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INTERVAL ESTIMATES FOR CLOSURE-PHASE AND CLOSURE-AMPLITUDE IMAGING IN RADIO ASTRONOMY

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Closure-phase and closure-amplitude imaging are methods for reconstructing a radioimage from the results of approximate measurements. If we know the measurements precision (i.e., the intervals for the measured values), what is the precision of the result of this reconstruction? This problem cannot be solved by standard interval methods because one of the measured quantities, the phase, takes its values on a circle, not on a real line.

In the present paper we give the desired estimates. The basic result is that if we measure the phase $\theta(\vec{x})$ with precision ϵ, so that the closure phase $\theta(\vec{x})+\theta(\vec{y})-\theta(\vec{x}+\vec{y})$ is known with precision 3ε, then from these measurements we can reconstruct θ with precision 6ε. Similar estimates are given for closure amplitude.

ИНТЕРВАЛЬНЫЕ ОЦЕНКИ ДЛЯ МЕТОДОВ ЗАМЫКАНИЯ ФАЗЫ И ЗАМЫКАНИЯ АМПЛИТУДЫ В РАДИОАСТРОНОМИИ

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Методы замыкания фазы и замыкания амплитуды – это методы для восстановления радиообраза по результатам приближенных измерений. Если нам известна точность измерений (т.е. интервалы для измеренных значений), то какова точность результата восстановления? Эта задача не может быть решена стандартными интервальными методами из-за того, что одна из измеряемых величин – фаза, свои значения принимает на окружности, а не на

вещественной прямой.

В настоящей работе мы получаем искомые оценки. Основной результат следующий. Если мы измеряем фазу $\theta(\vec{x})$ с точностью ε, и тем самым замыкание фазы известно нам с точностью 3ε, то из этих измерений мы можем восстановить θ с точностью 6ε. Аналогичные оценки получены для замыкания амплитуды.

1. Brief introduction to the physical problem

The specific feature of very distant objects is that, although they may be physically large, due to the enormous distance their angular (visible) size on the sky is extremely small. For example, quasars have details of milliarcsecond size (0.001 of an arc second). Normal telescopes are unable to see such tiny details, because their *angular resolution* $\Delta\phi$ (ability to distinguish nearby points) is limited. A good approximation to this resolution is given by: $\Delta\phi \approx \lambda/D$, where λ is the wavelength of interest and D is the diameter of the telescope. Thus, in order to improve the angular resolution, it is necessary to increase D. But to distinguish details of distant radio sources ($\lambda \approx 21$cm) we need D equal to several thousand kilometers. Of course it is technically impossible to build such a big telescope.

We can overcome this difficulty if we take into consideration the fact that in actual antennas the signal is received by several parts and we observe the superposition of the signals received by different parts. So, although we cannot build a single large antenna, we can simulate one if we place several antennas at different locations, collect the signals, send them to one place and there simulate the superposition by using a computer. This is called a Very Long Baseline Interferometry (VLBI).

As a result of this simulation, we have a signal that simulates what we would have received from a single large antenna. This signal is sinusoidal (with frequency equal to the frequency of observations) and can therefore be characterized by its amplitude and phase. These values are related to the function that describes the brightness $I(\vec{x})$ of the source that we are observing at point $\vec{x}$ (this function is called an *image*, or a *brightness distribution*). In the idealized situation, when we neglect noise and measurement error, the measured amplitude is proportional to the absolute value $A(\vec{b}) = |F(I)(\vec{b})|$ of the Fourier transform $F(I)$ of the desired image I, where $\vec{b}$ is equal to the projection of the vector $\vec{R}$ between two antennas onto the plane that is perpendicular to the source. Likewise the phase

$\theta(\vec{b})$ equals the phase of the (complex) Fourier transform. So in this case, if we observe the same source on different pairs of antennas with different $\vec{b}$, we obtain both the amplitude A and the phase θ of $F(I)$, and thus reconstruct the complex value of $F(I)$ for every $\vec{b}$ as $A\exp(i\theta)$. By applying the inverse Fourier transform, we determine the desired image I.

The above description applies only to the simplified case when we can neglect the errors. This is the case with bright sources. If we now try to analyze weak sources (e.g., the distant ones) for which the signal is much smaller, we cannot neglect the errors any longer. One of the methods to avoid the errors is called the *closure phase* method. Suppose that we observe a source simultaneously on 3 different antennas, whose coordinates in the plane that is perpendicular to the source are $\vec{r}_A, \vec{r}_B$ and $\vec{r}_C$. Ideally as a result of these observations we would obtain the three values $\theta(\vec{r}_A - \vec{r}_B)$, $\theta(\vec{r}_B - \vec{r}_C)$ and $\theta(\vec{r}_A - \vec{r}_C)$. Crudely speaking, these values correspond to the differences of phase ϕ of the signals received by these antennas; e.g., $\theta(\vec{r}_A - \vec{r}_B) = \phi_A - \phi_B$. On the path to each of these antennas there are variations that alters these phases ϕ. So, for example, antenna A actually receives the phase $\phi_A + n_A$, where n_A is caused by travel effects. As a result instead of $\phi_A - \phi_B = \theta(\vec{r}_A - \vec{r}_B)$ we have the altered value $(\phi_A+n_A)-(\phi_B+n_B) = \theta(\vec{r}_A-\vec{r}_B)+n_A-n_B$. So we measure three values $m_{AB} = \theta(\vec{r}_A - \vec{r}_B) + n_A - n_B$, $m_{BC} = \theta(\vec{r}_B - \vec{r}_C) + n_B - n_C$ and $m_{AC} = \theta(\vec{r}_A - \vec{r}_C) + n_A - n_C$. Since we do not know the values of n_A, n_B, n_C, each of these measured results gives no information about the phase. But what we can do is we can compute a sum of these three values that is free from such effects: $m_{AB} + m_{BC} - m_{AC} = \theta(\vec{r}_A - \vec{r}_B) + \theta(\vec{r}_B - \vec{r}_C) - \theta(\vec{r}_A - \vec{r}_C)$. This expression has the form $\theta(\vec{x}) + \theta(\vec{y}) - \theta(\vec{x} + \vec{y})$, where $\vec{x} = \vec{r}_A - \vec{r}_B$ and $\vec{y} = \vec{r}_B - \vec{r}_C$. This combination is called the *closure phase* (this idea appeared first in different radioimage processing problems [Jennison 1953, 1958] and was applied to VLBI in [Rogers et al 1974]). If we have a sufficiently dense net of antennas, then we can measure the closure phase for all $\vec{x}$ and $\vec{y}$, and then reconstruct θ and I.

For a current state of closure phase methods see [Perley et al 1989] (especially Chapters 9 and 19).

Closure phase eliminates only additive errors, but there are other types of errors that are not eliminated. Therefore we can measure the phase on each of the antennas only approximately (with some precision ε), and thus we compute the closure phase with a precision 3ε. The main problem is: *what is the resulting error in the reconstructed phase?*

This problem is solved for the case when we know the statistical characteristics of the errors [Lannes 1989, 1989a]; however, in reality we seldom know them. The only thing that we do know about the errors is the upper limit that is guaranteed by the manufacturer of the measuring device or by authors of the measuring procedure. In other words, the only thing that we know is the interval of possible values of an error. For this case the problem of how to estimate the resulting error in the image is still open.

This is a challenge for interval mathematics, because, first, it is an important problem and, second, traditional methods of error estimate (see, e.g., [Kreinovich et al 1991]) are not applicable, because these methods are applicable to the intervals of the real line, and here we have intervals on a *circle* (where the phases are defined).

We are going to meet this challenge in the present paper.

Similar ideas can be applied to diminish the error in the amplitude [Perley et al 1989]. For two antennas that are located in the points $\vec{r}_A$ and $\vec{r}_B$ the amplitude $A(\vec{r}_A-\vec{r}_B)$ is computed as a correlation between the signals received by these antennas. So, crudely speaking, an amplitude is proportional to the product of the signals: $A(\vec{r}_A-\vec{r}_B)=k_Ak_B$. These signals are extremely weak, so they have to be amplified. Even the best amplifiers have unpredictable fluctuations of their amplifying coefficient. So instead of k_Ak_B we actually measure the value $m_{AB}=k_Ak_Bg_Ag_B$, where g_A corresponds to this unknown change in amplification. As in the case of phases, if we observe the same source by four antennas, we can form a combination that contains no g: $m_{AC}m_{BD}/(m_{AB}m_{CD})$. This combination is called the *closure amplitude*. If we denote $\vec{x}=\vec{r}_A-\vec{r}_B$, $\vec{y}=\vec{r}_B-\vec{r}_C$, $\vec{z}=.\vec{r}_B-\vec{r}_D$, then we can express the closure amplitude as $A(\vec{x}+\vec{y})A(\vec{z})/(A(\vec{x})A(\vec{y}-\vec{z}))$.

For closure amplitude several problems occur that are similar to the problems of closure phase: this trick does not eliminate all possible errors; so the measured amplitude and hence closure amplitude contain some error. What is the resulting error in reconstructing A? An additional difficulty in this case is that since the errors are mainly multiplicative, the precision with which we measure A is not constant, but depends on the value that we are measuring: namely, it is proportional to the value of A. For example, if we know that the measured value $\tilde{A}$ equals gA, where $0.95 \le g \le 1.0$, then the maximum possible error $|\tilde{A} - A|$ is $0.05A$.

2. Closure-phase imaging: formulation of a mathematical problem and main results

2.1. Basic definitions

Definition 1 (image). By an *image*, or a *brightness distribution*, we mean a function $I(\vec{x})$ from the plane R^2 into the set of real numbers. The absolute value $|FT(I)|$ of its Fourier transform will be denoted by A and called an *amplitude function*, and its phase will be denoted by $\theta(\vec{x})$ and called a *phase function* of that image I. So $FT(I) = A(\vec{x})\exp(i\theta(\vec{x}))$, where A is an even function and θ is odd: $A(-\vec{x}) = A(\vec{x})$ and $\theta(-\vec{x}) = -\theta(\vec{x})$.

Comments.

1. If we know both the amplitude and the image functions, then we can reconstruct the image $I(\vec{x})$ by applying the inverse Fourier transform FT^{-1} to $A(\vec{x})\exp(i\theta(\vec{x}))$.

2. The phase is defined only modulo 2π.

Definition 2 (circle). By T we'll denote a circle with unit radius, i.e., the set of values $[0, 2\pi]$ where equality is defined so that $0 = 2\pi$, and the sum $A + B$ and difference $A - B$ of two values from T are defined as $A+B$ *mod* 2π and $A-B$ *mod* 2π. For every real number x by $p(x)$ we'll denote an element x *mod* 2π of T.

Example.If π and $3/2\pi$ are two elements of T, then their sum in T equals $1/2\pi$: first we add these values and obtain $5/2\pi$, then we divide the result by 2π and take the remainder $1/2\pi$.

Comment. This mapping p is a *homomorphism* in the sense that $p(x + y) = p(x) + p(y)$, $p(-x) = -p(x)$ and $p(0) = 0$.

Denotations. Elements of T will be denoted by capital letters, and real numbers by small ones.

Geometrically the values of T correspond to the rotations of a unit circle. A rotation by 2π radians leaves the circle intact, therefore $2\pi = 0$ in T. $+$ corresponds to the composition of two rotations, and $-A$ is a rotation in the opposite direction. Another possible representation is when we fix some point P on a circle and represent A by the point into which P turns when we rotate the circle by an angle A. This representation enables us to define the distance $\rho(A, B)$ on T as the length of the shortest of the two arcs that connect A and B. For example, for $A = 0.5$ and $B = 1$ the distance equals to 0.5 because the shortest of the two arcs is going from 0.5 to 1, and for $\pi/4$ and $3/2\pi$ the distance is $3/4\pi$, because the shortest arc goes from $3/2\pi$ through $2\pi = 0$ to $1/4\pi$.

Definition 3 (distance). By a *distance* $\rho(A, B)$ between the two values $A, B \in T$ we mean the value $\min(|A - B|, |A + 2\pi - B|, |B + 2\pi - A|)$.

Comments.

1. The maximum value of the distance is π, when the points are on opposite sides of the circle. If $d(A, B) \leq \varepsilon$, this means that we can get B from A by adding a value Δ from the interval $[-\varepsilon, \varepsilon]$.

2. It is easy to prove that this distance satisfies the usual properties of a distance: symmetry $\rho(A, B) = \rho(B, A)$, triangle inequality $\rho(A, C) \leq \rho(A, B) + \rho(B, C)$.

3. The relationship between this distance and the distance on a real line is simple: $\rho(p(x), p(y)) = \min_n |x - y - 2\pi n|$, where n runs over all integers. This means, on one hand, that for every two numbers x, y the distance between $p(x)$ and $p(y)$ is either the same as the distance between x and y (if the path from x to y is the shortest) or smaller, that is, $\rho(p(x), p(y)) \leq |x - y|$. On the other hand, it means that for every x, y there exists an integer n such that $|x - y - 2\pi n| = \rho(p(x), p(y))$.

We have mentioned that the phase θ is defined only modulo $2\pi n$ for an integer n. However, since an image is normally located in a bounded region, its Fourier transform is an analytical function, and therefore its phase $Im(\log(FT(I))$ is also analytical. So we can consider $\theta(\vec{x})$ to be a continuous function from R^2 to the set of real numbers R. So we arrive at the following definition.

Definition 4 (phases). By a *phase function* we mean a continuous odd function $\theta(\vec{x})$ from the plane R^2 into real numbers. By a *phase*, that corresponds to a phase function, we mean a function $\Theta(\vec{x}) = p(\theta(\vec{x}))$. By a *closure phase*, that corresponds to a phase function θ, we mean a function $\Phi(\vec{x}, \vec{y}) = \Theta(\vec{x}) + \Theta(\vec{y}) - \Theta(\vec{x} + \vec{y})$.

2.2. Closure-phase imaging in case of precise measurements

First we are going to describe to what extent closure phase determines a phase function. These results are in essence already known in closure-phase imaging, but we have added precise mathematical formulations and proofs because we'll use them to compute the error estimates. (All proofs are given in Section 4).

Proposition 1. *If two phase functions $\theta(\vec{x})$ and $\theta'(\vec{x})$ correspond to the same closure phase, then $\theta'(\vec{x}) = \theta(\vec{x}) + \vec{k}\vec{x}$ for some constant vector $\vec{k}$. Vice versa, if $\theta'(\vec{x}) = \theta(\vec{x}) + \vec{k}\vec{x}$ for some constant vector $\vec{k}$, then the corresponding closure phases coincide.*

Comment. In other words, closure phase determines phase function uniquely modulo the linear function. This non-uniqueness has a simple interpretation in terms of the image itself. Suppose that we know the amplitude $A(\vec{x})$. Then we can get an image $I(\vec{x})$ by applying the inverse Fourier transform to $A(\vec{x})\exp(i\theta(\vec{x}))$.

Proposition 2. *If $\theta'(\vec{x}) = \theta(\vec{x}) + \vec{k}\vec{x}$ for some constant vector $\vec{k}$, $A(\vec{x})$ is an amplitude function, $I(\vec{x}) = FT^{-1}(A(\vec{x})\exp(i\theta(\vec{x})))$ and $I'(\vec{x}) = FT^{-1}(A(\vec{x})\exp(i\theta'(\vec{x})))$, then $I'(\vec{x}) = I(\vec{x} + \vec{a})$ for some constant vector $\vec{a}$. Vice versa, if $I'(\vec{x}) = I(\vec{x} + \vec{a})$ for some constant vector $\vec{a}$, then for the corresponding phase functions $\theta'(\vec{x}) = \theta(\vec{x}) + \vec{k}\vec{x}$ for some constant vector $\vec{k}$.*

Comment. So the non-uniqueness of image reconstruction in the closure-phase imaging really means that from the closure phase we can reconstruct an image only modulo its absolute position; when we shift the image as a whole and thus change its coordinates, the closure phase does not change. In view of that it is reasonable to give the following definition.

Definition 5. We say that two phase function $\theta(\vec{x})$ and $\theta'(\vec{x})$ are *equivalent* if the difference between them is a linear function of $\vec{x}$. We say that two images $I(\vec{x})$ and $I'(\vec{x})$ are *equivalent* if $I'(\vec{x}) = I(\vec{x} + \vec{a})$ for some constant vector $\vec{a}$.

In these terms the above Propositions may be reformulated as follows:

Proposition 1′. *Two phase functions correspond to the same closure phase if and only if they are equivalent.*

Proposition 2′. *Two images $I(\vec{x})$ and $I'(\vec{x})$ are equivalent if and only if* *their amplitude functions coincides and their phase functions are equivalent.*

2.3. Closure-phase imaging in case of real measurements

Definition 6. Suppose a positive real number ε is fixed. We'll call it a *precision*. Suppose also that a phase function $\theta(\vec{x})$ is given, and $\Phi(\vec{x},\vec{y})$ is a corresponding closure phase. By a *result of measuring its closure phase with precision* ε we mean a function $\tilde{\Phi}(\vec{x},\vec{y})$ that satisfies the inequality $\rho(\tilde{\Phi}(\vec{x},\vec{y}),\Phi(\vec{x},\vec{y})) \leq 3\varepsilon$.

Comments.

1. This inequality has 3ε in the right-hand side, because in order to measure the closure phase one needs to measure three phases and add them. Since the maximum possible error resulting from measuring each of the three phases is ε, the maximum possible error of this sum is 3ε.

2. For each $\vec{x}$ and $\vec{y}$ the set of possible values of $\tilde{\Phi}(\vec{x},\vec{y})$ form an interval, but an interval on the circle. If only these measurements results $\tilde{\Phi}(\vec{x},\vec{y})$ are available, then the only thing that we know about the actual phase function θ is that it satisfies the inequality $\rho(\tilde{\Phi},\Phi) \leq 3\varepsilon$. Of course, there can be several functions θ that satisfy this inequality, and the measurements do not allow us to choose between them. We arrive at the following definition.

Definition 7. Suppose that a phase function $\theta(\vec{x})$ is given. We'll call it an *actual phase function*. Suppose also that $\tilde{\Phi}$ is a result of measuring its closure phase with precision ε. We say that a phase function $\theta'(\vec{x})$ is *reconstructed from the approximate closure phase* if $\rho(\tilde{\Phi}(\vec{x},\vec{y}),\Phi'(\vec{x},\vec{y})) \leq 3\varepsilon$, where Φ' is a closure phase corresponding to θ'.

2.4. Main problem

The main problem is: *how close is the result θ' of reconstructing the phase from approximate measurements to the actual phase function $\theta(\vec{x})$?*

2.5. Why is this problem difficult to solve?

This problem sounds like a problem of interval mathematics (see, e.g., [Moore 1966], [Moore 1979], [Alefeld et el 1983], [Ratschek et al 1984]): we have an algorithm (actually several different algorithms are known [Perley et al 1989]) that transforms the approximate values of the closure phase into a phase function; we know the intervals of possible values of the closure phase, and we need to know the interval of possible values of the result. However, for the following reasons it is difficult to apply the usual methods of interval mathematics to this problem:

1) All these intervals are the intervals on the circle, not on a real line, as in the usual interval mathematics.

2) Interval mathematics usually gives an estimate for some given algorithm. Although we are interested in the precision of some already existing algorithm, but we are also interested in the principal possibility of reconstructing the phase. So if there is an algorithm that would give the result with the better precision than all the existing methods, we would like to know that.

Since we cannot directly apply the standard methods, we must invent something new.

2.6. Solution

Theorem 1. *Suppose that $\theta(\vec{x})$ is the actual phase function, precision is $\varepsilon < \pi/6$ and $\theta'(\vec{x})$ is reconstructed from the approximate closure phase. Then there exists a function θ'' that is equivalent to θ and such that $|\theta''(\vec{x}) - \theta'(\vec{x})| \leq 6\varepsilon$ for all $\vec{x}$.*

Comments.

1. So closure-phase imaging allows us in principle to reconstruct the phase function with precision 6ε, where ε is the precision with which we measure the phases. This result is better than the previous known estimate 9ε [Kosheleva et al 1978].

2. The restriction $\varepsilon < \pi/6$ is really not very restrictive, because in actual measurements the precision is much better than $\pi/6 \approx 10^o$.

3. The following Theorem shows that 6ε is the best possible precision estimate in the sense that one cannot reconstruct the phase $\theta(\vec{x})$ with better precision:

Theorem 2. *Suppose that $\theta(\vec{x})$ is any phase function, and $0 < \varepsilon < \pi/6$. Then there exists a function $\theta'(\vec{x})$ that is reconstructed from an $\varepsilon-$approximate closure phase, such that for any function $\theta''(\vec{x})$ that is equivalent to $\theta(\vec{x})$, we have $\sup_{\vec{x}} |\theta''(\vec{x}) - \theta'(\vec{x})| \geq 6\varepsilon$.*

In other words, there exists a result $\tilde{\Phi}(\vec{x}, \vec{y})$ of measuring the closure phase with precision ε, and a function $\theta'(\vec{x})$ that can be reconstructed from this approximate closure phase $\tilde{\Phi}(\vec{x}, \vec{y})$ such that:

- this function $\theta'(\vec{x})$ differs from $\theta(\vec{x})$ by $\geq 6\varepsilon$ for some $\vec{x}$, and
- if we take any phase function $\theta''(\vec{x})$ that is equivalent to the actual phase $\theta(\vec{x})$, then $\theta''(\vec{x})$ will also differ from the reconstructed phase function $\theta'(\vec{x})$ by at least 6ε for some $\vec{x}$.

3. Precision of closure-amplitude imaging

3.1. Closure-amplitude imaging in case of precise measurements

Definition 8. By an *amplitude function* we mean a continuous even function $A(\vec{x})$ from the plane R^2 into the set of positive real numbers. By a *closure amplitude* that corresponds to an amplitude function A, we mean a function $C(\vec{x},\vec{y},\vec{z}) = A(\vec{x}+\vec{y})A(\vec{z})/(A(\vec{x})A(\vec{y}-\vec{z}))$.

Comment. This combination is possible only when A if different from 0, therefore we consider only the cases when the amplitude function never attains 0 and is thus always positive. For other functions closure amplitude methods are not applicable (at least not directly applicable). This restriction does not really limit our possibilities because, if for some actual image $A(\vec{x}) = 0$ for some $\vec{x}$, then by making an arbitrarily small perturbation (that will not be detectable by our measurement instruments because there is always a limitation of precision), we can always come to an image that leads to the same observational data and already has no 0 at this point $\vec{x}$.

Proposition 3. *If two amplitude functions $A(\vec{x})$ and $A'(\vec{x})$ correspond to the same closure amplitudes, then $A'(\vec{x}) = kA(\vec{x})$ for some constant k. Vice versa, if $A'(\vec{x}) = kA(\vec{x})$ for some constant k, then the corresponding closure amplitudes coincide.*

Comment. In other words, closure amplitude determines the amplitude function uniquely modulo a multiplicative constant. This non-uniqueness has a simple interpretation in terms of the image itself. Namely, suppose that we know the phase $\theta(\vec{x})$. Then we can obtain an image $I(\vec{x})$ by applying the inverse Fourier transform to $A(\vec{x})\exp(i\theta(\vec{x}))$.

Proposition 4. *If $A'(\vec{x}) = kA(\vec{x})$ for some constant k, $\theta(\vec{x})$ is a phase function, $I(\vec{x}) = FT^{-1}(A(\vec{x})\exp(i\theta(\vec{x})))$ and $I'(\vec{x}) = FT^{-1}(A'(\vec{x})\cdot \exp(i\theta(\vec{x})))$, then $I'(\vec{x}) = kI(\vec{x})$. Vice versa, if $I'(\vec{x}) = kI(\vec{x})$ for some constant k, then for the corresponding amplitude functions $A'(\vec{x}) = kA(\vec{x})$.*

Comment. So the non-uniqueness of image reconstruction in the closure-amplitude imaging really means that from the closure amplitude we can reconstruct an image only modulo its absolute value, because I and kI lead to the same closure amplitude. In view of this it is reasonable to give the following definition.

Definition 9. We say that two amplitude function $A(\vec{x})$ and $A'(\vec{x})$ are *equivalent* if their ratio is a constant: $A'(\vec{x}) = kA(\vec{x})$. We say that two images $I(\vec{x})$ and $I'(\vec{x})$ are *amplitude- equivalent* if $I'(\vec{x}) = kI(\vec{x})$ for some constant k.

In these terms the above Propositions may be reformulated as follows:

Proposition 3$'$. *Two amplitude functions correspond to the same closure amplitude if and only if they are equivalent.*

Proposition 4$'$. *Two images $I(\vec{x})$ and $I'(\vec{x})$ are amplitude-equivalent if and only if their phase functions coincide and their amplitude functions are equivalent.*

3.2. Closure-amplitude imaging in case of real measurements

Definition 10. Suppose a positive real number δ is fixed. We'll call it a *relative precision of amplitude measurement*, or a *precision* for short. Suppose also that an amplitude function $A(\vec{x})$ is given, and $C(\vec{x}, \vec{y}, \vec{z})$ is a corresponding closure amplitude. By a *result of measuring the closure amplitude with precision* δ we mean a function $\tilde{C}(\vec{x}, \vec{y}, \vec{z})$ that satisfies the following inequality:

$$\frac{(1-\delta)^2}{(1+\delta)^2} \le \frac{\tilde{C}(\vec{x}, \vec{y}, \vec{z})}{C(\vec{x}, \vec{y}, \vec{z})} \le \frac{(1+\delta)^2}{(1-\delta)^2}$$

Comments.

1. The reason for this inequality is as follows: when we measure A with relative precision δ, it means that the result $\tilde{A}$ of the measurement must be between $A - \delta A$ and $A + \delta A$, i.e., in other words, that $1 - \delta \le \tilde{A}/A \le 1 + \delta$. The closure amplitude is obtained by multiplying two amplitudes and dividing by two values. So the maximum possible value for the measured closure amplitude is when the multiplying terms take their maximum possible values ($A(1+\delta)$) and the dividing terms

take their minimal possible values $((1-\delta)A)$. In this case the resulting closure amplitude is twice increased by $1+\delta$ and twice divided by $1-\delta$, so, if the real value is C, the maximum possible measured value is $C(1+\delta)^2/(1-\delta)^2$. This explains the right inequality. The left inequality can be explained in a similar manner.

2. For each $\vec{x}, \vec{y}$ and $\vec{z}$, if the only thing we know is the result $\tilde{C}$ of measuring the closure amplitude with precision δ, then we actually know an interval of possible values of C: $[((1-\delta)^2/(1+\delta)^2)\tilde{C}, ((1+\delta)^2/(1-\delta)^2)\tilde{C}]$. So the only thing that we know about the actual amplitude function A is that the resulting closure amplitude lies in this interval. Of course, there can be several functions A that satisfy this inequality, and the measurements do not allow us to choose between them. So we arrive at the following definition.

Definition 11. Suppose that an amplitude function $A(\vec{x})$ is given. We'll call it an *actual amplitude function.* Suppose also that $\tilde{C}$ is a result of measuring its closure amplitude with precision δ. We say that an amplitude function $A'(\vec{x})$ is *reconstructed from the approximate closure amplitude* if

$$\frac{(1-\delta)^2}{(1+\delta)^2} \le \frac{\tilde{C}(\vec{x}, \vec{y}, \vec{z})}{C'(\vec{x}, \vec{y}, \vec{z})} \le \frac{(1+\delta)^2}{(1-\delta)^2}$$

where C' is a closure amplitude corresponding to A'.

3.3. Main problem

The main problem is: *how close is the result A' of reconstructing the amplitude from approximate measurements to the actual amplitude function $A(\vec{x})$?*

Comment. This problem is also non-trivial, because interval mathematics usually gives an estimate for some given algorithm, but now we are interested in the principal possibility of reconstructing the amplitude. So if there is an algorithm that would give the result with better precision than all the existing methods, we would like to know it.

3.4. Solution

Theorem 3. *Suppose that $A(\vec{x})$ is the actual phase function, precision is $\delta < 1$ and $A'(\vec{x})$ is reconstructed from the approximate closure amplitude.*

Then there exists a function A'' that is equivalent to A and such that $(1-\delta)^4/(1+\delta)^4 \le A''(\vec{x})/A'(\vec{x}) \le (1+\delta)^4/(1-\delta)^4$ for all $\vec{x}$.

Comments.

1. So closure-amplitude imaging allows us in principle to reconstruct the amplitude function with relative precision $(1+\delta)^4/(1-\delta)^4-1$, where δ is the relative precision, with which we measure the amplitudes. For small δ this relative error estimate is approximately equal to 8δ.

2. The restriction $\delta < 1$ is really not very restrictive because in actual measurements the precision is much better.

4. Proofs

Proof of Proposition 1. That θ and $\theta + \vec{k}\vec{x}$ correspond to the same closure phase can be checked by explicit computations. So let us prove that if θ and θ' correspond to the same closure phase, then they differ by a linear function.

Suppose that two phase functions $\theta(\vec{x})$ and $\theta'(\vec{x})$ correspond to the same closure phase $\Phi(\vec{x},\vec{y})$. According to the definition of the closure phase this means that $\Theta(\vec{x})+\Theta(\vec{y})-\Theta(\vec{x}+\vec{y}) = \Theta'(\vec{x})+\Theta'(\vec{y})-\Theta'(\vec{x}+\vec{y})$ for all $\vec{x}$ and $\vec{y}$, or, what is equivalent, $\Theta(\vec{x})+\Theta(\vec{y})-\Theta(\vec{x}+\vec{y})-\Theta'(\vec{x})-\Theta'(\vec{y})+\Theta'(\vec{x}+\vec{y})=0$. The values Θ are defined modulo 2π. In terms of θ and θ' this equality means that the corresponding difference equals $2\pi n$ for some integer n, i.e., that for every $\vec{x}$ and $\vec{y}$ there exists an integer $n(\vec{x},\vec{y})$ such that $\theta(\vec{x})+\theta(\vec{y})-\theta(\vec{x}+\vec{y})-\theta'(\vec{x})-\theta'(\vec{y})+\theta'(\vec{x}+\vec{y}) = 2\pi n(\vec{x},\vec{y})$. If we denote by $\delta\theta(\vec{x})$ the difference $\theta'(\vec{x})-\theta(\vec{x})$ between the two functions, we can rewrite this equality as $\delta\theta(\vec{x})+\delta\theta(\vec{y})-\delta\theta(\vec{x}+\vec{y}) = 2\pi n(\vec{x},\vec{y})$.

Both functions $\theta(\vec{x})$ and θ' are continuous (because they are phase functions, and we defined phase functions to be continuous). So their difference $\delta\theta$ is also a continuous function. Therefore the linear combination $\delta\theta(\vec{x})+\delta\theta(\vec{y})-\delta\theta(\vec{x}+\vec{y})$ is a continuous function of $\vec{x}$ and $\vec{y}$. But according to the above inequality this combination is equal to $2\pi n(\vec{x},\vec{y})$. So $2\pi n(\vec{x},\vec{y})$ is also a continuous function of $\vec{x}$ and $\vec{y}$, and hence $n(\vec{x},\vec{y})$ is a continuous function. But n is always an integer. So n cannot take two different values (else according to the intermediate value theorem it would take all intermediate values, including those non-integer ones). So n is a constant, i.e., $n(\vec{x},\vec{y}) = n$ for some n.

In order to compute this n let's consider the values $\vec{x} = \vec{y} = 0$. Since both θ and θ' are phase functions, they are both odd; in particular, $\theta(0) = \theta'(0) = 0$. Therefore $\delta\theta(0) = 0$, and therefore for $\vec{x} = \vec{y} = 0$: $\delta\theta(0) + \delta\theta(0) - \delta\theta(0) = 0$. But, on the other hand, this value is equal to $2\pi n$, and so $n = 0$.

Substituting $n = 0$ into the above equality, we conclude that $\delta\theta(\vec{x}) + \delta\theta(\vec{y}) - \delta\theta(\vec{x}+\vec{y}) = 0$ or, which is equivalent, $\delta\theta(\vec{x}+\vec{y}) = \delta\theta(\vec{x}) + \delta\theta(\vec{y})$ for all $\vec{x}$ and $\vec{y}$. Here $\delta\theta(\vec{x})$ is a continuous function, and continuous functions with this property are known to be linear (see, e.g., Section 5.1 of [Aczel 1966]). So $\theta'(\vec{x}) - \theta(\vec{x}) = \delta\theta(\vec{x}) = \vec{k}\vec{x}$ for some constant vector k, hence $\theta'(\vec{x}) = \theta(\vec{x}) + \vec{k}\vec{x}$. Q.E.D.

Proof of Proposition 2 directly follows from the well-known properties of Fourier transform: that a Fourier transform of a shift $I(\vec{x} + \vec{a})$ equals $\exp(i\vec{a}\vec{x})FT(I)$.

Proof of Theorem 1. The idea of this proof is as follows: We have already seen in the proof of Proposition 1 that the problem of uniqueness of image reconstruction can be reduced to solving a linear functional equation. We'll now use a similar reduction of our new problem to the problem of finding functions that are in some reasonable sense "almost linear".

Suppose that θ is an actual phase function, θ' is reconstructed from the approximate closure phase $\tilde{\Phi}$ and $\varepsilon < \pi/6$. By Φ and Φ' we'll denote the closure phases that correspond to θ and θ'.

Since $\tilde{\Phi}$ is the result of approximate measurement, then according to Definition 6 the following inequality is true for all $\vec{x}$ and $\vec{y}$: $\rho(\tilde{\Phi}(\vec{x},\vec{y}), \Phi(\vec{x},\vec{y})) \le 3\varepsilon$. Since θ' is reconstructed from $\tilde{\Phi}$, by Definition 7 the following inequalities are true: $\rho(\tilde{\Phi}(\vec{x},\vec{y}), \Phi'(\vec{x},\vec{y}) \le 3\varepsilon$. Applying the triangle inequality for the distance ρ, we conclude that $\rho(\Phi(\vec{x},\vec{y}), \Phi'(\vec{x},\vec{y})) \le \rho(\tilde{\Phi}(\vec{x},\vec{y}), \Phi(\vec{x},\vec{y})) + \rho(\tilde{\Phi}(\vec{x},\vec{y}), \Phi'(\vec{x},\vec{y}) \le 3\varepsilon + 3\varepsilon = 6\varepsilon$.

By the definition $\Phi(\vec{x},\vec{y}) = \Theta(\vec{x}) + \Theta(\vec{y}) - \Theta(\vec{x}+\vec{y}) =$ (by the definition of Θ) $= p(\theta(\vec{x})) + p(\theta(\vec{y})) - p(\theta(\vec{x}+\vec{y})) =$ (since p is a homomorphism) $= p(\theta(\vec{x}) + \theta(\vec{y}) - \theta(\vec{x}+\vec{y}))$. Similarly $\Phi'(\vec{x},\vec{y}) = p(\theta'(\vec{x}) + \theta'(\vec{y}) - \theta'(\vec{x}+\vec{y}))$. So the previous inequality may be rewritten as follows:

$$\rho(p(\theta(\vec{x}) + \theta(\vec{y}) - \theta(\vec{x}+\vec{y})), p(\theta'(\vec{x}) + \theta'(\vec{y}) - \theta'(\vec{x}+\vec{y})) \le 6\varepsilon.$$

According to the properties of ρ (see above comments after the definition of a distance) $\rho(p(a), p(b)) = |a - b - 2\pi n|$ for some integer n. So in our case for every $\vec{x}$ and $\vec{y}$ there exists an integer $n(\vec{x}, \vec{y})$ such that $|\Delta(\vec{x}, \vec{y}) - 2\pi n(\vec{x}, \vec{y})| \leq 6\varepsilon$, where by $\Delta(\vec{x}, \vec{y})$ we denoted the difference between the two arguments of p: $\Delta(\vec{x}, \vec{y}) = (\theta(\vec{x}) + \theta(\vec{y}) - \theta(\vec{x}+\vec{y})) - (\theta'(\vec{x}) + \theta'(\vec{y}) - \theta'(\vec{x}+\vec{y}))$. Denoting the difference between θ' and θ by $\delta\theta$ (as in the proof of Proposition 1), we conclude that $\Delta(\vec{x}, \vec{y}) = \delta\theta(\vec{x}) + \delta\theta(\vec{y}) - \delta\theta(\vec{x} + \vec{y})$. Just as in that proof, $\delta\theta$ is a continuous function, so Δ is also a continuous function of both its arguments $\vec{x}$ and $\vec{y}$.

The inequality $|\Delta(\vec{x}, \vec{y}) - 2\pi n(\vec{x}, \vec{y})| \leq 6\varepsilon$ means that $\Delta(\vec{x}, \vec{y})$ belongs to the interval $[2\pi n - 6\varepsilon, 2\pi n + 6\varepsilon]$. Since $\varepsilon < \pi/6$, we conclude that $6\varepsilon < \pi$, so this interval, in its turn, belongs to the open interval $((2n - 1)\pi, (2n + 1)\pi)$. From this we can conclude that Δ cannot attain the values $(2n + 1)\pi$.

We know that the value of a function $\Delta(\vec{x}, \vec{y})$ for each $\vec{x}$ and $\vec{y}$ belongs to one of these intervals. Let's prove that for all $\vec{x}$ and $\vec{y}$ the interval is the same, i.e., that $n(\vec{x}, \vec{y})$ does not depend on $\vec{x}$ and $\vec{y}$ and is actually a constant. Let's prove it by reduction to a contradiction. Indeed, suppose that a function n takes at least two different values, $m < n$. This means that Δ takes a value from $((2m - 1)\pi, (2m + 1)\pi)$ and a value from the interval $((2n - 1)\pi, (2n + 1)\pi)$. According to the intermediate value theorem a continuous function Δ must take all the intermediate values, in particular, the value $(2m + 1)\pi$, but this value does not belong to any of the intervals, in which the value of Δ is located, and therefore cannot be the value of Δ. The contradiction proves that n is a constant. For $\vec{x} = \vec{y} = 0$ we have $\Delta(0, 0) = 0$, so $n = 0$.

Substituting 0 into the above inequality for Δ, we conclude that $|\Delta(\vec{x}, \vec{y})| \leq 6\varepsilon$, and, substituting the definition of Δ in terms of $\delta\theta$, that $|\delta\theta(\vec{x}) + \delta\theta(\vec{y}) - \delta\theta(\vec{x} + \vec{y})| \leq 6\varepsilon$. For this particular inequality it is known [Hyers 1941, Ulam 1960] that there exists a vector $\vec{k}$ such that $|\delta\theta(\vec{x}) - \vec{k}\vec{x}| \leq 6\varepsilon$. Since we defined $\delta\theta$ as $\theta - \theta'$, we conclude that $|\theta(\vec{x}) - \theta'(\vec{x}) - \vec{k}\vec{x}| \leq 6\varepsilon$. This inequality can be rewritten as $|\theta''(\vec{x}) - \theta'(\vec{x})| \leq 6\varepsilon$, where the function $\theta''(\vec{x}) = \theta(\vec{x}) - \vec{k}\vec{x}$ is equivalent to θ (see Proposition 1$'$). So we have a function for which the desired inequality is true. Q.E.D.

Proof of Theorem 2. Let us define the following auxiliary function: $f(x) = x$, if $|x| \leq 1$, $f(x) = 1$ if $x > 1$, and $f(x) = -1$ if $x < -1$. Let us

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prove that $|f(x)+f(y)-f(x+y)| \leq 1$ for all x and y. To prove it let us consider all possible cases.

Since the function $f(x)$ is odd (i.e., $f(-x)=-f(x)$), without losing generality, we can consider only the case when $x+y \geq 0$ (because the case when $x+y<0$ can be treated in a similar manner). Also, since the desired expression is symmetric with respect to x and y, we can, again without loosing generality, assume that $x \geq y$. In this case, $2x \geq x+y \geq 0$, hence $x \geq 0$. So, it is sufficient to consider only the cases when $x \geq 0$, $x \geq y$, and $x+y \geq 0$. Let us enumerate all such cases. Each of the numbers x and $x+y$ can either belong to the interval [0,1], or to the semi-line $(1, \infty)$. Combining the two pairs of alternatives, we get four possible cases. In each of these cases, we have three subcases depending on whether y is <-1, between -1 to 1, or >1. We will prove that the desired inequality is true in all these subcases.

1. $|x| \leq 1$, $|x+y| \leq 1$. Here, since $y \leq x$, only two subcases are possible: $|y| \leq 1$ and $y<-1$.

   1a. If $|y| \leq 1$, then $f(x)+f(y)-f(x+y)=x+y-(x+y)=0$, so the desired inequality is evidently true.

   1b. If $y<-1$, then $|f(x)+f(y)-f(x+y)|=|x-1-(x+y)|=|-1-y|=|1+y|$. From $x+y \geq -1$ and $x \leq 1$ we conclude that $y=(x+y)-x \geq (-1)-1 \geq -2$. Therefore, $-2 \leq y \leq -1$, $-1 \leq 1+y \leq 0$, and $|1+y| \leq 1$.

2. $|x| \leq 1$, $x+y>1$. Here, since $x+y>1>x$, we have $y=(x+y)-x>0$, and from $y \leq x$, we conclude that $0<y \leq 1$. In this case, $|f(x)+f(y)-f(x+y)|=|x+1-(x+y)|=|1-y|$. From $0<y \leq 1$ it follows that $0 \leq 1-y<1$, so $|1-y| \leq 1$.

3. $x>1$, $|x+y| \leq 1$. Here, $x+y \leq 1<x$, so $x+y<x$, $y<0$, and so, we have only two possible subcases: $|y| \leq 1$ and $y<-1$.

   3a. $|y| \leq 1$. In this subcase, $|f(x)+f(y)-f(x+y)|=|1+y-(x+y)|=|1-x|$. Since $x+y \leq 1$ and $y \geq -1$, we conclude that $x=(x+y)-y \leq 2$. Hence, $1<x \leq 2$, $-1 \leq 1-x \leq 0$, and $|1-x| \leq 1$.

   3b. $y<-1$. In this subcase, $|f(x)+f(y)-f(x+y)|=|1-1+f(x+y)|=|f(x+y)|$, and from $|f(z)| \leq 1$ we conclude that $|f(x)+f(y)-f(x+y)|=|f(x+y)| \leq 1$.

4. $x > 1$, $x + y > 1$. In this case, $f(x) = f(x + y) = 1$, hence $|f(x) + f(y) - f(x + y)| = |f(y)|$, and $|f(y)| \leq 1$ follows from the definition of the function $f(x)$.

So, in all cases, $|f(x) + f(y) - f(x + y)| \leq 1$. Now, we are ready to produce the desired θ': let us take $\tilde{\Phi}(\vec{x}, \vec{y}) = (\theta(\vec{x}) + 3\varepsilon f(x_1)) + (\theta(\vec{y}) + 3\varepsilon f(y_1)) - (\theta(\vec{x} + \vec{y}) + 3\varepsilon f(x_1 + y_1))$ (where x_1 denotes the first coordinate of the vector $\vec{x}$), and $\theta'(\vec{x}) = \theta(\vec{x}) + 6\varepsilon f(x_1)$. In this case, $|\tilde{\Phi}(\vec{x}, \vec{y}) - \Phi(\vec{x}, \vec{y})| = 3\varepsilon|f(x_1) + f(y_1) - f(x_1 + y_1)|$, so from the above-mentioned property of the function $f(x)$ we conclude that $|\tilde{\Phi}(\vec{x}, \vec{y}) - \Phi(\vec{x}, \vec{y})| \leq 3\varepsilon$. Therefore, $\tilde{\Phi}(\vec{x}, \vec{y})$ is a possible result of measuring closure phase with precision ε. Likewise, $|\tilde{\Phi}(\vec{x}, \vec{y}) - \Phi'(\vec{x}, \vec{y})| \leq 3\varepsilon$, therefore, $\theta'(\vec{x})$ is reconstructed from the approximate closure phase.

Now, for $x_1 \geq 1$, we have $\theta'(\vec{x}) - \theta(\vec{x}) = 6\varepsilon$. Suppose that $\theta''(\vec{x})$ is equivalent to $\theta(\vec{x})$. According to Definition 5, this means that $\theta''(\vec{x}) = \theta(\vec{x}) + k_1 x_1 + k_2 x_2$ for some real numbers k_1 and k_2. If $k_1 = 0$, then for $x_1 > 1$ and $x_2 = 0$, we have $|\theta''(\vec{x}) - \theta'(\vec{x})| = |\theta(\vec{x}) - \theta'(\vec{x})| = 6\varepsilon$.

If $k_1 \neq 0$, then for $x_2 = 0$ and $x_1 \to \infty$, we have $\theta''(\vec{x}) - \theta'(\vec{x}) = (\theta''(\vec{x}) - \theta(\vec{x})) + (\theta(\vec{x}) - \theta'(\vec{x})) = k_1 x_1 - 6\varepsilon \to \pm\infty$ ($\to +\infty$ if $k_1 > 0$, and $\to -\infty$ if $k_1 < 0$). Therefore, for sufficiently big x_1, we have $|\theta''(\vec{x}) - \theta'(\vec{x})| \geq 6\varepsilon$.

So, in both cases, we proved that there exists an $\vec{x}$ such that $|\theta''(\vec{x}) - \theta'(\vec{x})| \geq 6\epsilon$. Therefore, $\sup_{\vec{x}} |\theta''(\vec{x}) - \theta'(\vec{x})| \geq 6\varepsilon$. Q.E.D.

Proof of Proposition 3. That A and kA lead to the same closure amplitude is easy to check. So it is sufficient to show that if A and A' corresponds to the same closure amplitude, then $A' = kA$ for some constant k.

By the definition of the closure amplitude the equality means that

$$A(\vec{x} + \vec{y})A(\vec{z})/(A(\vec{x})A(\vec{y} - \vec{z})) = A'(\vec{x} + \vec{y})A'(\vec{z})/(A'(\vec{x})A'(\vec{y} - \vec{z})).$$

Dividing both sides of this equation by the left-hand side, we come to the following equality in terms of the ratio $k(\vec{x}) = A'(\vec{x})/A(\vec{x})$: $k(\vec{x} + \vec{y})k(\vec{z})/(k(\vec{x})k(\vec{y} - \vec{z})) = 1$. This means that the numerator and the denominator of this fraction are equal, i.e., $k(\vec{x} + \vec{y})k(\vec{z}) = k(\vec{x})k(\vec{y} - \vec{z})$. In particular, for $\vec{z} = \vec{0}$ we conclude that $k(\vec{x} + \vec{y})k(\vec{0}) = k(\vec{x})k(\vec{y})$. This is almost a functional equation $k(\vec{x}+\vec{y}) = k(\vec{x})k(\vec{y})$, for which the solutions are known. To reduce to this equation let's divide both sides of our equation by $k^2(\vec{0})$. Then $(k(\vec{x}+\vec{y})/k(\vec{0}) = (k(\vec{x})/k(\vec{0})(k(\vec{y})/k(\vec{0}))$. So for the function $\tilde{k}(\vec{x})$, defined as $k(\vec{x})/k(\vec{0})$, we derived the desired functional equation $\tilde{k}(\vec{x}+\vec{y}) = \tilde{k}(\vec{x})\tilde{k}(\vec{y})$.

By the definition the amplitude function is continuous, therefore k is also continuous as a ratio of two continuous functions, and therefore $\tilde{k}$ is also continuous. The continuous solutions of the above functional equation are enumerated in [Aczel 1966], Section 5.1: they are $\tilde{k}(\vec{x}) = \exp(\vec{a}\vec{x})$ for some constant vector $\vec{a}$. But by definition an amplitude function is even; therefore k and $\tilde{k}$ are also even, i.e., $\exp(\vec{a}\vec{x}) = \exp(-\vec{a}\vec{x})$ for all $\vec{x}$. This is possible only if $\vec{a} = \vec{0}$. Substituting $\vec{a} = \vec{0}$, we conclude that $\tilde{k} = 1$ and therefore $k(\vec{x}) = k(\vec{0})\tilde{k}(\vec{x}) = k(\vec{0})$ is a constant. So $A' = kA$ for some constant k. Q.E.D.

Proof of Proposition 4 follows from well-known properties of the Fourier transform.

Proof of Theorem 3. Suppose that A is an actual amplitude function, A' is reconstructed from the approximate closure amplitude $\tilde{C}$ and $\delta < 1$. By C and C' we'll denote the closure amplitudes that correspond to A and A'.

Since $\tilde{C}$ is the result of approximate measurement, then according to Definition 10 the following inequality is true for all $\vec{x}, \vec{y}$ and $\vec{z}$:

$$\frac{(1-\delta)^2}{(1+\delta)^2} \leq \frac{\tilde{C}(\vec{x},\vec{y},\vec{z})}{C(\vec{x},\vec{y},\vec{z})} \leq \frac{(1+\delta)^2}{(1-\delta)^2}$$

Since A' is reconstructed from $\tilde{C}$, by Definition 11 it means that

$$\frac{(1-\delta)^2}{(1+\delta)^2} \leq \frac{C'(\vec{x},\vec{y},\vec{z})}{\tilde{C}(\vec{x},\vec{y},\vec{z})} \leq \frac{(1+\delta)^2}{(1-\delta)^2}$$

¿From these two inequalities we can draw some conclusions about the possible value of $C'/C = (C'/\tilde{C})(\tilde{C}/C)$: since it is a product of two positive numbers, and we know upper and lower estimates for both factors, then the maximum value of this product is the product of two upper estimates and the lower estimate for the product is the product of two lower estimates (unlike the phase case, this is a typical interval estimate). So we conclude that $Z^{-1} \leq C'/C \leq Z$, where $Z = (1+\delta)^4/(1-\delta)^4$.

Now let's perform some transformations similar to the ones we did in the proof of Proposition 3. Substituting the expressions for C and C' in terms of A and A' into these inequalities and denoting the ratio $A'(\vec{x})/A(\vec{x})$ by $k(\vec{x})$, we conclude, that

$$Z^{-1} \leq k(\vec{x}+\vec{y})k(\vec{z})/(k(\vec{x})k(\vec{y}-\vec{z})) \leq Z$$

for all $\vec{x}, \vec{y}$ and $\vec{z}$. In particular, if we substitute $\vec{z} = \vec{0}$ and denote $\tilde{k}(\vec{x}) = k(\vec{x})/k(\vec{0})$, we conclude that $Z^{-1} \leq \tilde{k}(\vec{x}+\vec{y})/(\tilde{k}(\vec{x})\tilde{k}(\vec{y})) \leq Z$ for all $\vec{x}$ and $\vec{y}$.

This inequality looks similar to the one that we obtained in the proof of Theorem 1. The only difference is that there we had $+$ and $-$, and here we have multiplication and division. So to reduce this new inequality to the one that we already know how to solve let's apply the standard trick that reduces multiplication to $+$ and division to $-$: logarithms. If we introduce $K(\vec{x}) = \ln(\tilde{k}(\vec{x}))$, then for K the above inequality turns into the following one: $-\ln Z \leq K(\vec{x}) + K(\vec{y}) - K(\vec{x}+\vec{y}) \leq \ln Z$, or, that is equivalent, $|K(\vec{x}) + K(\vec{y}) - K(\vec{x}+\vec{y})| \leq \ln Z$. This is precisely the inequality that we know how to solve [Hyers 1941]: the solution is that $|K(\vec{x}) - \vec{a}\vec{x}| \leq \ln Z$ for some constant vector $\vec{a}$.

Let's now use the fact that k, $\tilde{k}$ and therefore K are even functions, i.e., that $K(-\vec{x}) = K(\vec{x})$. Applying the above inequality for $\vec{x}$ and $-\vec{x}$ and using this evenness, we conclude that $|K(\vec{x}) - \vec{a}\vec{x}| \leq \ln Z$ and $|K(\vec{x}) - (-\vec{a}\vec{x})| \leq \ln Z$. Therefore by the triangle inequality $|\vec{a}\vec{x} - (-\vec{a}\vec{x})| \leq |K(\vec{x}) - \vec{a}\vec{x}| + |K(\vec{x}) - \vec{a}\vec{x}| \leq \ln Z + \ln Z = 2\ln Z$. In other words, $|2\vec{a}\vec{x}| \leq 2\ln Z$ for all $\vec{x}$. In particular, if we substitute $\vec{x} = \lambda\vec{a}$, we conclude that $\lambda\vec{a}^2 \leq \ln Z$ for every positive λ, therefore $\vec{a}^2 = 0$ and $\vec{a} = \vec{0}$.

Substituting $\vec{a} = \vec{0}$ into the above inequality, we conclude that $|K(\vec{x})| \leq \ln Z$, i.e., that $-\ln Z \leq K(\vec{x}) \leq \ln Z$ for all $\vec{x}$. We want to deduce some inequalities for $\tilde{k}$. Since $K = \ln\tilde{k}$, $\tilde{k} = \exp(K)$. And because exp is a monotonic function, we get the following inequalities for $\tilde{k}$: $\exp(-\ln Z) = Z^{-1} \leq \tilde{k}(\vec{x}) \leq \exp(\ln Z) = Z$. But $\tilde{k}(\vec{x}) = k(\vec{x})/k(\vec{0})$ and $k(\vec{x}) = A'(\vec{x})/A(\vec{x})$. So the resulting inequality means that $Z^{-1} \leq A'(\vec{x})/(A(\vec{x})k(\vec{0})) \leq Z$, and so for the amplitude function $A''(\vec{x}) = k(\vec{0})\cdot$

$A(\vec{x})$, that is evidently equivalent to A, we get the desired inequality. Q.E.D.

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