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ESTIMATING A RANGE OF VALUES OF FUNCTIONS USING EXTENDED INTERVAL ARITHMETICS

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In this paper, we describe several methods of estimating a range of values of a function that use various generalizations of intervals. We also describe results that describe the accuracy of the resulting estimates.

ОЦЕНИВАНИЕ МНОЖЕСТВА ЗНАЧЕНИЙ ФУНКЦИЙ С ИСПОЛЬЗОВАНИЕМ ОБОБЩЕННЫХ ИНТЕРВАЛЬНЫХ АРИФМЕТИК

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В работе рассматривается оценивание множества значений функции с применением ее естественного расширения и различных обобщений понятия “интервал”. Приводятся результаты, касающиеся точности такого оценивания.

In this paper, we consider the problem of estimating the range of values of a function. The problem is as follows: We are given a function $f(x) : \theta \to \mathbb{R}$, $\theta \subseteq \mathbb{R}^n$ and a set X, $X \subseteq \mathbb{R}^n$, $X \in \Omega_1$ and we are interested in estimating the range of f, i.e., in finding a set $Y \subseteq \mathbb{R}$ such that

$$x \in X \Longrightarrow f(x) \in Y.$$

It is often required also that the set Y be of definite type, that is $Y \in \Omega_2$.

In different applications, the function f is specified differently. In this paper, we consider the case that f is given *analytically*, that is, in the form of a term composed by applying superposition to a given finite set of functions which we will call *basic*. The set of basic functions will be denoted $\mathbb{F} = \{f_1, f_2, \ldots, f_n\}$. The set of functions that can be obtained by such a composition will be denoted by $S(\mathbb{F})$.

We can consider different classes Ω_1. In the simplest case, the set Ω_1 is a set of n-dimensional intervals [1] for different n. In a more complicated case, intervals can be replaced by other sets (examples will be given later). Notice that, as a rule, no single set Ω_1 is considered, but a family of sets Ω^k, $\Omega^k \subseteq 2^{\mathbb{R}^k}$. In the estimation problem, it is usually assumed that $\Omega_2 = \Omega^1$. We will denote Ω_1 by Ω.

The algorithm that is used in interval computations to estimate the range of f can be generalized to the general case of arbitrary sets $\mathbb{F}$, Ω_1 as follows. To every basic function $f_i : \theta \to \mathbb{R}, \theta \subseteq \mathbb{R}^m$, its extension $F_i : \Omega^m \to \Omega$ is associated, satisfying the following condition: for any $x \in \theta$ and $X \in \Omega^m$,

$$x \in X \Longrightarrow f_i(x) \in F_i(X).$$

Next, in the term that specifies a function f, all basic functions f_i are replaced by corresponding extensions F_i. The resulting term describes the desired function F which guarantees estimating a range of values of the function f, since clearly the following condition holds:

$$x \in X \Longrightarrow f(x) \in F(X).$$

The function F obtained in this way is called a *natural extension* of the function f.

A range $\overline{f}(X)$ of a function $f(x)$ (that we are going to estimate) is defined as follows:

$$\overline{f}(X) = \{y \mid \exists\, x \in X \ y = f(x)\}.$$

The range is also called a *joint extension*.

When we use this method to estimate a range of a function, we encounter two types of difficulties.

First, it is not always possible to find reasonable extensions F_i for all basic functions $f_i \in \mathbb{F}$. In the simplest case of intervals, ranges $\overline{f}_i$ are usually taken as the desired extensions. However, this may not be possible for some functions $f_i \in \mathbb{F}$. For example, let Ω^i be a set of usual i-dimensional intervals. Consider the function $d(x_1, x_2) = x_1/x_2$. If $0 \in X_2$ then $\overline{d}(X_1, X_2) \notin \Omega$.

The second type of difficulties is caused by the fact that the above algorithm gives only an "exterior" estimate for the range, and often this estimate is an "overshoot".

In this paper, we are principally concerned with the first type of difficulty.

When we solve a problem of estimating the range of a given function f for a given set X, then we have the following choices.

- First, we can change (to some extent) the collection of basic functions $\mathbb{F}$. Even though the given function f is already described as a composition of some basic functions, we can apply appropriate equivalent transformations, and represent this very f as a composition of some other functions f_j. For example, we can thus eliminated undesirable basic functions by replacing them with a composition of other functions that are easier to estimate.
- Second, we can choose different extensions for basic functions.
- Finally, we can also change the classes Ω^i (that contain possible approximation sets).

In this paper, we will mainly exploit the third choice.

It is worth noting that these three choices are not independent. For example, the choice of a collection of basic functions influences the choice of approximating sets Ω^i which in turn, influences the choice of extensions of basic functions, and so on.

The main criterion for all these choices is the accuracy of the estimates thereby obtained. By an *accuracy*, we mean some numerical measure that describes the set $F(X) \setminus \overline{f}(X)$.

Let us illustrate this idea by examples.

We have already mentioned that in the simplest case the set Ω^n is a set of usual n-dimensional intervals [1], that is,

$$\Omega^n = \{([a_1, b_1], \ldots, [a_n, b_n]),\ a_i, b_i \in \mathbb{R},\ a_i \le b_i\}.$$

Usually, the set $\mathbb{F}$ of basic functions includes all four arithmetic operations. However, as we have already observed, in the case of (standard) intervals, the interval extension of division can't be defined if an interval divisor contains 0. For definitions of other arithmetical functions, as well as for elementary functions such as e^x, $\sin x$, $\cos x$ and $\arctan x$ there are no difficulties. Another example of a function for which interval extension is difficult to define is $\ln([a, b])$ for $a < 0$, $b > 0$.

In view of that, in interval computations, usually we consider a case when there are no divisions by an interval that contains 0, and no other problematic operations (like $\ln([a, b])$ for $a < 0 < b$) are applied.

For such cases, we can apply the above-described strategy, and thus compute $F(X)$. Let us denote $F(X) = [\alpha, \beta]$ and $\overline{f}(X) = [a, b]$. Then, according to the above result, $[a, b] \subseteq [\alpha, \beta]$. So, $F(X)$ is an estimate for $\overline{f}(X)$. As the accuracy of this estimate, it is natural to consider either the total length of the different set, i.e., the quantity $q = \alpha - a + b - \beta$, or the ratio of this quantity to the diameter of the interval X. The following theorem is well known (see, e.g., [1]): *Assume that a function f satisfies the Lipschitz condition. Then, there exists a constant k such that $q \le k\, wid(X)$, where $wid(X)$ denotes the size of an interval X.*

To overcome the fact that the set of usual intervals is not closed under these operations (division and ln), the following well known approach is used: to complete the set of standard intervals by adding infinite intervals. Different authors propose different generalizations of the concept of an interval. For example, one can add intervals infinite at one or both sides. A survey of different possibilities can be found in [2]. The most general case is when we take for the set Ω^n the following set:

$$\Omega^n = \left\{([a_1, b_1], \ldots, [a_n, b_n]),\ a_i, b_i \in \hat{\mathbb{R}}\right\},\ \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}. \qquad (1)$$

Here, by a nonstandard interval $[a, b]$, $a > b$, we mean a set $(-\infty, b] \cup [a, +\infty)$ and by intervals $[\infty, b], [a, \infty]$ and $[-\infty, \infty]$ we mean the sets $(-\infty, b], [a, +\infty)$ and $(-\infty, \infty)$, respectively.

It is easy to see that for $d(x_1, x_2) = x_1/x_2$, $D(X_1, X_2) \in \Omega$ for any X_1 and X_2. Apart from arithmetic functions, we can now all elementary functions into the set of basic functions, including such elementary functions as $\ln x$, $\sin x$, $\cos x$, $\tan x$, $\operatorname{cotan} x$. As a typical example, let us consider the function $e(x) = e^x$. As an interval extension $E(X)$ of this function one can't consider a joint extension since if $X = [a, b]$, $a > b$, then $e([a, b]) = (0, e^b] \cup [e^a, \infty) \notin \Omega$ (where Ω is the set of generalized intervals defined in (1)). Instead of the set $(0, e^b] \cup [e^a, \infty)$ one can consider as an extension either the generalized interval $[e^a, e^b]$, or the generalized interval $[0, \infty]$. The specific choice of one of them depends on the application. Depending on this choice, we will end up with one of the following definitions of interval extension for e^x:

$$E([a,b]) = \begin{cases} [e^a, e^b], & a, b \in \mathbb{R} \\ [0, e^b], & a = \infty, b \in \mathbb{R} \\ [e^a, \infty], & a \in \mathbb{R}, b = \infty \\ [0, \infty], & a, b = \infty \end{cases}$$

or

$$E([a,b]) = \begin{cases} [e^a, e^b], & a, b \in \mathbb{R}, a \le b\,. \\ [0, e^b], & a = \infty, b \in \mathbb{R} \\ [e^a, \infty], & a \in \mathbb{R}, b = \infty \\ [0, \infty], & a, b = \infty \text{ or} \\ & a, b \in \mathbb{R}, a > b\,. \end{cases}$$

It is also possible to have a "hybrid" definition, when for some intervals $[a, b]$, $a > b$, the extension is defined according to the first formula, and for other intervals, the second formula is used.

This example illustrates a general method to overcome the situation when the set of generalized intervals is not closed with respect to some basic function. In this case, we propose to extend the range of values of a basic function to a minimal (in some sense) generalized interval that contains this range.

Another possible method (that we will consider later on) consists of introducing so-called *multi-intervals.*

The accuracy of estimating a range of values of a function using generalized intervals is discussed in [3]. As the size of a generalized interval, one can't take its length, since for infinite intervals, length is always infinite. It is proposed instead to fix some monotone increasing function

$\tau: (-\infty,+\infty) \to (-1,1)$ (for example, $\tau(x) = \frac{2}{\pi}\arctan x$), and as a diameter of one-dimensional intervals, take the following quantity:

$$\overline{q}([a,b]) = \begin{cases} \tau(b)-\tau(a), & a,b\in\mathbb{R},\ b\geq a \\ \tau(b)-\tau(a)+2, & a,b\in\mathbb{R},\ b<a \\ 1-\tau(a), & a\in\mathbb{R},\ b=\infty \\ 1+\tau(b), & b\in\mathbb{R},\ a=\infty \\ 2, & a=b=\infty\ . \end{cases} \qquad (2)$$

As a *diameter* of a multivariate generalized interval, one can then take takes the largest of the diameters that correspond to each coordinate.

For above-described generalized intervals, one can prove an analogue of the above-given theorem (that the accuracy q of an interval estimate is limited by a expression $k\,wid(X)$ that is linear in the diameter $wid(X)$ of an interval X on which the estimate is performed). This analogue is given in [3]. Assume that we are estimating a function f on a generalized interval X. Let f satisfy the Lipschitz condition with respect to the diameter of the generalized interval introduced above. Then the estimating error for a range of values of the function is bounded by an expression that is linear in terms of the on the diameter of a (generalized) interval on which the estimation is done.

An alternative method of extending the range of values of a basic function consists of introducing *multi-intervals* [4]. By a multi-interval, we mean a union of a finite number of (standard or generalized) intervals. Practically for all the functions f that are used in applications, if $X_1,...,X_k$ are intervals (finite or infinite), then $f(X_1,...,X_k)$ is a multi-interval. Therefore, if $X_1,...,X_k$ are multi-intervals, the range will also be a multi-interval.

As a "diameter" of a multi-interval, it is natural to take the sum of diameters of its components. So, for an arbitrary function f, to estimate to what extent the resulting multi-interval $F(X)$ is a good estimate for an actual range, we can take the (thus defined) diameter of the set $F(X)$: the smaller this value, the better the approximation.

The main disadvantage of multi-intervals is the (relative) complexity of their implementation on a computer.

In conclusion we remark that there is yet another way to generalize the concept of interval, namely, as boundary points of an interval, one

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can take not arbitrary real numbers, but numbers representable on a particular computer; however this topic goes beyond the frameworks of this paper.

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