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ON SOME TWO-SIDED METHODS FOR SOLVING SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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In this paper, we discuss ideas to overcome the so-called "wrapping effect". The methods are based on a posteriori error estimates, bounding the solutions by polygons, pretransformation of the ODE's, and sensitivity analysis. An illustrative example is given.

НЕКОТОРЫЕ ДВУСТОРОННИЕ МЕТОДЫ РЕШЕНИЯ СИСТЕМ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

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В работе рассматриваются различные подходы для построения двусторонних решений, пути преодоления "эффекта упаковывания". Рассматриваются методы, основанные на апостериорных оценках погрешности, оценках многогранниками множества решений, преобразовании систем ОДУ, анализе чувствительности.

Introduction

In this paper, some ideas connected with overcoming the so-called "wrapping effect" are discussed. In the literature on the subject, the wrapping effect is often also called the bootstrapping effect or Moore's effect. This effect manifests itself in extreme growth of the width of the two-sided solution of a system of differential equations relative to the true solution. Various methods have been proposed to overcome this effect. Among these are:

- automatic transformation of coordinates [11];
- transformation of a system of ODE's to a convenient form [17];
- bounding the solution set by parallelepipeds, ellipsoids and domains of other forms [3], [13];
- Analytical solution of a system of ODE's and subsequent construction of an interval extension [16];

The wrapping effect is also discussed in [10], [12], [14], [15], [17], [18] and other works.

A detailed survey on interval methods for the solution of systems of ODE's is contained in [15]. The monograph [8] considers two-sided methods based on the decomposition of an operator into isotone and antitone components. Also in [8], two-sided methods based on a posteriori estimates of an error are considered.

1. The problem statement and principal definitions

Let R^n be a space of n-dimensional vectors. In what follows, we denote interval numbers $\mathbf{a} = [\underline{a}, \bar{a}]$ with bold font: **a**, **b**, **c**, **f**. Similarly, $\mathbf{R}^n$ is a space of n-dimensional interval vectors, while $\mathrm{wid}(\mathbf{x}) = \bar{x} - \underline{x}$, $|\,(\mathbf{x}) = (x + \underline{x})/2$ and $\mathrm{abs}(\mathbf{x}) = \max(|\bar{x}|, |\underline{x}|)$.

Consider the following system:

$$
\begin{aligned}
x' &= f(t, x, k), \quad t \in (0, l), \\
x(0) &= x_0,
\end{aligned}
\tag{1.1}
$$

where $f = \{f_i\}_{i=1}^n, f_i = f_i(t, x, k)$

$x_0 \in R^n$	–	the initial values vector, $x_0 \in \mathbf{x}_0$;
$k \in R^m$	–	the vector of parameters, $k \in \mathbf{k}$;
$x \in R^n$	–	the vector of unknown variables.

We shall assume that x is a function of t, k and x_0:

$$x = x(t, k, x_0). \tag{1.2}$$

Denote by $\gamma^*(t)$ the set of solutions of a system of ODE's

$$\gamma^*(t) = \{x(t, k, x_0) | x_0 \in \mathbf{x}_0, k \in \mathbf{k}\}.$$

Definition 1.1. *A two-sided solution $\mathbf{x}^*$ of the problem (1.1) with minimal width is called optimal.*

Let the real functions [9] $F_i^l(y_1, \dots, y_n, z_1, \dots, z_n), l = 1, 2$ **exist such that the following inequalities hold for** $\underline{y} \le \bar{y}$, $\underline{z} \le \bar{z}$:

$$F_i^\nu(t, \underline{y}, \bar{z}) \le F_i^\nu(t, \bar{y}^i, \underline{z}), \quad \nu = 1, 2, \quad i = 1, \dots, n \tag{1.3}$$

where $\bar{y}^i = (\bar{y}_1, \bar{y}_{i-1}, \dots, \underline{y}_i, \bar{y}_{i+1}, \dots, \bar{y}_n)$. Also assume that the inequalities

$$F_i^1(t, x, x) \le f_i(t, x, k) \le F_i^2(t, x, x) \tag{1.4}$$

hold for these functions.

Theorem 1.1. [9]. *Suppose that the vector functions $\underline{x}, \bar{x} \in R^n$ satisfy the following relations.*

$$\begin{aligned} &\underline{x}' \le F^1(t, \underline{x}, \bar{x}), \quad t \in (0, l), \\ &\bar{x}' \ge F^2(t, \bar{x}, \underline{x}), \\ &\underline{x}(0) \le \underline{x}_0, \\ &\bar{x}(0) \ge \bar{x}_0. \end{aligned} \tag{1.5}$$

Then any solution x of the system (1.1) with initial condition $\underline{x}_0 \le x(0) \le \bar{x}_0$ satisfies the estimates

$$\underline{x}(t) \le x(t) \le \bar{x}(t), \quad t \in (0, 1).\blacksquare$$

Note that according to (1.3)

$$F^1(t, \underline{x}, \bar{x}) \le \inf f(t, x, k), \quad F^2(t, \bar{x}, \underline{x}) \ge \sup f(t, x. k).$$

Thus, we can take the bounds of interval extensions of functions f_i for F_i^l, provided those extensions are inclusion monotonic. In fact, let $\mathbf{f}(t, \mathbf{x}_j, \mathbf{k}) = [\underline{f}(t, \mathbf{x}, \mathbf{k}), \bar{f}(t, \mathbf{x}, \mathbf{k})]$ be an interval extension of the function f, where $\mathbf{x} = [\underline{x}, \bar{x}]$. Then the functions $\underline{f}$ and $\bar{f}$ fully satisfy conditions (1.3) and (1.4). Condition (1.3) holds since the interval extension is inclusion monotonic, while (1.4) holds because $\mathbf{f}$ is an interval extension.

Definition 1.2. *A function f is called isotone if the following condition holds:*

$$x \leq y \Rightarrow f(x) \leq f(y).$$

It is called antitone if

$$x \leq y \Rightarrow f(x) \geq f(y).$$

A function is called monotone if it is isotone or antitone.

Remark 1.1. Let the function $f(x, y)$ be isotone with respect to x and antitone with respect to y. Then the interval extension $\mathbf{f}(\mathbf{x}, \mathbf{y})$ has the following form:

$$\underline{f}(\mathbf{x}, \mathbf{y}) = f(\underline{x}, \bar{y}) \quad \text{and} \quad \bar{f}(\mathbf{x}, \mathbf{y}) = \bar{f}(\bar{x}, \underline{y}).$$

We can rewrite the system (1.5) in the following form:

$$\begin{aligned} &\underline{x}_i' \leq \underline{f}_i(t, \mathbf{x}^{[\underline{x}_i]}, \mathbf{k}), \quad t \in (0, l), \\ &\bar{x}_i' \geq \bar{f}_i(t, \mathbf{x}^{[\bar{x}_i]}, \mathbf{k}), \quad i = 1, 2, \ldots, n, \\ &\underline{x}(0) \leq \underline{x}_0, \\ &\bar{x}(0) \geq \bar{x}_0, \end{aligned} \tag{1.6}$$

where $\mathbf{x}^{[x_i]}$ means $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{i-1}, x_i, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n)$.

Generally, solving the above system gives a two-sided solution that is wider than optimal. However, in some special cases, the solution of the system (1.6) furnishes the optimal bounds. Consider the following cases.

Let the system (1.6) be of the following form:

$$\begin{aligned} &\underline{x}' = f(t, \underline{x}, k^1), \quad t \in (0, l), \\ &\bar{x}' = f(t, \bar{x}, k^2), \\ &\underline{x}(0) = \underline{x}_0, \\ &\bar{x}(0) = \bar{x}_0. \end{aligned} \tag{1.7}$$

where $k_i \in \mathbf{k}$. In this case $\underline{x}$ and $\bar{x}$ are particular solutions of the initial system, and are therefore optimal.

Let us formulate sufficient conditions for representing the system (1.6) in the form (1.7).

Lemma 1.1. *Let the following conditions hold:*

$$\frac{\partial f_i}{\partial x_j}(x,k) \geq 0, \qquad i \neq j, \quad i,j = 1,2,\ldots,n. \tag{1.8}$$

$$\text{sign}\,\frac{\partial f_i}{\partial k_j}(x,k) = C_j, \qquad j : wid(\mathbf{k}_j) \neq 0, k \in \mathbf{k}, x \in \mathbf{x}. \tag{1.9}$$

Then the system (1.6) has the form (1.7).

Proof. In fact, conditions (1.8) and (1.9) combined with the monotonicity of f with respect to x and k guarantee that an interval extension of the function f can be represented in the form (1.7). ■

Consider the example:

$$x_1' = -k_1 x_1, \qquad x_1(0) = 1,$$
$$x_2' = k_1 x_1 - k_2 x_2, \quad x_2(0) = 0.$$

This system of ODE's simulates a simple chemical reaction. Conditions (1.8) hold for chemical kinetic equations [2]. But conditions (1.9) are not valid as a rule. The system of ODE's for a two-sided solution has the following form:

$$\underline{x}_1' = -\bar{k}_1 \underline{x}_1, \qquad \underline{x}_1(0) = 1,$$
$$\underline{x}_2' = \underline{k}_1 \underline{x}_1 - \bar{k}_2 \underline{x}_2, \quad \underline{x}_2(0) = 0,$$
$$\bar{x}_1' = -\underline{k}_1 \bar{x}_1, \qquad \bar{x}_1(0) = 1,$$
$$\bar{x}_2' = \bar{k}_1 \bar{x}_1 - \underline{k}_2 \bar{x}_2, \quad \bar{x}_2(0) = 0.$$

The constructed system decomposes into independent subsystems, but since the condition does not hold, the two-sided solution is wider than the true one. In this example, the width of the two-sided solution depends on wid($\mathbf{k}_1$), and we get the optimal bounds for the set of solutions if wid($\mathbf{k}_1$) = 0.

In general, it is impossible to solve the system (1.6) analytically. Hence, we use numerical methods to find $\underline{x}$ and $\bar{x}$. In cases in which errors in the integration method may be neglected, the system (1.6) may be solved by the method most convenient for the given problem. If guaranteed estimates are needed, one can use the available a posteriori error estimates for the numerical solution.

2. A posteriori error estimates

Let us formulate the problem (1.6) in the following form:

$$\begin{aligned} &\underline{x}_i' = \underline{f}_i(t, \mathbf{x}^{[\underline{x}_i]}, \mathbf{k}), \quad t \in (0, l), \\ &\bar{x}_i' = \bar{f}_i(t, \mathbf{x}^{[\bar{x}_i]}, \mathbf{k}), \quad i = 1, 2, \dots, n, \\ &\underline{x}(0) = \underline{x}_0, \\ &\bar{x}(0) = \bar{x}_0. \end{aligned} \tag{2.1}$$

Suppose that the problem (2.1) is solved numerically using some integration method with accuracy p on the mesh

$$\omega_h = x_j, \quad j = 1, 2, \dots, N, \quad N \text{ an integer.}$$

This results in the approximate solution $\underline{x}_i^n(x)$, $\bar{x}_i^n(x)$, $x \in \omega_n$ at the mesh points. Using equation (2.1), one can calculate approximate values of the derivatives $x_i^1(x)$ at the nodes of ω_n. Using the calculated values, we construct Hermitian splines $\underline{S}_i$ and $\bar{S}_i$ of degree $r-1$ passing through the points $\underline{x}_n^h(x)$, $\bar{x}_n^h(x)$, $x \in \omega_h$.

Theorem 2.1. [11]. *Let S interpolate the approximate solution x^h. Then there exists a constant C independent of h and x^h such that*

$$\| d^\nu (x^h - S)/dx^\nu \|_{c[0,l]} \leq C(h^{r-\nu} \| x \|_{W^r_\infty [0,l]} + h^p), \ \nu = 0, 1, 2. \blacksquare$$

We shall search for the solution of the system (1.6) in the following form:

$$\begin{aligned} \underline{x} &= \underline{S} + \underline{a}\underline{S}_1 \\ \bar{x} &= \bar{S} + \bar{a}\bar{S}_1 \end{aligned} \tag{2.2}$$

where $\underline{a}$ and $\bar{a}$ are constants.

$\underline{S}_1$ and $\bar{S}_1$ are the splines interpolating a numerical solution of the following system of ODE's:

$$\begin{aligned} &\underline{z}_i' = \sum_{j=1}^{n} (\partial \underline{f}_i(\mathbf{S}, \mathbf{k}) \partial \underline{x}_j) \underline{z}_j + (\underline{f}_i(\mathbf{S}, \mathbf{k}) - \underline{S}_i')_- \\ &\bar{z}_i' = \sum_{j=1}^{n} (\partial \bar{f}_i(\mathbf{S}, \mathbf{k}) \partial \bar{x}_j) \bar{z}_j + (\bar{f}_i(\mathbf{S}, \mathbf{k}) - \bar{S}_i')_+ \\ &\underline{z}(0), \bar{z}(0) = 0 \end{aligned}$$

where

$$(f)_+ = \begin{cases} f \text{ if} & f \geq 0 \\ 0 \text{ otherwise} \end{cases}, \qquad (f)_- = \begin{cases} f \text{ if} & f \leq 0 \\ 0 \text{ otherwise.} \end{cases}$$

Substituting (2.2) into (1.6), we see that $\underline{a}$ and $\bar{a}$ must satisfy the following system of inequalities:

$$\begin{aligned}
&(\underline{S} + \underline{a}\underline{S}_1)'_i \leq \underline{f}_i([\underline{S} + \underline{a}\underline{S}_1, \bar{S} + \bar{a}\bar{S}_1]^{[\underline{x}_i]}, \mathbf{k}), \quad t \in (0, l) \\
&(\bar{S} + \bar{a}\bar{S}_1)'_i \geq \bar{f}_i([\bar{S} + \bar{a}\bar{S}_1, \underline{a}\underline{S}_1]^{[\bar{x}_i]}, \mathbf{k}), \quad i = 1, 2, \dots, n \\
&(\underline{S} + \underline{a}\underline{S}_1)(0) \leq \underline{x}_0 \\
&(\bar{S} + \bar{a}\bar{S}_1)(0) \geq \bar{x}_0.
\end{aligned}$$

This system of inequalities can be rewritten as the system of nonlinear equations with respect to $\underline{a}$ and $\bar{a}$:

$$\begin{aligned}
\underline{a} &= \Phi_1(\underline{a}; \bar{a}) \\
\bar{a} &= \Phi_2(\underline{a}, \bar{a}).
\end{aligned}$$

The system of nonlinear equations constructed in this way is easily solved by the simple iteration method:

$$\begin{aligned}
\underline{a}_{i+1} &= \Phi_1(\underline{a}_i, \bar{a}_i) \\
\bar{a}_{i+1} &= \Phi_2(\underline{a}_i, \bar{a}_i).
\end{aligned} \tag{2.3}$$

The initial approximation can be chosen as follows:

$$\underline{a}_0, \quad \bar{a}_0 = 1.0. \tag{2.4}$$

Lemma 2.1. *The iteration process* (2.3) *with the initial approximation* (2.4) *converges for l small enough.*■

The following inequality gives a bound on the width of the constructed two-sided solution for $\mathbf{k} = k$ and $\underline{x}_0 = \bar{x}_0$ [4].

$$\begin{aligned}
\rho(t) \leq C \max_{i=1,\dots,n} \{&(h^{r-1} \parallel \underline{x} \parallel_{W^r_\infty[0,1]} + h^p)\underline{S}_1(t), \\
&(h^{r-1} \parallel \bar{x} \parallel_{W^r_\infty[0,1]} + h^p)\bar{S}_1(t)\}.
\end{aligned} \tag{2.5}$$

Theorem 2.2. *Let the system of ODE's (1.6) be represented in the form (1.7), and let* $\mathbf{x}^*$ *be the optimal solution. Then*

$$|\underline{x}(t) - \underline{x}^*(t)| \le Ch^\sigma$$
$$|\bar{x}(t) - \bar{x}^*(t)| \le Ch^\sigma,$$

where $\sigma = \min(r-1, p)$.

Proof. The proof follows directly from (2.5).■

This allows us to avoid the wrapping effect if the system (1.6) is representable in the form (1.7).

3. Application of sensitivity analysis

The main idea of this approach consists of analysis of the partial derivatives of the solution with respect to the parameters. This approach mostly coincides with the standard sensitivity analysis, but its realization requires the use of interval analysis techniques. Therefore, we shall call it the method of interval sensitivity analysis [6].

Suppose that we want to evaluate $\bar{x}^{(i)}$, the upper bound γ^* with respect to the i-th coordinate of $\bar{x}_i^{(i)} \ge x_i$ for all $x \in \gamma^*$.

To do this, consider the system of ODE's

$$\begin{aligned} \bar{x}^{(i)} &= f(t, \bar{x}^{(i)}, k^{(i)}) \\ \bar{x}^{(i)}(0) &= x_0^{(i)}. \end{aligned} \tag{3.1}$$

Here

$$k_j^{(i)} = \begin{cases} \bar{k}_j & \text{if } \mathbf{x}_{ij}^k(t) > 0 \\ \underline{k}_j & \text{if } \mathbf{x}_{ij}^k(t) < 0 \\ \mathbf{k}_j & \text{if } \mathbf{x}_{ij}^k(t) \ni 0 \end{cases}$$

and

$$x_0^{(i)} = \begin{cases} \bar{x}_{0j} & \text{if } \mathbf{x}_{ij}^0(t) > 0 \\ \underline{x}_{0j} & \text{if } \mathbf{x}_{ij}^0(t) < 0 \\ \mathbf{x}_0 & \text{if } \mathbf{x}_{ij}^0(t) \ni 0, \end{cases}$$

where $\mathbf{x}_{ij}^k(t)$ is an interval extension of $\frac{\partial x_i}{\partial k_j}(t, \mathbf{k}, \mathbf{x}_0)$,and $\mathbf{x}_{ij}^0(t)$ is an interval extension of $\frac{\partial x_i}{\partial x_{0j}}(t, \mathbf{k}, \mathbf{x}_0)$. If $\mathbf{x}_{ij}^k(t)$ and $\mathbf{x}_{ij}^0(t)$ do not contain zeros, then the system (3.1) does not contains interval parameters and may be solved by various interval and two-sided methods. For example, if $[\underline{x}^{(i)n}, \bar{x}^{(i)n}]$ is a two-sided solution obtained by the method described in Sect. 2 with cubic Hermitian splines, then

$$|\bar{x}^{(i)} - \bar{x}^{(i)h}| \leq Ch^3, \quad t \in (0, l).$$

The interval functions $\mathbf{x}_{ij}^k(t)$ and $\mathbf{x}_{ij}^0(t)$ may be found by simultaneously solving (1.1) and the system of ODE's

$$\dot{x}_{ij}^k = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_l}(t, x, k) x_{il}^k + \frac{\partial f_i}{\partial k_j}(t, x, k)$$
$$x_{ij}^k(0) = 0 \tag{3.2}$$

$$\dot{x}_{ij}^0 = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_l}(t, x, k) x_{il}^0$$
$$x_{ij}^k(0) = \delta_{ij}, \tag{3.3}$$

where δ_{ij} is the Kronecker symbol.

Theorem 3.1. *Let*

$$0 \notin \frac{\partial f_i}{\partial k_j}(t, \mathbf{x}^0, \mathbf{k}), \quad i = 1, \ldots, n, \quad j = 1, \ldots, m, \text{ and}$$

$$0 \notin \frac{\partial f_i}{\partial x_k}(t, \mathbf{x}^0, \mathbf{k}), \quad i, k = 1, \ldots, n, \quad k \neq i.$$

Then there exists $t_0 > 0$ such that the following conditions hold:

$$0 \notin \mathbf{x}_{ij}^k(t), \quad 0 \notin \mathbf{x}_{ij}^0(t), \quad t \in (0, t_0). \blacksquare$$

Consider a numerical example:

$$\begin{aligned} x_1 &= kx_2, & x_1(0) &= x_{01} \in [-0.1, 0.1] \\ x_2 &= -kx_1, & x_2(0) &= x_{02} \in [0.9, 1.1], \quad k \in \mathbf{k} = [1.0, 2.0] \end{aligned} \tag{3.4}$$

In Fig.3.1, the two-sided solution obtained by the method of interval sensitivity analysis and the exact solution are compared. As this comparison shows, the proposed method gives optimal bounds in practice for the set of solutions, up to time $t \approx 0.71$.

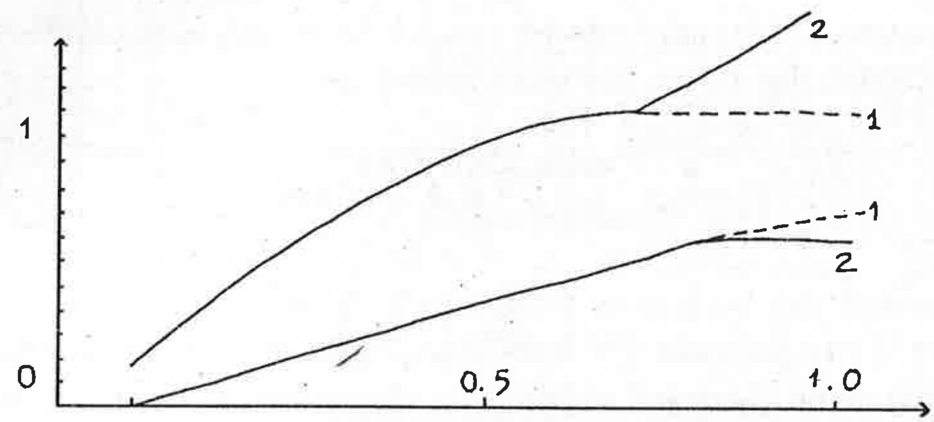

Fig.3.1
1 – exact solution, 2 – method of interval sensitive analysis

4. Construction of domains containing the sets of solutions

Denote by $\Re$ the set of all n-dimensional domains. The elements of this set will be denoted the same way as interval numbers. We shall consider only those $\mathbf{x} \in \Re$ that can be uniquely described with a finite set of parameters. For example, ellipsoids, polyhedra and any parallelepiped may be given by some vertex O and n edges emanating from it: $e_i, i = 1, 2, \ldots, n$. Therefore, a vector of parameters xp is associated with each such element $\mathbf{x} \in \Re$.

To construct the map $\mathbf{x}(t) : R \to \Re$ containing the set of all solutions, it is essential to construct the map γ^*:

$$\gamma^*(t, \tau, \mathbf{k}, x_0) \supseteq \{x(t + \tau, k, x_0) \mid k \in \mathbf{k}\}.$$

This map can be constructed approximately using numerical integration methods, such as Euler's method

$$\gamma(t,\tau,\mathbf{k},x_0) \supseteq \{x^h(t+\tau,k,x_0)|x^h(t+\tau,k,x_0) = x_0+\tau f(t,x_0,k), \quad k \in \mathbf{k}\}.$$

In this case, the bounds γ^* and γ differ by $O(\tau)$. Therefore, knowing $\mathbf{x}(t,\mathbf{k},\mathbf{x}_0)$ at some time t, one may construct the domain $\mathbf{x}(t+\tau,\mathbf{k},\mathbf{x}_0)$

$$\mathbf{x}(t+\tau,\mathbf{k},\mathbf{x}_0) \supseteq \{\gamma(t,\tau,\mathbf{k},z_0)|z_0 \in \mathbf{x}(t,\mathbf{k},\mathbf{x}_0)\}.$$

Given the vectors $xp(t)$ and $xp(t+\tau)$ respectively, one can construct the system of ODE's describing the behaviour of xp:

$$\begin{aligned} xp' &= fp(t,x,k) \\ xp(0) &= xp_0. \end{aligned} \tag{4.1}$$

If we construct the system of ODE's (4.1) in such way that one can reconstruct the trajectories $x(t,k,x_0)$ from $xp(t)$, we can try to establish optimal bounds on the set of solutions.

Consider some numerical examples. The following system describes a simple chemical reaction [2].

$$\begin{aligned} x_1' &= -k_1 x_1, & x_1(0) &= 1, \\ x_2' &= k_1 x_1 - k_2 x_2, & x_2(0) &= 0, \\ k_1 \in \mathbf{k}_1 &= [0.5, 1.0], & k_2 \in \mathbf{k}_2 &= [1.5, 2.0]. \end{aligned}$$

Fig. 4.1 displays bounds on the set of solutions at different time points by parallelepipeds.

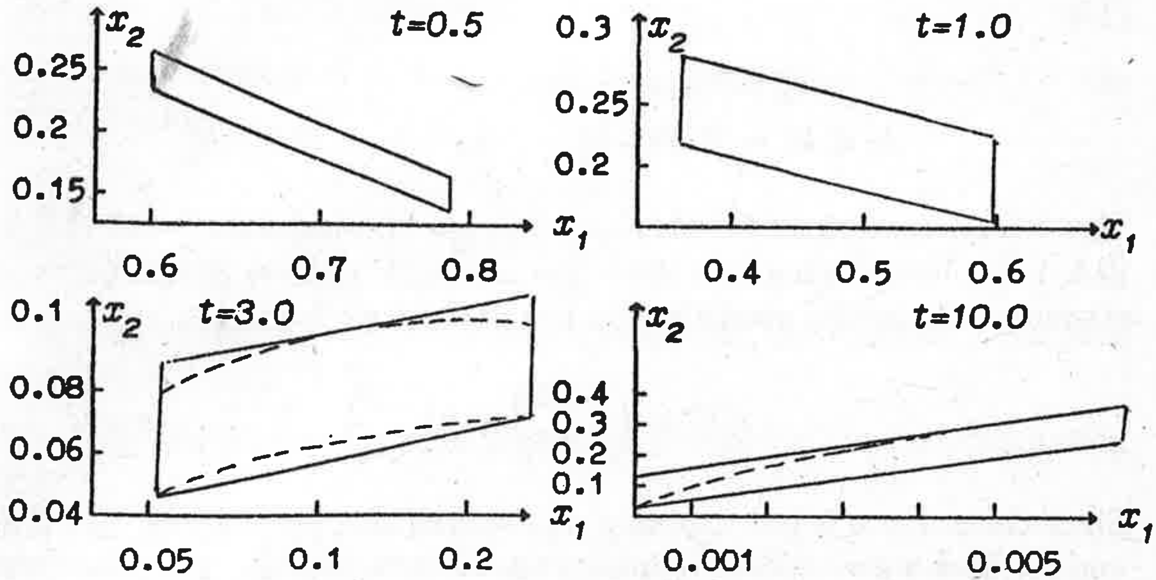

Fig.4.1

In Fig 4.2 bounds for problem (3.4) by parallelepipeds are shown.

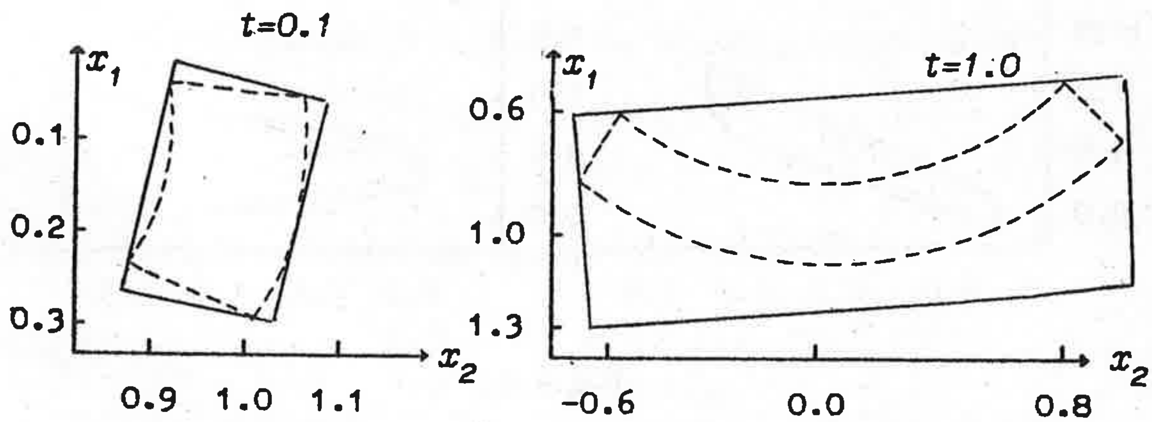

Fig.4.2

We illustrate how polyhedral domains are used with the following system of ODE's.

$$x_1' = x_1,$$
$$x_2' = -k_1x_1 - k_2x_2,$$
$$k_1 \in \mathbf{k_1} = [0.25, 1.0], \qquad k_2 \in \mathbf{k_2} = [0.0, 0.6].$$

The initial conditions for this problem are the intervals $[-0.5, 0.5]$ and $[0.5, 1.5]$. Introducing one more parameter $k \in \mathbf{k} = [0.0, 1.0]$, we can represent the initial conditions in the parametric form

$$x(0) = \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix} + k \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix}.$$

Since the domain is not uniquely represented as a polyhedron, two different polyhedra are used to represent the set of solutions. They are chosen in such a way that some their vertices always lie on the bound of the set of solutions. Thus the intersection of these polyhedra follows the set of solutions. In Fig. 4.3, the intersections of the polyhedra at times $t = 0.5$ and $t = 1.0$ are shown.

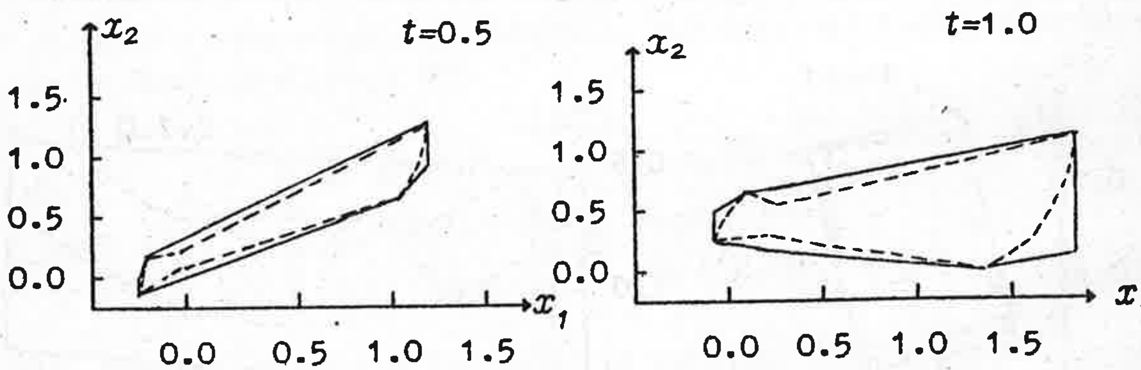

Fig.4.3

As Fig 4.3 shows, although each polyhedron has a large size, their intersection has no wrapping effect.

5. Transformation of a system of ODE's

As was shown in Section 1, the best case occurs when the system (1.6) is representable in the form (1.7). This can be attained in different ways. In particular, one of possibilities is a change of variables.

Let

$$x = x(y), \quad k = k(p). \tag{5.1}$$

Then, by substitution of these relations into the system (1.6), we get the transformed system of ODE's

$$\begin{aligned} y' &= g(y, p), \\ y(0) &= y_0. \end{aligned}$$

The dependencies (5.1) must be chosen such that the function g satisfies conditions (1.8) and (1.9).

The following theorem concerns a possible choice of the transformation (5.1).

Theorem 5.1. [1]. *In a sufficiently small neighbourhood of a non-degenerate point x_0, the coordinate system $(y_1, y_2, \dots, y_n)$ may be chosen such that, in this system, equation (1.1) would be written in the form*

$$\begin{aligned} y_1' &= 1, \\ y_2', y_3', \dots, y_n' &= 0. \quad \blacksquare \end{aligned}$$

Consider the example (3.4). Make the following transformation:

$$\begin{aligned} x_1 &= r\cos(\phi), \\ x_2 &= r\sin(\phi). \end{aligned}$$

Then, in the new variables, the system (3.4) becomes

$$\begin{aligned} \phi' &= \mathbf{k}, \qquad & \phi(0) &= \mathbf{c}_0, \\ r' &= 0, & r(0) &= \mathbf{r}_0. \end{aligned}$$

It is easy to see that the transformed system has no wrapping effect, though its solution is slightly wider than the exact one.

Conclusion

The examples show that it is possible to overcome the wrapping effect in some cases. Combination of the proposed methods successfully defeats the wrapping effect.

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