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Computation of Integrals of Uncertain Vector Functions

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The paper deals with the problem of numerical integration of a function $[0,1] \to \mathbf{R}^n$ for which only a set-membership description is known. The set of values of the integrals of all possible realizations of the uncertain function is considered as a guaranteed result of the integration. The problem is to approximate this set, with a prescribed accuracy ε, by polyhedral sets, in particular, to enclose it in an n-dimensional interval which is ε-minimal.

In the first part of the paper we focus on quadrature formulae for set-valued integrals and the estimation of their errors. In the second part we sketch the idea of their implementation.

Вычисление интегралов неопределенных векторных функций

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Рассматривается задача численного интегрирования функции $[0,1] \to \mathbf{R}^n$, о которой известна только ее принадлежность некоторому множеству. Множество значений интегралов всех возможных реализаций неопределенной функции рассматривается как гарантированный результат интегрирования. Задача состоит в аппроксимации этого множества с заданной точностью ε с помощью многогранных множеств и, в частности, в заключении его внутрь n-мерного интервала, являющегося ε-минимальным.

В первой части статьи мы рассматриваем квадратурные формулы для интегралов, значениями которых являются множества, уделяя внимание оценке их погрешностей. Во второй части мы даем набросок схемы их применения.

1 Introduction

We begin with the following problem of integration of an uncertain function $f : [0,1] \mapsto \mathbf{R}^n$. Suppose that the only information about $f(\cdot)$ is that $f(t) \in \mathrm{conv}\{f_1(t), \ldots, f_r(t)\}$, $t \in [0,1]$, where $f_1(\cdot), \ldots, f_r(\cdot)$ are known functions. Then $\int_0^1 f(t)\, dt$ may, in principle, take any value from the set

$$I = \left\{ \int_0^1 f(t)\, dt;\ f(t) \in \mathrm{conv}\{f_1(t), \ldots, f_r(t)\},\ f(\cdot) - \text{integrable} \right\}.$$

The problem is to approximate the convex compact set I with any given accuracy.

The above problem is well understood in the scalar case $f(\cdot) : [0,1] \to \mathbb{R}^1$, $f(t) \in [f_1(t), f_2(t)]$, where

$$I = \left[\int_0^1 f_1(t)\, dt, \int_0^1 f_2(t)\, dt \right]$$

provided that $f_1(t) \leq f_2(t)$ on $[0,1]$. In higher dimensions, however, the problem is more complicated. In particular, even if the uncertainty is one-dimensional (that is, $f(t) \in [f_1(t), f_2(t)] = \mathrm{conv}\{f_1(t), f_2(t)\} \subset \mathbf{R}^n$), the set I can be a body in $\mathbf{R}^n$ (that is, with nonempty interior).

Since the uncertain function f can be presented as

$$f(t) = \sum_{i=1}^{r} w_i(t) f_i(t), \qquad w_i(t) \geq 0,\ \ \sum_{i=1}^{r} w_i(t) = 1$$

we have that

$$\int_0^1 f(t)\, dt = \int_0^1 F(t) w(t)\, dt$$

where F is a known $(n \times r)$-matrix function (with columns f_i) and $w(\cdot) = (w_1(\cdot), \ldots, w_r(\cdot))^*$ is unknown, but bounded in a convex and compact set W $(= \{w\,;\ w_i \geq 0,\ \sum w_i = 1\})$. Thus the problem we started with is a particular case of the more general problem of approximation of the set

$$I = \int_0^1 F(t) W\, dt = \left\{ \int_0^1 F(t) w(t)\, dt\,;\ w(t) \in W,\ w(\cdot) - \text{integrable} \right\} \tag{1}$$

where $F(\cdot)$ is an $(n \times r)$-matrix function and W is a convex and compact set in $\mathbf{R}^r$. The above set is known as Aumann's integral of the set-valued mapping $F(\cdot)W$ [1].

Two interrelated issues arise: 1) to develop the theory of quadrature formulae for set-valued integrals as in (1); 2) to choose tools for constructive representation/approximation of convex sets in $\mathbf{R}^n$, in order to implement these formulae. These two issues are considered in the two sections below.

2 Set-valued quadrature formulae

In this section we formulate certain set-valued analogues to well-known quadrature formulae and estimate the corresponding error. Linear composite quadrature formulae of first and second order accuracy are known (see [2] and [3], respectively). There is a principle barrier, however, towards higher order approximations of set-valued integrals by linear quadrature formulae, which can be overcome only in exceptional cases. The main result in this section is a quadrature formula of third order accuracy, that artificially involves nonlinearity and is applicable mainly to the case when W is an r-dimensional interval (that is, a box).

As a measure of the deviation of a set $J \subset \mathbf{R}^n$ from the integral I we use the Hausdorff distance

$$H(I, J) = \max \big\{ \max_{x \in I} \mathrm{dist}(x, J), \max_{x \in J} \mathrm{dist}(x, I) \big\}.$$

Further we denote also

$$|W| = \max \big\{ |w| \, ; \; w \in W \big\}, \quad |F(t)| = \max \big\{ |F(t) l| \, ; \; |l| = 1 \big\}$$

where $|\cdot|$ is the Euclidean norm in $\mathbf{R}^n$.

Let $t_0 = 0$, $t_1 = h$, $\ldots$, $t_N = Nh = 1$ be the N-points uniform grid in $[0, 1]$. The following simple fact is well-known (see e.g. [4]).

Proposition 1. *Let $F(\cdot)$ be of bounded variation. Consider the rectangle formula*

$$J_N = \sum_{i=0}^{N-1} F(t_i) W$$

where the sum of sets is in Minkowski sense. Then

$$H(I, J_N) \leq |W| \bigvee_0^1 F(\cdot)/N. \tag{2}$$

In the above inequality and further $\bigvee_0^1 F(\cdot)$ denotes the variation of F on $[0, 1]$ with respect to the distance in the space of matrices induced by the operator norm (defined above).

Now let us consider the more general linear composite quadrature formula

$$h \sum_{i=0}^{N-1} \sum_{j=1}^{p} a_j g(t_i + h\tau_j) \tag{3}$$

where $a_j \geq 0$ and $0 \leq \tau_1 \leq \ldots \leq \tau_p \leq 1$ are parameters. The following theorem is given in [3] (see also [5]).

Theorem 1. *Let $F(\cdot)$ be Lipschitz continuous with constant L_0 and let the derivative $\dot{F}(\cdot)$ be also Lipschitz continuous with constant L_1. Suppose that the quadrature formula (3) is exact for polynomials of degree one. Then the composite set-valued quadrature formula*

$$J_N = h \sum_{i=0}^{N-1} \Big(\sum_{j=1}^{p} a_j F(t_i + h\tau_j) \Big) W \tag{4}$$

provides accuracy

$$H(I, J_N) \leq \big(1 + \sum_{j=1}^{p} a_j\big)(2L_0 + 3L_1)|W|/N^2. \tag{5}$$

Notice that the set W multiplies the whole inner sum in (4), rather than the summands separately (as [2] suggests).

We stress the following principial difference between Proposition 1 and Theorem 1. If we denote

$$I^i = \int_{t_i}^{t_{i+1}} F(t)W \, dt, \quad J_N^i = hF(t_i)W$$

then

$$\sum_{i=1}^{N} H(I^i, J_N^i) \leq |U| \bigvee_0^1 F(\cdot)/N$$

which means that (2) can be obtained by summing up the local errors of integration. In contrast to that, if now we denote

$$J_N^i = \Big(\sum_{j=1}^{p} a_j F(t_i + h\tau_j) \Big) W$$

one can verify that even for very simple examples (like $F(t) = (1, t)^*$ and $W = [-1, 1] \subset \mathbb{R}^1$)

$$H(J^i, J_N^i) \geq \mathrm{const}/N^2 \tag{6}$$

for any particular quadrature formula of the type of (4). Thus the sum of the local errors is proportional to $1/N$, despite that the global error is of order $1/N^2$, according to Theorem 1. An effect of *nonaccumulation of errors* is behind the second order estimate (5), as explained in more details in [6].

The inequality (6) also implies that higher than second order approximations to Aumann's integrals cannot be provided in general by linear quadrature formulae like (4). In a more precise way this is claimed by the next proposition.

Proposition 2. *(A negative result.) Let $W \in \mathbb{R}^1$ be a nondegerate interval, let $F : [0, 1] \to \mathbb{R}^2$ be continuously differentiable with Lipschitz continuous derivative and let rank$\big\{F(t), \dot{F}(t)\big\} = 2$ for every $t \in [0, 1]$. Then for every linear quadrature formula (4) there is a constant c such that*

$$H(J, J_N) \geq c/N^2,$$

for any N.

Proof. According to Theorem 3.1 (see also Remark 3.1) in [7] the integral I in this case is a strongly convex set in $\mathbb{R}^2$ (that is, there is $\alpha > 0$ such that for any points $x, y \in I$ and every unit vector $l \in \mathbb{R}^2$ the point $(x + y)/2 + \alpha|x - y|^2 l$ also belongs to I). Hence, the Hausdorff distance $H(I, P_k)$ between I and any polygon P_k with k vertices is not less than c/k^2, where the constant c is independent of k and the polygon P_k. On the other hand the number of vertices of J_N in (4) is proportional to N, which proves the claim of the proposition.

A different point of view on the "barrier" towards higher than second order quadrature formulae is presented in [4].

We mention that approximations to the integral I of third order accuracy are proposed in [2] and in a more elaborate form also in [4], but under assumptions stronger than twice differentiability of the support function of the integrand. This assumption is rather restrictive for many applications and in particular does not hold in the situation described by the above proposition.

Below we present a quadrature formula for the Aumann's integral I, which is of different type and provides third order approximation without any assumptions concerning the support function of the integrand (and applicable also for integrals as in Proposition 2). The approach is taken from [5], relays again on the effect of nonaccumulation of errors, but is applicable essentially for sets W that are boxes. The computational aspects of the nonlinearities that this type of quadrature formulae involve will be discussed in the next section.

Consider again the Aumann's integral (1), supposing now that W is a box, that is, without loss of generality, $W = [-1,1]^r$. Since in this case we have

$$I = \sum_{k=1}^{r} \int_0^1 f_k(t)[-1,1]\,dt$$

where f_k is the k-th column of F, it is enough to cope with one dimensional uncertainty: $r = 1$. Thus we consider

$$I = \int_0^1 f(t)[-1,1]\,dt. \tag{7}$$

Theorem 2. *Let $f(\cdot)$ be twice differentiable and let the second derivative $\ddot{f}(\cdot)$ be bounded by a constant M and be Lipschitz continuous with constant L. Let N be an integer, h and $t_i, i = 0,\ldots,t_N$ be defined as above and the vectors f_i^0, f_i^1 and f_i^2 satisfy the inequalities*

$$|f_i^0 - f(t_i)| \le c_0 h^3, \;\; |f_i^1 - \dot{f}(t_i)| \le c_1 h^2, \;\; |f_i^2 - \ddot{f}(t_i)| \le c_2 h.$$

Denote

$$S = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 \, ; \, \left| \begin{array}{lcl} y_1 &=& 1 - 2\alpha, \\ y_2 &=& \frac{1}{2} - \alpha^2, \\ y_3 &=& \frac{1}{6} - \frac{1}{3}\alpha^3, \end{array} \quad \alpha \in [0,1] \right. \right\}.$$

Then the quadrature formula

$$J_N = h \operatorname{co} \sum_{i=1}^{N-1} \left\{ \pm [f_i^0 \; f_i^1 \; f_i^2] S \right\} \tag{8}$$

provides accuracy

$$H(I, J_N) \leq \left(M + \frac{7}{3} L + c_0 + \frac{c_1}{2} + \frac{c_2}{3} \right) / N^3 \tag{9}$$

(here "co" denotes the convex hull).

Proof. In the proof we use the more general argument from [5], attributing more attention to the estimation of the involved constants.

Denote by $\mathcal{W}_1(t_i, t_{i+1})$ the set of all piece-wise constant functions on $[t_i, t_{i+1}] \to [-1, 1]$ with at most one jump. Denote

$$I_N = \sum_{i=1}^{N-1} \left\{ \int_{t_i}^{t_{i+1}} f(t) w(t) \, dt \, ; \; w(\cdot) \in \mathcal{W}_1(t_i, t_{i+1}) \right\}. \tag{10}$$

Taking into account that $I_N \subset I$ one can estimate

$$H(I, I_N) = \sup_{|l|=1} \left(\max_{x \in I} \langle x, l \rangle - \max_{y \in I_N} \langle y, l \rangle \right)$$

$$= \sup_{|l|=1} \sum_{i=0}^{N-1} \left(\int_{t_i}^{t_{i+1}} |f(t)| \, dt - \max_{w(\cdot) \in \mathcal{W}_1(t_i, t_{i+1})} \int_{t_i}^{t_{i+1}} \langle f(t), l \rangle w(t) \, dt \right). \tag{11}$$

Fix an arbitrary $|l| = 1$. For those i for which the function $\langle f(t), l \rangle$ has at most one zero in (t_i, t_{i+1}) the corresponding summands in (11) are equal to zero. Denote by K_2 the rest of indexes i in (11). According to [5, Lemma 3]

$$\sum_{i \in K_2} \left(\int_{t_i}^{t_{i+1}} |f(t)| \, dt \right) \leq h^3 \left(\|\ddot{f}(\cdot)\|_C + 2 \bigvee_0^1 \ddot{f}(\cdot) \right) \leq h^3 (M + 2L).$$

Hence, taking $w(t) = 0$ for the corresponding members in (11), we estimate

$$H(I, I_N) \leq (M + 2L)/N^3. \tag{12}$$

Now let us estimate $H(I_N, J_N)$. Using the Taylor expansion of f up to second order terms on each interval $[t_i, t_{i+1}]$ in (10) we obtain that

$$H\left(I_N, \mathrm{co}\left(\sum_{i=1}^{N-1}\left\{\int_{t_i}^{t_{i+1}}(f_i^0 + (t-t_i)f_i^1 + 0.5(t-t_i)^2 f_i^2)w(t)\,dt\,;\ w(\cdot) \in \mathcal{W}_1(t_i, t_{i+1})\right\}\right)\right)$$

$$\leq \left(c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \frac{L}{3}\right)h^3.$$

Simple calculation often used in linear optimal control theory implies that the second argument of H coincides with J_N (the parameter α in the definition of the set S in (8) corresponds to the position of the only jump of $w(\cdot)$ in each of the subintervals).

Then the claim of the theorem follows from the above inequality and (12).

Notice that if the exact values of $f(t_i)$ and of the derivatives up to second order are known, one may use them instead of f_i^j and take $c_0 = c_1 = c_2 = 0$ in (9).

We mention that the above theorem reveals again the effect of nonaccumulation of errors, mentioned above in the case of second order schemes. Namely, in general the Hausdorff distance between each summand in (8) and the corresponding piece of the integral I is of order $1/N^3$, however, the error of the sum J_N remains $1/N^3$, in contrast with the sum of the errors, which is of second order.

We mention also that the set S in (8) is an one dimensional curve in $\mathbb{R}^3$ and can be considered as a nonlinear image of the set $W = [-1, 1]$, while the second order approximations (4) do not involve nonlinear operations. This nonlinearity is the price of the higher order accuracy.

3 Constructive implementation

Since the Aumann's integrals in the form of (1) are convex and compact sets, we chose in the present context to associate with any such set its support function

$$\rho(l|I) = \max_{x \in I} \langle l, x \rangle, \quad l \in \mathbf{R}^n.$$

We suppose that the set W, which is a part of the available data, is simple enough, so that its support function can be evaluated approximately with accuracy ε for any $l \in \mathbb{R}^r$:

$$\tilde{\rho}(l) = \rho(l|W) + \varepsilon(l), \quad |\varepsilon(l)| \leq \varepsilon.$$

For example, in the "standard" case

$$W = [a_1, b_1] \times \ldots \times [a_r, b_r] \tag{13}$$

we have

$$\rho(l|W) = \sum_{i=1}^{r} l_i z_i, \quad \begin{cases} z_i = a_i & \text{if } l_i \leq 0 \\ z_i = b_i & \text{if } l_i > 0 \end{cases}$$

where $l = (l_1, \ldots l_r)^*$ and $\varepsilon(l)$ is the error in the computation of the above sum. If W is an ellipsoid given as

$$W = \left\{ w \in \mathbf{R}^n ;\ \langle E^{-1}(x - e), x - e \rangle \leq 1 \right\}$$

where $e \in \mathbf{R}^r$, E is a symmetric positive definite matrix, then

$$\rho(l|W) = \langle l, e \rangle + \langle El, l \rangle^{1/2}.$$

If the conditions of Theorem 1 are fulfilled, then one can evaluate

$$\rho(l|I) \approx h \sum_{i=0}^{N-1} \sum_{j=1}^{p} a_j \tilde{\rho}\big(F(t_i + h\tau_j)^* l\big) \tag{14}$$

with accuracy $\varepsilon \sum_{j=1}^{p} a_j + H(I, J_N)$, where $H(I, J_N)$ is estimated by (5). The requirement about this accuracy could be met by choosing appropriately the number N. The impact of the computational error in the linear operations in (14) can also be taken into account (observe that only dot products are involved in the case (13), which can be performed with very high and controlled accuracy).

In the papers [8, 9] one can find more details concerning computer realization with controlled accuracy of the above results, including the case of matrix $F(\cdot)$ resulting from an uncertain system of linear differential equations.

If the conditions of Theorem 2 are fulfilled, then

$$\rho(l|I) \approx h \sum_{i=0}^{N-1} \max_{\alpha \in [0,1]} \left\{ \pm \Big(\langle f_i^0 + \frac{1}{2} f_i^1 + \frac{1}{6} f_i^2, l \rangle - 2 \langle f_i^0, l \rangle \alpha - \langle f_i^1, l \rangle \alpha^2 - \frac{1}{3} f_i^2 \alpha^3 \Big) \right\}$$

with accuracy $H(I, J_N)$ given by (9). Thus the calculation involves solving of $2N$ one-dimensional maximization problems for cubic functions on $[0,1]$, which problems seem to be still tractable with result verification. Computer implementation of the above third order scheme is subject to further work.

We mention that the opportunity to calculate with prescribed accuracy the values of the support function of Aumann's integrals makes it possible to solve with result verification a number of problems involving uncertain integrals or linear control/uncertain systems. In particular, one can enclose the integral in a minimal box (within the given accuracy) by calculating the value of the support function with this accuracy along the coordinate directions and their opposite ones. More generally, one can enclose the integral (or its projection on a specified subspace) in a polyhedron which does not differ in Hausdorff sense from the integral more than required. More details about the implementation with result verification of the above approach can be found in [8].

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