blems", Bishkek, 1991,

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SOLVING SYSTEMS OF SPECIAL FORM NONLINEAR EQUATIONS BY MEANS OF SOME MODIFICATIONS OF RUNGE TYPE INTERVAL ITERATIVE METHOD

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An interval iterative method without estimating of inverse matrices is developed and investigated. Effective interval methods which take into account the specifics of considered particular system are proposed. The convergence theorems for developed interval methods are proved.

РЕШЕНИЕ СИСТЕМ НЕЛИНЕЙНЫХ УРАВНЕНИЙ СПЕЦИАЛЬНОГО ВИДА НЕКОТОРЫМИ МОДИФИКАЦИЯМИ ИНТЕРВАЛЬНОГО ИТЕРАЦИОННОГО МЕТОДА ТИПА РУНГЕ

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Развит и исследован интервальный итеративный метод без оценивания обратной матрицы. Предложены эффективные интервальные методы, которые учитывают специфику рассматриваемой частной системы. Доказываются теоремы сходимости для предложенных интервальных методов.

Solving the systems of nonlinear equations

$$f(x) = 0, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, is usually reduced to solving a sequence of systems of equations obtained by means of the construction of interval extensions for Taylor-series expansions of the left-hand side of system (1). However, if more than two terms in the Taylor-series expansion of $f(x)$ are used, one sees the problem complicated by the necessity of solving high-degree matrix equations with interval coefficients. Hence, system (1), as a rule, is resolved by the Newton interval method or by its modifications differing one from other by the technique of solving corresponding system of linear equations with interval coefficients and by the way of the approximation of an interval expansion of the matrix $f'(x)$.

To resolve the system of equations (1), we have constructed and studied in [1] the interval iterative Runge type method

$$K(X^{(k)}) = x^{(k)} - C^{(k)} f(x^{(k)}) + (I - C^{(k)} F'(X^{(k)}))(X^{(k)} - x^{(k)}), \tag{2}$$

$$X^{(k+1)} = K(X^{(k)}) \cap X^{(k)}, \ k = 0, 1, 2, \ldots, \tag{3}$$

where $X^{(0)}$ is an initial interval; $C^{(k)}$ is an approximated inversion of a center of the matrix $F'(X^{(k)})$; I is an identity matrix; $x^{(k)} = m(X^{(k)})$ (that is, the midpoint of the interval $X^{(k)}$);

$$F'(X^{(k)}) = \tfrac{1}{4} f'(x^{(k)}) + \tfrac{3}{4} f'(x^{(k)} + \tfrac{2}{3}(X^{(k)} - x^{(k)})).$$

The construction of this method was based on Runge's idea of the solution of the Cauchy problem for ordinary differential equations. We also used the behavior of "midpoints" of residual terms in Lagrange's form of generalized Taylor series [3] for the mapping $f(x)$ and relations between these points of Taylor-series expansions for the mapping $f(x)$ and its derivative.

Under some sufficiently simple natural conditions, the method (2), (3) has no less than 3 order of convergence (see, for example, [1]).

However, to resolve some kind of nonlinear systems of equations of the special form the following method occurs to be more effective:

$$\begin{cases} y^{(k+1)} = & y^{(k)} - P^{(k)} f(y^{(k)}); \\ x^{(k+1)} = & x^{(k)} - P^{(k)} f(x^{(k)}), \ k = 0, 1, 2, \ldots; \\ P^{(k+1)} = & P^{(k)}(I + (I - B(x^{(k+1)}, y^{(k+1)})P^{(k)}) \times \\ & \times (2I - B(x^{(k+1)}, y^{(k+1)})P^{(k)})). \end{cases} \tag{4}$$

Theorem 1. *Let* $(x,y) \in [x^{(0)}, y^{(0)}] \subset D \subset \mathbb{R}^n, (x \le y)$ *and suppose that*

(1) *the mapping* $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ *is twice continuously Frechet differentiable and* $f(x^{(0)}) \le 0 \le f(y^{(0)})$;

(2) *the mapping*

$$A : \{(x,y) | [x^{(0)}, y^{(0)}] \times [x^{(0)}, y^{(0)}]\} \to M_{nn}(\mathbb{R}),$$

where

$$f(x) - f(y) = A(x,y)(x-y);$$

(3) *the continuous mapping*

$$B : \{(x,y) | [x^{(0)}, y^{(0)}] \times [x^{(0)}, y^{(0)}]\} \to M_{nn}(\mathbb{R}),$$

where

$$B = B(x,y) = \frac{1}{4} f'(y) + \frac{3}{4} f'(y + \frac{2}{3}(x-y)),$$

satisfies the relations

a) $B(\overline{x}, \overline{y}) \le B(x,y)$, *if* $x \le \overline{x} \le \overline{y} \le y$;

b) $A(x,y) \le B(x,y)$;

c) there is $B(x,y)^{-1}$ *and* $B(x,y)^{-1} \ge 0$;

(4) *the matrix* $P^{(0)} \in M_{nn}(\mathbb{R})$ *is nonsingular and*

a) $B(x^{(0)}, y^{(0)}) P^{(0)} \le I$;

b) $P^{(0)} B(x^{(0)}, y^{(0)}) \le I$;

c) $P^{(0)} \ge 0$.

Then the statement of the theorem holds also for all $x^{(k)}, y^{(k)}$ $(k = 0, 1, \ldots)$, *obtained from (4) and*

$$a)\ x^* = \lim_{k \to \infty} x^{(k)} = \lim_{k \to \infty} y^{(k)} = y^*;$$

$$b)\ P^* = \lim_{k \to \infty} P^{(k)} = B(x^*, y^*)^{-1} = f'(x^*)^{-1};$$

$$c)\ f(x^*) = f(y^*) = 0.$$

If, furthermore, there is the third Frechet derivative of the mapping $f(x)$ *that satisfies the Lipschitz condition, then the order of convergence* *of sequences obtained from (4) to a solution of system (1) of equations is no less than 3.*

Clearly, the method (4) can be considered as a perturbed method (2), (3). Hence Theorem 1 contains stability conditions of the method (2), (3) and perturbation bounds that do not disturb the order of convergence of this method.

It is easy to see that the statement of Theorem 1 holds, in particular, if B is a M-matrix and as $P^{(0)}$, one chooses one of the following matrices:

$$a)\ P^{(0)} = (D^{(0)})^{-1},$$
$$b)\ P^{(0)} = (D^{(0)} - L^{(0)})^{-1},$$
$$c)\ P^{(0)} = (D^{(0)} - U^{(0)})^{-1},$$

where $B = D - L - U$; D is a diagonal matrix, L and U are strictly lower and strictly upper triangular matrices respectively.

We obtain system satisfying conditions in the statement of Theorem 1, in particular, when resolving boundary-value problems for ordinary differential equations

$$p(x)y'' + q(x)y' + r(x)y = f(x, y); \tag{5}$$
$$y(a) = c;\ y(b) = d, \tag{6}$$

by the discretization method.

In fact, in the special case of boundary conditions (6) $y(0) = \alpha$; $y(1) = \beta$ we obtain the system of equations

$$c_i y_{i-1} + a_i y_i + b_i y_{i+1} = -h^2 d_i\ (i = 1, \dots, n),$$

where h is a discretization step, $a_i = 2p_i - r_i h^2$; $b_i = -p_i - q_i h/2$; $c_i =$

$-p_i + q_i h/2$; $d_i = f_i$, that is,

$$\begin{pmatrix} a_1 & b_1 & & & & \\ c_2 & a_2 & b_2 & & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & c_{n-1} & a_{n-1} & b_{n-1} \\ & 0 & & & c_n & a_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = -h^2 \begin{pmatrix} d_1 + c_1\alpha/h^2 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n + b_n\beta/h^2 \end{pmatrix}$$

Let $h \to 0$. Then for $p(x) > 0$ there is h such that $a_i > 0$, $b_i < 0$, $c_i < 0$ $(i = 1, \dots, n)$, that is the matrix B is a M-matrix; therefore, under above choice of the matrix $P^{(0)}$, the statement of Theorem 1 is satisfied.

The numerical solution by discretization method [2] of a boundary-value problem for the elliptic equation

$$-F(x, y, u, u_x, u_y, u_{xx}, u_{yy}) = 0 \tag{7}$$

in $D \subset R^2$, D being a simple, connected, bounded domain, and

$$u(x, y)|_{\partial D} = g(x, y) \tag{8}$$

(∂D is the boundary of the domain D; $Fu_{xx} \geq m_1 > 0$; $Fu_{yy} \geq m_2 > 0$) requires resolving system of nonlinear equations with dominating main diagonal.

Let F be of the form

$$-F(x, y, u, u_x, u_y, u_{xx}, u_{yy}) = -(A(x, y)u_x)_x - (C(x, y)u_y)_y + f(x, y, u),$$

where $A \geq m_1 > 0$; $C \geq m_1 > 0$, $f_u \geq 0$, and D is a rectangular domain. Using the approximation of partial derivatives up to h^2 accuracy and denoting

$$A_{i\pm 1/2} = A(x \pm h/2, y),\ C_{i,j\pm 1/2} = C(x, y \pm h/2),$$
$$K_{i,j} = A_{i+1/2,j} + A_{i-1/2,j} + C_{i,j+1/2} + C_{i,j-1/2},$$

we reduce solving this problem to solving the system of nonlinear equations

$$Hz + \Phi(z) = 0, \tag{9}$$

where

$$H = \begin{pmatrix} S_1 & -R_1 & & & 0 \\ -R_2 & S_2 & & & \\ & \vdots & \vdots & \vdots & \\ & & & & -R_{p-1} \\ 0 & & & -R_{p-1} & S_p \end{pmatrix},$$

$$R_j = \begin{pmatrix} C_{1,j+\frac{1}{2}} & & & & 0 \\ & C_{2,j+\frac{1}{2}} & & & \\ \vdots & \vdots & \vdots\ \vdots\ \vdots & & \vdots \\ 0 & & & & C_{m,j+\frac{1}{2}} \end{pmatrix},$$

$$S_j = \begin{pmatrix} K_{1,j} & -A_{\frac{3}{2},j} & & & 0 & \\ -A_{\frac{3}{2},j} & K_{2,j} & -A_{\frac{5}{2},j} & & & \\ & \vdots & \vdots & \vdots & \vdots & \\ & & & -A_{m-\frac{1}{2},j} & K_{m-1,j} & -A_{m+\frac{1}{2},j} \\ & 0 & & & -A_{m+\frac{1}{2},j} & K_{m,j} \end{pmatrix}.$$

$$\Phi(z) = h^2(f(h,h,z_1),\dots,f(mh,mh,z_n))^{\mathsf{T}} -$$
$$-(u_{01}+u_{10},u_{20},\dots,u_{m+1,m}+u_{m,m+1})^{\mathsf{T}},$$
$$z = (u_{11},u_{21},\dots,u_{mm})^{\mathsf{T}}.$$

The system of this form can be resolved by means of the following modification of the method (2), (3):

$$Y^{(k)} = x^{(k)} - \widetilde{D}^{(k)}(U(X^{(k)})(X^{(k)} - x^{(k)}) -$$
$$- L(X^{(k)})(Y^{(k)} - x^{(k)}) + f(x^{(k)})) \tag{10}$$
$$X^{(k+1)} = Y^{(k)} \cap X^{(k)},\ k = 0,1,2,\dots \tag{11}$$

where

$$F'(X^{(k)}) = D(X^{(k)}) - L(X^{(k)}) - U(X^{(k)}),\ \widetilde{D}(X^{(k)}) = D^{-1}(X^{(k)}).$$

Theorem 2. *Let $x^* \in X^{(0)} \subseteq D$ and let all $F'(X^{(0)})$ be M-matrices. Then $x^* \in X^{(k)}$ $(k = 0, 1, \ldots)$ and $\lim_{k \to \infty} X^{(k)} = x^*$, where x^* is a solution of system (1).*

Since $A \geq m_1 > 0$; $C \geq m_1 > 0$, $f_u \geq 0$, the derivative of system (9) is a M-matrix; therefore, system (9) satisfies the statement of Theorem 2, that is, the method (10), (11) can be applied to it.

The theoretical conclusions obtained above are verified by numerical experiments.

References

1. Vengersky, P.S. and Senio, P.S. *An interval method for solving system of nonlinear equations based on limit mean value theorems.* L'vov, 1990, deposited in UkrNIINTI 20.09.90, N1612-Uk90 (in Russian).
2. Dobronets, B.S. and Shaydurov, V.V. *Two-sided numerical methods.* Nauka, Novosibirsk, 1990 (in Russian).
3. Ortega, J. and Rheinboldt, W. *Iterative solution of nonlinear equations on several variables.* McGraw-Hill, New York, 1970.