)*

3. *Verification:*
   Calculate verified $I_1 = F([\lambda_1])$, $I_2 = F([\lambda_2])$, $I_3 = F(I_0)$, *using an*

interval evaluation of F at the point intervals $[\lambda_i]$ and the interval I_0 by employment of a machine matrix interval arithmetic and verified inclusion of the matrix continued fraction $C(\lambda)$ (cf. [11]).

(a) *If $\sup\big(F([\lambda_1]) \cdot F([\lambda_2])\big) < 0$ and I_3 is bounded, it follows for reason of the continuity of $F(\lambda)$:*
In the interval $I_0 = [\lambda_1,\ \lambda_2]$ lies at least one zero of $F(\lambda)$, and so the existence of at least one eigenvalue of the recurrence inside I_0 follows from Theorem 1.

(b) *If $0 \in I_1$ or $0 \in I_2$, no statements about eigenvalue inclusion are possible. In this case we enlarge ϵ of Step 2 with the rule $\epsilon := \epsilon \cdot b$. If $\epsilon < 1$, we repeat Step 3; otherwise we terminate the algorithm and the result is not verified.*

(c) *If I_3 is not bounded, $F(\lambda)$ may have a pole inside I_0, and so no statement about eigenvalue inclusion is possible.*

Using this algorithm, we have calculated the verified inclusions given below. The calculations are turned out with interval variables in *PASCAL–XSC*. To calculate the results in Section 5 we have truncated the continued fraction $C(\lambda)$ at the index r, with r between 100 and 4000 depending on k, so that the assumptions of Theorem 1 are fulfilled. The verification of the continued fraction was turned out as described above.

In the complex case we only consider the sinusoidal phase detector characteristic $g(\varphi) = \sin\varphi - \gamma$ and utilize the following theorem proved in [9].

All non-vanishing eigenvalues of (1) together with $2\pi M$-periodic boundary conditions are solutions of the nonlinear equation

$$D(\lambda) = C \cdot C^* - 1 = 0,\ M > 1, \quad C = 0,\ C^* = 0,\ M = 1.$$

C^* denotes the conjugate complex of the continued fraction (4). To verify a complex eigenvalue we use *Rouchés* theorem and calculate the change of the argument of $D(\lambda)$ on a rectangle including the eigenvalue. To this behind we must decompose the four sides into 10^t pieces with length δ, t, δ depending on the parameter p.

As an example we have verified the following eigenvalues with the parameters $p = 10^{-1.2}$, $\gamma = \sin(\pi/36)$, $\nu = 1$:

$$0.9619652477^{90}_{75} \pm 1.86^{80}_{60} \cdot 10^{-9} i,\ \ M = 1,\ \ t = 5,\ \ \delta = 10^{-15},$$
$$0.96193946^{30}_{00} \pm 1.067^{80}_{60} \cdot 10^{-5} i,\ \ M = 4,\ \ t = 6,\ \ \delta = 10^{-15}.$$

4 Characteristics with infinite Fourier series

Now we turn our attention to phase detector characteristics with infinite *Fourier* series expansions such as the 2π-periodic triangular

$$g(\varphi) = \begin{cases} 2\varphi/\pi, & -\pi/2 \leq \varphi \leq \pi/2 \\ \operatorname{sgn}(\varphi)(2 - 2|\varphi|/\pi), & \pi/2 \leq |\varphi| \leq \pi \end{cases}$$

sawtooth or tanlock characteristic [1, 2]

$$g(\varphi) = \sqrt{1-a^2}\frac{\sin\varphi}{1 + a\cos\varphi}, \quad 0 \leq a < 1 \tag{5}$$

and are interested in determining the truncation error which arises if we calculate verified inclusions for the eigenvalues of problem (2) with a trigonometric polynomial $g_f(\varphi)$ as a few term approximation of $g(\varphi)$.

In order to find an absolute error bound for the eigenvalues we utilize the self-adjoint form

$$u'' + \left(\frac{g'(\varphi)}{2p} - \frac{g^2(\varphi)}{4p^2}\right) u = -\frac{\lambda}{p}\, u\,. \tag{6}$$

Now we take $g_f(\varphi)$ as a few term approximation of $g(\varphi)$. If the estimation

$$E(g, g_f) := \left| \frac{g'(\varphi)}{2} - \frac{g^2(\varphi)}{4p} - \frac{g_f'(\varphi)}{2} + \frac{g_f^2(\varphi)}{4p} \right| \leq \epsilon \tag{7}$$

holds uniformly for all φ, we get an absolute error bound for all eigenvalues of (1). This argument is a consequence of a well-known comparison theorem which is given below in a more general form. Here we consider two eigenvalue problems $pu'' + (f + \lambda)u = 0$, $pu_t'' + (f_t + \lambda_t)u_t = 0$ and the same boundary conditions. Then it holds:

If $f \geq (\leq) f_t$ then u oscillates faster (slower) than u_t and for corresponding eigenvalues leading to the same number of zeros of the eigenfunctions $u(\varphi, \lambda)$, $u_t(\varphi, \lambda_t)$ we have $\lambda \leq (\geq) \lambda_t$.

Thus we are able to prove the following theorem using inclusions containing only one eigenvalue guaranteed by the asymptotic relations described in Section 1:

Theorem 2. *Let ϵ be a uniform bound for (7) and $[\lambda_l,\ \lambda_u]$ an eigenvalue inclusion of the boundary value problem*

$$py_1'' + (g_f y_1)' = -\lambda y_1, \quad y_1(-2\pi) = y_1(2\pi), \quad y_1'(-2\pi) = y_1'(2\pi). \tag{8}$$

Then the interval $[\lambda_l - \epsilon,\ \lambda_u + \epsilon]$ created through ϵ-inflation is an eigenvalue inclusion for at least one eigenvalue of

$$py_2'' + (gy_2)' = -\lambda y_2, \quad y_2(-2\pi) = y_2(2\pi), \quad y_2'(-2\pi) = y_2'(2\pi). \tag{9}$$

Proof: If the corresponding λ_2 $\big($i.e. y_1 and y_2 have the same number of zeros in $[-2\pi, 2\pi)\big)$ is outside the expanded interval the eigenfunction y_2 oscillates faster (slower) and has more (less) zeros than y_1, which is a contradiction.

Especially for the calculation of verified results for small eigenvalues we get better results if we use estimations of relative error instead of absolute error. These estimations are possible using *Sturm's* comparison theorem:

Let $P \geq P_1 > 0$, $Q \geq Q_1$, P, P_1, Q, Q_1 be continuous functions on (a, b). Assume that $u(\varphi)$ and $u_1(\varphi)$ are real solutions of the equations

$$-(Pu')' + Qu = 0, \quad -(P_1 u_1')' + Q_1 u_1 = 0$$

respectively on (a, b). Then between any two consecutive zeros φ_1, φ_2 of $u(\varphi)$ there exists at least one zero of u_1 (i.e. u_1 oscillates faster than u).

To apply the theorem we take

$$\begin{aligned} P_1 &= p \exp\left(-\frac{1}{p} \int_0^{\varphi} g_f(\theta) d\theta \right), \quad Q_1 = -\frac{\lambda}{p} P_1, \\ P &= p \exp\left(-\frac{1}{p} \left(\int_0^{\varphi} g(\theta) d\theta \pm \Delta_{1,2} \right) \right), \quad Q = -\frac{\lambda}{p} P \exp\left(\pm \frac{\Delta}{p} \right) \end{aligned} \tag{10}$$

and construct constants Δ_1, $\Delta_2 > 0$, $\Delta := \Delta_1 + \Delta_2$ to obtain the inequalities $P_1 \geq (\leq) P$, $Q_1 \geq (\leq) Q$ as in the assumption of *Sturm's* theorem. Now we evaluate the defining integrals and constants in (10):

$$g(\varphi) \ := \ \sum_{m=1}^{\infty} (-1)^{m-1} \beta_m \sin m\varphi,$$

$$
\begin{aligned}
g_f(\varphi) &:= \sum_{m=1}^{k-1}(-1)^{m-1}\beta_m \sin m\varphi, \quad \beta_m \geq 0, \\
-\Delta_2 &= -\sum_{m=k}^{\infty}\frac{\beta_m}{m}\big((-1)^m+1\big) \leq \int_0^{\varphi}\big(g(\theta)-g_f(\theta)\big)d\theta \\
&\leq \sum_{m=k}^{\infty}\frac{\beta_m}{m}\big(1-(-1)^m\big) = \Delta_1.
\end{aligned}
$$

Furthermore,

$$\Delta := \Delta_1 + \Delta_2 = 2\sum_{m=k}^{\infty}\frac{\beta_m}{m}.$$

This error bound Δ ensures by an application of *Sturm's* theorem an interval inclusion $[\lambda_l \exp(-\Delta/p),\ \lambda_u \exp(\Delta/p)]$ for each eigenvalue λ with an eigenvalue inclusion $[\lambda_l,\ \lambda_u]$ of the boundary value problem (8).

5 Numerical results

In this section we give numerical results for the eigenvalues of (2). As phase characteristic we use the tanlock (5) and smooth approximations, introduced by *Rosenkranz* [13], for triangular and sawtooth characteristic to avoid the well-known overshooting destructive to every approximation by finite sums.

$$g_{ta}(\varphi) = \sqrt{1-a^2}\,\frac{2}{a}\sum_{m=1}^{\infty}(-1)^{m-1}\left(\frac{1}{a}-\sqrt{\frac{1}{a^2}-1}\right)^m \sin m\varphi \quad \text{(tanlock)} \tag{11}$$

$$g_{tr}(\varphi) = \frac{8}{\pi^2}\sum_{m=0}^{\infty}(-1)^m \mu_{2m+1}\frac{\sin(2m+1)\varphi}{(2m+1)^2} \quad \text{(triangular)} \tag{12}$$

$$g_s(\varphi) = \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m-1}\mu_m\frac{\sin m\varphi}{m} \quad \text{(sawtooth)} \tag{13}$$

$$\mu_m = \sqrt{\frac{\pi y}{2}}e^{-y}\left\{I_{\frac{m-1}{2}}(y) + I_{\frac{m+1}{2}}(y)\right\},$$

$$I_m(y) = \sum_{\nu=0}^{\infty}\frac{(y/2)^{2\nu+m}}{\nu!\;\Gamma(m+\nu+1)}, \quad y = \frac{SNR}{2}.$$

If we now choose $g_{f(\cdot)}(\varphi)$ as the $(k-1)$-th partial sum of $g_{(\cdot)}(\varphi)$ we have to estimate the term in formula (7) for absolute error bounds. Using monotonicity arguments for the above defined values μ_k and estimations for the *Bessel* functions of first kind and the series remainder we get the following estimations:

$$E(g_{tr}, g_{f(tr)}) \;\le\; \frac{2e^{c-(1/(2p))}\, p^{-k-0.5}}{\pi^{1.5}\,(2k+1)\,4^k k!}\left(1+\frac{1}{4p\,(k+1)}\right)\frac{1}{1-4p\,c}\left(1+\frac{\max_\varphi |g_{tr}+g_{f(tr)}|}{2p\,(2k+1)}\right),$$

$$E(g_s, g_{f(s)}) \;\le\; \frac{y^{k/2}\, e^{c-y}}{\sqrt{\pi}\; 2^{k/2}\,\Gamma\Big((k+1)/2\Big)}\,\frac{1+y/(k+1)}{1-\sqrt{y/(k+1)}}\left(\frac{2}{\sqrt{\pi}}+\frac{G_0}{p\,k}\right), \quad c=\frac{y^2}{2k+2},$$

$$E(g_{ta}, g_{f(ta)}) \;\le\; \frac{a^k}{2}\left(\sqrt{\frac{1+a}{1-a}}+\frac{k}{2}+\frac{1}{p}\right), \quad k=2\kappa>0.$$

The constant G_0 in the sawtooth case is defined as

$$G_0 := \mu_1 \sup_n \max_\varphi |G_n(\varphi)|$$

with $0 \le G_n(\varphi) = \frac{2}{\pi}\sum_{m=1}^{n}(-1)^{m-1}\frac{\sin m\varphi}{m} \le \frac{2}{\pi}\sum_{m=1}^{n}\frac{1}{m}\sin\frac{m\pi}{n+1} \le 1.1799$ (*Gibbs* constant). Now we give the first eigenvalues of (2), for $M=2$, $k=10,12,14$, $p=0.1$, and $g(\varphi)=\sum_{m=1}^{k-1}(-1)^{m-1}\beta_m \sin m\varphi$ in verified inclusion form:

$\lambda_\nu^{(2)}$ $k=10$, 12, 14	Triangular: $y=5,\ \beta_{2m}=0$, $(-1)^{m-1}\beta_{2m-1}=\frac{2^{5/2}}{\pi^{3/2}}\cdot \frac{e^{-y}\sqrt{y}}{(2m-1)^2}\Big(I_{m-1}(y)+I_m(y)\Big)$	Sawtooth: $y=0.5\cdot 10^{0.7}$, $\beta_m=\sqrt{\frac{2y}{\pi}}\frac{e^{-y}}{m}\cdot \Big(I_{\frac{m-1}{2}}(y)+I_{\frac{m+1}{2}}(y)\Big)$	Tanlock: $a=0.5$, $\beta_m=\sqrt{1-a^2}\cdot \frac{2}{a}\left(\frac{1}{a}-\sqrt{\frac{1}{a^2}-1}\right)^m$
$\nu=0$	$8.554091^{218}_{081}E{-}8$	$1.03011637^{674}_{049}E{-}6$	$3.36160^{997}_{694}E{-}9$
	$8.554091^{948}_{879}E{-}8$	$1.030013^{649}_{510}E{-}6$	$3.36160^{979}_{440}E{-}9$
	$8.554091^{964}_{881}E{-}8$	$1.03000^{804}_{578}E{-}6$	$3.36160^{936}_{289}E{-}9$
$\nu=1$	$0.63205848158^{207}_{105}$	$0.3180201346718^{252}_{200}$	$0.5756273872517^{576}_{344}$
	$0.63205821393^{442}_{340}$	$0.3180202096899^{429}_{020}$	$0.5756273765253^{365}_{133}$
	$0.63205821694^{890}_{738}$	$0.31802021137^{411}_{379}$	$0.5756273765059^{872}_{409}$

To verify the eigenvalues of the characteristics with infinite *Fourier* series expansions we find in the above given cases the following estimations for $E(g, g_f)$:

triangular characteristic	$4c = \frac{y^2}{k+1}, \quad p = 0.1, \quad y = \frac{1}{2p}$			
k	4	5	6	7
ϵ	0.0306	0.00744	0.001821	0.0004294
tanlock characteristic	$a = 0.5, \quad p = 0.1$			
k	8	10	12	14
ϵ	0.0013158	0.0001068	$8.55316E{-6}$	$6.7766E{-7}$

In the case of relative error bounds we get the following expressions. Triangular:

$$\exp\left(\pm\frac{\Delta}{p}\right) = \exp\left(\pm\frac{2^{3.5-k}y^{k+0.5}\exp\left(\frac{y^2}{4k+4}-y\right)(2k+2+y)}{\pi^{1.5}\,p\,k!\,(2k+1)^3\,(2k+2-y)}\right)$$

Sawtooooth:

$$\exp\left(\pm\frac{\Delta}{p}\right) = \exp\left(\pm\frac{2^{2-k/2}y^{k/2}\exp\left(\frac{y^2}{2k+2}-y\right)\left(1+y/(k+1)\right)}{\sqrt{\pi}p\,k^2\,\Gamma\left(\frac{k+1}{2}\right)\left(1-\sqrt{y/(k+1)}\right)}\right)$$

Tanlock:

$$\exp\left(\pm\frac{\Delta}{p}\right) = \exp\left(\pm\frac{4}{p\,k}\frac{\left(\frac{1}{a}-\sqrt{\frac{1}{a^2}-1}\right)^k}{1-\sqrt{\frac{1-a}{1+a}}}\right)$$

In a numerical example this leads to the following values:

triangular $p = 0.1, \quad y = 0.5$		sawtooth $p = 0.1, \quad y = 0.5 \cdot 10^{0.7}$		tanlock $p = 0.1, \quad a = 0.5$	
k	Δ/p	k	Δ/p	k	Δ/p
4	0.00716	8	0.023392	8	0.0003144
5	0.00129	10	0.0033952	10	0.00001806
6	0.000244	12	0.0004654	12	$1.08026E{-6}$
7	0.00004581	14	0.00005924	14	$6.648E{-8}$

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Received: September 1, 1993
Revised version: March 1, 1994

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