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INTERVAL CO-INTEGRATION OF DIFFERENTIAL EQUATIONS CONNECTED BY A SUBSTITUTION OF THE VARIABLE

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On each step of interval integration a mutual correction is executed: the interval extension of the solution of a given equation is intersected by the transformed interval extension of a connected equation solution and vice versa.

ИНТЕРВАЛЬНОЕ СОВМЕСТНОЕ ИНТЕГРИРОВАНИЕ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ, СВЯЗАННЫХ ЗАМЕНОЙ ПЕРЕМЕННЫХ

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На каждом шаге интервального интегрирования выполняется взаимная коррекция: интервальное расширение решения данного уравнения пересекается с трансформированным интервальным расширением связанного уравнения и наоборот.

The different versions of interval calculation of any number ω give different intervals $\Omega_i \ni \omega$. Thus, in general, their intersection determines a narrower interval for ω.

This property may be applied to the calculation of the initial value problem for first order ordinary differential equations

$$y' = f(x, y) \tag{1}$$

under initial condition

$$y(x) = y_0. \tag{2}$$

Suppose the equation

$$u' = g(x, u) \tag{3}$$

is connected with (1) so that for some function φ, and for each solution $y(x)$ of (1), the function

$$u(x) = \varphi(y(x)). \tag{4}$$

satisfies (3). Further suppose that φ is a continuous and strictly monotonic function that transforms $R =]-\infty, +\infty[$ into R, so that

$$|\varphi(x)| \to +\infty \qquad (x \to \pm\infty).$$

The interval calculation of $y(x)$ and $u(x)$ gives the intervals $Y(x_k) \ni y(x_k)$ and $U(x_k) \ni u(x_k)$ at points $x_1, x_2, \ldots$. Relation (4) provides the opportunity of using the corresponding intervals for mutual correction. Suppose the integration process comes to the point x_k. (We do not specify an integration method because our idea can be applied to any method.) Let us integrate equation (1) with initial point x_k; as a result we compute the interval $Y^0(x_{k+1}) \ni y(x_{k+1})$. The same process for (3) gives the interval $U^0(x_{k+1}) \ni u(x_{k+1})$. We then correct the first one:

$$Y(x_{k+1}) = Y^0(x_{k+1}) \cap \Phi^{-1}(U^0(x_{k+1})) \tag{5}$$

and the second one:

$$U(x_{k+1}) = U^0(x_{k+1}) \cap \Phi(Y^0(x_{k+1})). \tag{6}$$

Here Φ and Φ^{-1} denote the interval extensions for φ and φ^{-1} respectively. The intervals $Y(x_{k+1})$ and $U(x_{k+1})$ are the results of this step of interval co-integration.

This co-integration of equation (1) and (3) begins with the calculation $u_0 = \varphi(y_0)$ or the interval $U_0 = \Phi(Y_0)$. The algorithm also contains a self-check: the empty intersection indicates the program error.

Sometimes the interval $Y^0(x_{k+1})$ and $U^0(x_{k+1})$ may be unbounded. An example occurs when solutions $y(x)$ or $u(x)$ have singular points.

Additionally, unboundness may be caused by the nature of the integration formula, in particular when $\Delta x = x_{k+1} - x_k$ is too large. Suppose, for example, that $U^0(x_{k+1}) = R$. Then $\varphi^{-1}(U^0(x_{k+1})) = R$ and $Y(x_{k+1}) = Y^0(x_{k+1})$. In this case, we get no correction. In this situation it is desirable to use interval arithmetic with unlimited intervals.

The preceeding assumption on φ is not the only one possible: another very interesting case is $\varphi(x) = 1/x$. Then equation (3) is of the form

$$u' = u^2 f(x, 1/u).$$

In this case the two-sided approximation of the mobile (latent) singular point of (1) is available [1].

Now we consider the application of this mutual interval correction idea to computing a set $\{(x, y)\}$ which encloses the integral curve $y = y(x)$ at interval $X_k = [x_k, x_{k+1}]$. We name this set the enclosing set (ES).

The simplest estimate for this set is a rectangle $X_k \times Y^*$. Here $Y^* \supseteq Y(x)$ for all $x \in X_k$. The case when $Y(x)$ is unknown may be treated by taking $Y^* = R$.

In [2] a method is described of finding the ES based on fixing Δx and estimating Y^* by the interval iterative process

$$Y_{i+1}^* = Y(x_k) + H\,F(X_k, Y_i^*), \tag{7}$$

where $H = [0, \Delta x]$. The initial value of Y_i^* is taken as $Y_0^* = Y(x_k)$. If the mapping $F(x, y)$ is monotonic relative to Y, then, as was proven in [2], process (7) gives an increasing sequence of intervals $Y_0^* \subseteq Y_1^* \subseteq Y_2^* \subseteq \ldots$.

If this sequence is bounded, then the computer realization of (7) is terminated upon convergence, i.e., by the equality $Y_{n+1}^* = Y_n^*$.

But if the sequence Y_i^* is unbounded (or "practically unbounded") then process (7) leads to arithmetic overflow. If the computer realization of interval arithmetic deals with unbounded intervals, then the overflow is treated as $Y_n^* = R$. Thus process (7) will give $Y_{n+1}^* = R$ on the next iteration. Formally, we have the convergence again. Under certain conditions convergence is rapid [2].

Now we can estimate the ES based on the inclusion

$$Y(x) \subseteq Y(x_k) + (x - x_k)F(X_k, Y^*), \tag{8}$$

where $x \in X_k$. From this inclusion we see that the narrower interval $Y(x_{k+1})$ corresponds to narrower Y^*.

Now we consider the problem of the application of our mutual correction algorithm to the ES.

We denote $Y^{\&} = Y^* \cap \Phi^{-1}(U^*)$ for equation (1) and $U^{\&} = U^* \cap \Phi(Y^*)$ for equation (3). We substitute these intervals in place of Y^* and U^* in the estimate of the remainder of the integration formula. By means of this estimate, the interval extensions of the remainders for (1) and (3) are evaluated.

For example, if the main integration process is based on formula (8), i.e.,

$$Y(x_{k+1}) = Y(x_k) + \Delta x\, F(X_k, Y^*),$$

then the equalities

$$Y^0(x_{k+1}) = Y(x_k) + \Delta x\, F(X_k, Y^{\&}),$$
$$U^0(x_{k+1}) = U(x_k) + \Delta x\, G(X_k, U^{\&})$$

are used. Obviously, this part of the integration process is carried out for equations (1) and (3) separately.

Finally, the information obtained on ES gives the additional element of mutual correction. Instead of (5) and (6) narrower intervals are obtained:

$$U(x_{k+1}) = U^0(x_{k+1}) \cap \Phi(Y^0(x_{k+1})) \cap U^{\&}. \tag{9}$$
$$Y(x_{k+1}) = Y^0(x_{k+1}) \cap \Phi^{-1}(U^0(x_{k+1})) \cap Y^{\&}. \tag{10}$$

In closing, note that the iterative process can also be applied to finding $Y^{\&}$ and $U^{\&}$. In this case denote

$$Y^{\&}_{i+1} = Y^{\&}_i \cap \Phi^{-1}(U^{\&}_i), \qquad U^{\&}_{i+1} = U^{\&}_i \cap \Phi(Y^{\&}_i)$$

and put $Y^{\&}_0 = Y^*$, $U^{\&}_0 = U^*$.

It is interesting to explore this process when Φ and Φ^{-1} are optimal interval extensions for φ and φ^{-1}. Write

$$Y^*_2 = Y^*_1 \cap \varphi^{-1}(U^{\&}_1), \qquad \varphi^{-1}(U^{\&}_1) = \varphi^{-1}(U^* \cap \varphi(Y^*)).$$

Due to the continuity and strict monotonity of function φ,

$$\varphi^{-1}(U_1^{\&}) = \varphi^{-1}(U^*) \cap \varphi^{-1}(\varphi(Y^*)) = \varphi^{-1}(U^*) \cap Y^* = Y_1^*.$$

So, $Y_2^{\&} = Y_1^{\&}$ and this iterative process is not useful.

But generally, for a machine computation, Φ and Φ^{-1} are not optimal and this iterative process may lead to improvements in the accuracy of co-integration.

References

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