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SYNTHESIS OF OPTIMAL CONTROL UNDER INTERVAL UNCERTAINTY IN MODELS

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The problem of synthesis of optimal control of linear dynamic plants under interval uncertainty conditions is considered. It is required to find a control law ensured the extremum of the quadratic test of optimality for every realization of nonmeasurable and uncertain factor. The solution of such a problem is obtained: it represents a self=organizing regulator implemented the randomizing control law

СИНТЕЗ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ В УСЛОВИЯХ ИНТЕРВАЛЬНОЙ НЕОПРЕДЕЛЕННОСТИ В МОДЕЛЯХ

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Рассматривается задача синтеза оптимального управления линейными динамическими объектами в условиях интервальной неопределенности в моделях, в которой требуется найти закон управления, обеспечивающий экстремум квадратического критерия оптимальности для каждой реализации неизмеряемого и неопределенного фактора. Получено решение такой задачи, которое представляет самоорганизующийся регулятор, реализующий рандомизированный закон управления.

The initial formulation of an optimal control problem under interval uncertainty condition implies a search of such a control which is optimal one under every realization of uncertain factors. In the control theory, there is no method of obtaining the exact solution of this problem. Introducing those or others hypotheses about uncertain factors and treatments of the solution concept, one transforms usually this problem to others kinds of problems for which there are developed corresponding methods of a search of minimax, robust, pareto-optimal, dominant, statistically optimal, and adaptive controls [2]–[3]. Under realization of these solutions, they lead to losses of a control quality compared with the solution of the initial problem either because of uncoinciding assumptions about uncertain parameters with their real values or because of nonzero expenses of the time required for adjustment of an adaptive regulator. Therefore, all known approaches can be considered as indirect approximate methods of solving the initial problem. This paper deals with the direct method of solving the problem of synthesis of optimal control under interval uncertainty conditions.

Formulation of the problem. Let the description of a plant be given:

$$x(t) = A(t) \cdot x(t) + B(t) \cdot u(t), \quad t \in [0, t_k], \qquad x(0) = x^0 \tag{1}$$

where $x(t)$ is the n-dimensional state vector; $u(t)$ is the m = dimensional control vector; $A(t)$; $B(t)$ are, respectively, $(n \times n)$- and $(n \times m)$-matrices whose entries are continuous functions of time. Concerning to these matrices it is known that only the bounds of their variation are given as

$$A(t) \in [A^{\min}, A^{\max}] \quad t \in [0, t_k], \tag{2}$$

$$B(t) \in [B^{\min}, B^{\max}] \quad t \in [0, t_k]. \tag{3}$$

Also there is given the quadratic criterion of optimality

$$J = \int_0^{t_k} [x^T Q(t) x + u^T R(t) u] dt, \tag{4}$$

where $Q(t)$, $R(t)$ are, respectively, positive semi-definite $(n \times n)$-matrix and positive definite $(m \times m)$-matrix. It is requiredd to find the control law

$$u(t) = \varphi[x(t), t] \quad t \in [0, t_k], \tag{5}$$

providing the minimal value for the criterion of optimality (4) for every realization of matrices $A(t)$, $B(t)$ satisfying conditions (2), (3).

Method of solving. Let us denote the realization of the matrices $A(t)$, $B(t)$ satisfying conditions (2), (3) by $A^r(t)$, $B^r(t)$. If this realization of the matrices were given in advance, then the solution of problem (1)–(5) should be represented by the well-known regulator [4]

$$u^r(t) = \varphi^r[x(t), t] = -R^{-1}(t) \cdot [B^r(t)]^T \cdot S^r(t) \cdot x(t), \tag{6}$$

where symmetric $(n \times n)$-matrix $S^r(t)$ is the solution of the nonlinear matrix differential Riccati equation

$$\begin{aligned} - S^r(t) &= S^r(t) \cdot A^r(t) + [A^r(t)]^T \cdot S^r(t) - \\ &- S^r(t) \cdot B^r(t) \cdot R^{-1}(t) \cdot [B^r(t)]^T \cdot S^r(t) + Q(t) \end{aligned} \tag{7}$$

provided

$$S^r(t_k) = 0.$$

The regulation (6)–(8) we call ideal one. This regulator cannot be realized. According to problem (1)–(5), the realization of the matrices $A^r(t)$, $B^r(t)$ is not defined in advance — there is an information about only the bounds of the domain of their variation in form (2),(3). Any other regulator different from (6)–(8) lead to losses of control quality which can be measured. That value will be called the expenses on teaching of a regulator. We consider a set of k pairs of different matrices

$$\{A^i(t), B^i(t), \quad t \in [0, t_k], \; i = 1, 2, \ldots, k\}, \tag{9}$$

satisfying conditions (2), (3) and also conditions written as

$$\begin{aligned} \min_i \{ \max [&\max_{t\in[0,t_k]} \|A^i(t) - A^r(t)\|, \\ &\max_{t\in[0,t_k]} \|B^i(t) - B^r(t)\|, \quad i = 1, 2, \ldots, k\} \le \epsilon \end{aligned} \tag{10}$$

$$\begin{aligned} \min \{ &\min_{t\in[0,t_k]} \|A^i(t) - A^j(t)\|, \\ &\min_{t\in[0,t_k]} \|B^i(t) - B^j(t)\|\} > \epsilon, \quad \forall i \ne j \; (i = 1, 2, \ldots, k) \end{aligned} \tag{11}$$

for any admissible realization of the matrices $A^r(t)$, $B^r(t)$, where $\epsilon > 0$. As the result, we obtain the set of linearly quadratic (LQ) problems generated by the set of the matrices $A^i(t)$, $B^i(t)$ from (9). The solution of these problems can be represented as the set of control laws

$$M_\varphi = \{u^i(t) = \varphi^i[x(t), t], \quad t \in [0, t_k],\ i = 1, 2, \dots, k\} \tag{12}$$

having form (6)–(8) in which the matrices with the superscript i are used in place of the matrices with the superscript r. The set of problems of the LQ type generated by the set of k different pairs of the matrices of form (9) will be called equivalent to the initial problem (1)–(5) if for arbitrary unknown in advance matrices $A^r(t)$, $B^r(t)$ satisfying conditions (2), (3), there is a control law in set (12) ensuring the fulfillment of the condition

$$\begin{aligned} &\min\{\|(\varphi^i - \varphi^r) \cdot x(t)\|_R, \quad i = 1, 2, \dots, k\} \le \delta \\ &\forall t \in [t_\alpha, t_\beta] \subseteq [0, t_k],\ t_\alpha < t_\beta, \end{aligned} \tag{13}$$

where δ is a positive number. If control law (6)–(8) is indefinite since the realizations of matrices $A^r(t)$, $B^r(t)$ is not specified, then the control laws in set (12) are completely defined. We prescribe in advance the set of matrices (9)–(11) according to the information about the bounds of definition domain (2), (3) and given values of parameters ϵ, δ. For the realization on a plant, from set (12) of control laws, we shall choose that law whose number is the solution of the following problem of optimization:

$$\begin{aligned} &\sigma = \arg\min\{\|x^i(t_{\rho+1}) - x^r(t_{rho+1})\|_Q, \quad i = 1, 2, \dots, k\}, \\ &\rho = 0, 1, 2, \dots, N-1;\ t_\rho = \rho \cdot \Delta t;\ \Delta t = t_k/N, \end{aligned} \tag{14}$$

where $x^i(t)$ is the state of plant model (1) for $A(t) = A^i(t)$, $B(t) = B^i(t)$; $x^r(t)$ is the state of the real plant.

The regulator will be called self-organized if it realizes σ th control law from set (12) according to algorithm (14) at every time moment t_ρ ($\rho = 1, 2, \dots, N-1$) and at the moment $t = 0$, the control $u(t_0)$ is obtained from algorithm

$$u(t_0) = [u^{\max} - u^{\min}]/2, \tag{15}$$

where

$$\begin{aligned} u^{\max} &= \max\{u^i = \varphi^i[x(t), t]\big|_{t=0}, \quad i = 1, 2, \dots, k\}; \\ u^{\min} &= \min\{u^i = \varphi^i[x(t), t]\big|_{t=0}, \quad i = 1, 2, \dots, k\}; \end{aligned}$$

$\varphi^i[x(t), t]$ has form (6)–(8).

The uniqueness of the solution of problem (14) is provided by the choice of the value $u^{\,sigma}(t_\rho)$ from the simultaneous consideration of conditions (15) for $\rho = 0$ or (12) for $\rho = 1, 2, \ldots, N-1$ and the inequality

$$\Delta t \cdot \big|\, \|[A^i(t_\rho) - A^j(t_\rho)] \cdot x^r(t_\rho)\| - \|[B^i(t_\rho) - B^j(t_\rho)] \cdot u^\sigma(t_\rho)\| \,\big| > \delta,$$
$$\forall i \neq j \ (i = 1, 2, \ldots, k) \tag{16}$$

Definition. A nonnegative value ΔI^σ is considered as expenses on teaching the regulator; the value ΔI^σ is calculated from formulae

$$\Delta I^\sigma = I^\sigma - T^* \tag{17}$$

$$I^\sigma = \int_0^{t_k} \{\|x^\sigma\|_Q + \|u^\sigma\|_R\} dt, \tag{18}$$

$$I^* = \int_0^{t_k} \{\|x^*\|_Q + \|u^r\|_R\} dt, \tag{19}$$

where

$$\|x\|_Q = \sqrt{x^T Q x}; \quad \|u\|_Q = \sqrt{u^T R u};$$

$x^\sigma(t), x^*(t)$ is a state of plant (1) closed, respectively, by the self-organized and ideal (6)–(8) regulators.

Estimate of expenses on teaching the self-organized regulator. We analyse expenses on teaching a self-organized regulator. Let the teaching of the regulator be occured in time Δt. This assumption is not principial one and is necessary only for obtaining more simple relations allowing to illustrate obtained results more descriptive. Taking into account the property of additivity of functionals (18), (19) and applying any formula of numerical integration we get

$$\begin{aligned} \Delta I^\sigma =& [\|x^\sigma(\Delta t)\|_Q - \|x^*(\Delta t)\|_Q] \cdot \Delta t \\ &+ [\|u^\sigma(\Delta t)\|_R - \|u^r(\Delta t)\|_R] \cdot \Delta t. \end{aligned} \tag{20}$$

Replacing equation (1) by a finite-difference one according to the Euler cheme, for corresponding matrices $A(t)$, $B(t)$, we obtain the following relations:

$$\begin{aligned} ||x^\sigma(\Delta t) - x^r(\Delta t)||_Q &= ||[A^\sigma(\Delta t) - A^r(\Delta t)] \cdot x^0 \\ &+ [B^\sigma(\Delta t) - B^r(\Delta t)] \cdot u^\sigma(\Delta t)||_Q \cdot \Delta t \\ &\leq ||A^\sigma(\Delta t) - A^r(\Delta t)|| \cdot ||x^0||_Q \cdot \Delta t \\ &+ ||B^\sigma(\Delta t) - B^r(\Delta t)|| \cdot ||u^\sigma(\Delta t)||_R \cdot \Delta t; \end{aligned} \tag{21}$$

$$||x^r(\Delta t) - x^*(\Delta t)||_Q \leq ||B^r(\Delta t)|| \cdot ||u^\sigma(\Delta t) - u^r(\Delta t)||_R \cdot \Delta t. \tag{22}$$

Then with regard to (10), (11), (16), we get

$$\Delta I^\sigma \leq \{[||x^0||_Q + ||u^{\max}||_R] \cdot \Delta t^2 + ||B^{\max}|| \cdot \Delta t^2 + \Delta t\} \cdot \max(\sigma, \epsilon). \tag{23}$$

Therefore, the value of expenses on teaching a self-organized regulator does not exceed a value proportional to the value $\max(\sigma, \epsilon)$ of parameters given under design of a mechanism of self-organization. Analogously, one can obtain the estimate of an error of the solution of problem (1)–(5) in the self-organized regulator of the form

$$\Delta J^\sigma = J^\sigma - J^*, \tag{24}$$

where J^σ, J^* are the values of test of optimality (4), respectively, for a self-organized and ideal regulators. After corresponding transformations, we obtain

$$\begin{aligned} \Delta J^\sigma \leq \zeta^{\max}(\Delta t) &\cdot \{[||x^0||_Q + \max(\delta, \epsilon) + ||u^{\max}||_R] \cdot \Delta t^2 \\ &+ ||B^{\max}|| \cdot \Delta t^2 + \Delta t\} \times \max(\delta, \epsilon)\{||x^{\max}||_Q \\ &+ ||B^{\max}|| \cdot \max(\delta, \epsilon) \cdot \Delta t + ||x^0||_Q \cdot \max(\delta, \epsilon) \cdot \Delta t \\ &+ ||u^{\max}||_R \cdot \max(\delta, \epsilon) \cdot \Delta t + ||B^{\max}|| \cdot ||u^{\max}||_R \cdot \Delta t \\ &+ ||x^0||_Q \cdot ||u^{\max}|| \cdot \Delta t + ||u^{\max}||_R^2 \cdot \Delta t\} \cdot \max(\delta, \epsilon), \end{aligned} \tag{25}$$

where

$$\zeta^{\max}(\Delta t) = 2 \cdot (||x^{\max}||_Q + ||u^{\max}||_R).$$

It follows from the formula (25) that the error of the solution of synthesis problem (1)–(5) in a self-organized regulator does not exceed a finite value proportional to $\max(\sigma, \epsilon)$ of parameters given under design of a mechanism of self-organization. For $\Delta t \to 0$ and $\max(\sigma, \epsilon) \to 0$, a self-organized regulator provides a control quality coincided still more with that for an ideal regulator.

In other words, in the limit case, the self-organized regulator turned out to be invariant with respect to interval uncertainty under prescribing of plant models.

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