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A NEW APPLICATION OF INTERVAL MATHEMATICS

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A new application of the interval mathematics to physical theories is considered. Interval numbers form the two-dimensional pseudo-euclidian space. The interval Lorentz group is studied. The interval Klein-Gordon equation of a scalar field motion is obtained.

НОВОЕ ПРИМЕНЕНИЕ ИНТЕРВАЛЬНОЙ МАТЕМАТИКИ

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Рассматривается новое применение интервальной математики к физическим теориям. Интервальные числа образуют двумерное псевдоевклидово пространство. Изучается интервальная группа Лоренца. Получено интервальное уравнение Клейна-Гордона движения скалярного поля.

On the set of positive interval numbers,

$$I^+ = \{[a^-, a^+] \mid a^-,\ a^+ \in \mathbb{R},\ a^+ \geq a^- > 0\}$$

the usual addition and multiplication operations (see, for example, [1]) are written as

$$\begin{aligned} [a^-, a^+] + [b^-, b^+] &= [a^- + b^-, a^+ + b^+] \\ [a^-, a^+] \cdot [b^-, b^+] &= [a^- b^-, a^+ b^+]. \end{aligned} \tag{1}$$

Let us consider the operations (1) in the case, where an interval is defined by its center and semiwidth: $[a^-, a^+] = a \pm \Delta a$, where $a = \frac{a^+ + a^-}{2}$, $\Delta a = \frac{a^+ - a^-}{2}$.

Then we have

$$\begin{aligned} &I^+ = \{a \pm \Delta a | a, \Delta a \in \mathbb{R},\ a > \Delta a \geq 0\} \\ &(a \pm \Delta a) + (b \pm \Delta b) = (a + b) \pm (\Delta a + \Delta b) \qquad (2) \\ &(a \pm \Delta a) \cdot (b \pm \Delta b) = (ab + \Delta a \Delta b) \pm (a \Delta b + b \Delta a). \end{aligned}$$

The division of interval numbers is written as

$$\frac{a \pm \Delta a}{b \pm \Delta b} = \frac{ab + \Delta a \Delta b}{b^2 - \Delta b^2} \pm \frac{a \Delta b + b \Delta a}{b^2 - \Delta b^2} \qquad (3)$$

Formulas (2), (3) imply

1. $$\frac{a \pm \Delta a}{b \pm \Delta b} = \frac{(a \pm \Delta a)(b \pm \Delta b)}{b^2 - \Delta b^2},$$
   that is, the interval numbers division is reduced to their multiplication with subsequent division by a real number.
2. $$\frac{1}{b \pm \Delta b} = \frac{b \pm \Delta b}{b^2 - \Delta b^2},$$
   that is, the division of 1 by an interval number is reduced to the division by a real number.
3. If $b^2 - \Delta b^2 = 1$, then $\frac{1}{b \pm \Delta b} = b \pm \Delta b$.
4. Formulas (2), (3) resemble the multiplication and division of complex numbers, the difference is in signs at $\Delta a \Delta b$ and Δb^2.

Therefore, there is a possibility of introducing the structure of a pseudo-euclidian space on interval numbers [2].

Let us call the number

$$|a \pm \Delta a| = \sqrt{a^2 - \Delta a^2}$$

by the length of the interval number $a \pm \Delta a$.

The equations

$$\begin{aligned} &|(a \pm \Delta a)(b \pm \Delta b)| = |a \pm \Delta a| \cdot |b \pm \Delta b| \\ &|a \pm \Delta a| + |b \pm \Delta b| \leq |(a \pm \Delta a) + (b \pm \Delta b)| \end{aligned}$$

hold; second one is the reversed triangle inequality.

Every interval number of length 1 admits the writing in the form

$$a(v)=\frac{1}{\sqrt{1-v^2}}\pm\frac{v}{\sqrt{1-v^2}},\quad \text{where}\quad v<1. \tag{4}$$

The multiplication by this number looks formally as the Lorentz transformation:

$$(t\pm x)\cdot a(v)=\frac{t+xv}{\sqrt{1-v^2}}\pm\frac{x+tv}{\sqrt{1-v^2}}.$$

The multiplication of two numbers of form (4) corresponds to the relativistic rule of the velocity addition

$$a(v_1)\cdot a(v_2)=a(v),$$

where $v=\frac{v_1+v_2}{1+v_1v_2}$.

It is possible to write the interval number $a\pm\Delta a$ in the hyperbolic form

$$a\pm\Delta a=r(\cosh\alpha\pm\sinh\alpha),$$

where $r=|a\pm\Delta a|$, $\tanh\alpha=\frac{\Delta a}{a}$, $\cosh\alpha,\sinh\alpha,\tanh\alpha$ are the hyperbolic cosine, sine, tangent.

The multiplication of two interval numbers in the hyperbolic form is proceeded according to the formula, analogous to the Moivre formula for complex numbers

$$(r_1(\cosh\alpha\pm\sinh\alpha))\cdot(r_2(\cosh\beta\pm\sinh\beta))=r_1r_2(\cosh(\alpha+\beta)\pm\sinh(\alpha+\beta)).$$

The analogy of the Euler formula

$$e^{\pm\alpha}=\cosh\alpha\pm\sinh\alpha$$

is true.

Then we consider the set

$$I=\{a\pm\Delta a\mid a,\ \Delta a\in\mathbb{R}\}$$

with addition and multiplication operations (2).

Let us define the interval number, intervally conjugate to the interval number $a \pm \Delta a$:

$$\overline{a \pm \Delta a} = a \pm (-\Delta a).$$

By analogy with the definition of the Hermitian scalar product in the unitary space we shall introduce the scalar product of two interval numbers by the formula

$$(a \pm \Delta a, b \pm \Delta b) = (a \pm \Delta a) \cdot \overline{(b \pm \Delta b)} = (ab - \Delta a \Delta b) \pm (-a\Delta b + b\Delta a). \quad (5)$$

Then we note that the extension of operations (2) from the set I^+ to the set I leads to the distinguishing the obtained operations from the usual ones [1] for the intervals which contain zero. Doing so we have in mind the further applications to physical theories, where the homogeneity of the space-time coördinates with respect to shifts and rotations leads to the presence of the conservation laws.

As the hypothesis, let as assume that breaking operations (2) near zero in the interval mathematics corresponds to the Heisenberg uncertainty principle.

We consider the four-dimensional vector space V over I with the scalar product

$$((t_1, x_1, y_1, z_1), (t_2, x_2, y_2, z_2)) = (t_1, t_2) - (x_1, x_2) - (y_1, y_2) - (z_1, z_2),$$

where the scalar product of two interval numbers is defined by equations (5).

Theorem. [3] *The group* $\mathrm{O}(1, 3, I)$ *of the isometries of the space* V *is the 16-parametric group isomorphic to the group* $\mathrm{GL}(4, \mathbb{R})$.

Corollary. *There are 16 real conservation laws, from which 12 correspond to 6 interval laws of the conservation of angular momentum and 4 do not have the analogy in the classic mechanics.*

Theorem. [3] *A movement of a closed system is characterized by the conservation of the following quantities:*

$$\sum (E\Delta t - t\Delta E); \ \sum (p_x \Delta x - x \Delta p_x);$$
$$\sum (p_y \Delta y - y \Delta p_y); \ \sum (p_z \Delta z - z \Delta p_z);$$

Here the summation is carried over all particles of the system, $E \pm \Delta E$ is the energy of the particle, $p \pm \Delta p$ is the momentum of the particle.

The physical space-time is a 4-dimensional space. The space V have the dimension 4 over the set I, but the dimension of V over $\mathbb{R}$ is 8, the space V has a signature $(+ \; - \; - \; + \; - \; + \; - \; +)$. For the construction of physical theories, we use the notions of fractional differentiation and integration (see, for example, [4], §7), assigning the dimension 1/2 to every real coordinate of the space V.

Applying the developed computational technique to the field theory we have a possibility to write the basic concepts of the theory, for example the Lagrangian and the motion equations, in two forms:

– in 4-dimensional space over I with the fractional differentiation and integration;

– in 8-dimensional space over $\mathbb{R}$ with the usual differentiation and integration;

As the example we consider the Klein-Gordon equation [4, 5]

$$(\Box - m^2)\varphi(x) = 0$$

for the real scalar field; here $\Box$ is the 4-dimensional D'Alembert operator.

Theorem. *The equation of the interval scalar field motion is written as*

$$(\Box^3 - m^2)\varphi(x) = 0$$

where $\Box$ is the 8-dimensional D'Alembert operator.

Corollary. *In the interval quantum electrodynamics, there are no familiar divergences of the classical quantum electrodynamics.*

For example, the convergence integral in interval electrodynamics

$$I(k) = \int \frac{d^8 p}{(m^2 - p^6 - i\epsilon)(m^2 - (p-k)^6 - i\epsilon)}$$

corresponds to the simplest divergence integral in the quantum electrodynamics

$$I(k) = \int \frac{d^4 p}{(m^2 - p^2 - i\epsilon)(m^2 - (p-k)^2 - i\epsilon)}.$$

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