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OPTIMAL SOLUTION OF INTERVAL LINEAR ALGEBRAIC SYSTEMS. I

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For an interval linear algebraic system $\mathbf{A}x = \mathbf{b}$, the problem of component-wise evaluating the united solution set $X^*(\mathbf{A}, \mathbf{b}) = \{A^{-1}b \mid A \in \mathbf{A}, b \in \mathbf{b}\}$ is considered. An iterative *PSS-algorithm* is introduced to compute optimal (exact) component-wise $X^*(\mathbf{A}, \mathbf{b})$ estimates. Some possible generalizations are pointed out, concerning interval linear algebraic systems with tied coefficients in particular.

ОПТИМАЛЬНОЕ РЕШЕНИЕ ИНТЕРВАЛЬНЫХ СИСТЕМ ЛИНЕЙНЫХ АЛГЕБРАИЧЕСКИХ УРАВНЕНИЙ. I

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Для интервальной системы линейных алгебраических уравнений $\mathbf{A}x = \mathbf{b}$ рассматривается задача покоординатного оценивания объединённого множества решений $X^*(\mathbf{A}, \mathbf{b}) = \{A^{-1}b \mid A \in \mathbf{A}, b \in \mathbf{b}\}$. Предложен итеративный *PSS-алгоритм* для вычисления оптимальных (точных) покоординатных оценок $X^*(\mathbf{A}, \mathbf{b})$. Указаны некоторые из его обобщений, касающиеся, в частности, интервальных линейных алгебраических систем со связанными коэффициентами.

Several interval problems are well-known that naturally generalize the familiar linear algebraic system

$$A\,x \;=\; b. \tag{1}$$

All of them are usually denoted by one formal notation

$$\mathbf{A}\,x \;=\mathbf{b}, \tag{2}$$

– interval linear algebraic system (ILAS) with interval matrix $\mathbf{A} \ni A$ and interval right-hand side vector $\mathbf{b} \ni b$. The most popular, and historically first of these generalizations, is the problem of finding outer componentwise estimates for the *united solution set* (USS)

$$X^*(\mathbf{A},\mathbf{b}) \;=\; X^* \;=\; \{\, x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})\ (\exists b \in \mathbf{b})\ (Ax = b)\}$$

– the solutions set of all real linear algebraic systems contained in (2). It is often formulated as follows:

$$\begin{gathered}\text{find an interval vector } \mathbf{V} \text{ that contains}\\ \text{the united solution set of the given ILAS.}\end{gathered} \tag{3}$$

If the components of $\mathbf{V}$ have the least possible length, i.e. coincide with the projections of $X^*(\mathbf{A},\mathbf{b})$ onto the coordinate axes, then $\mathbf{V}$ is called the *optimal interval solution* of the problem (3) and the corresponding componentwise estimates $X^*(\mathbf{A},\mathbf{b})$ are called *optimal* ones. When referring to this problem one sometimes talks about *solving interval linear algebraic systems* or even *solving interval linear equations* [8,23]. To my mind, the wide-ranging term *outer problem for interval linear algebraic systems* is more suitable for this case.

A large number of papers are devoted to problem (3), see for example [1,3–5,8,11,13–17,23–25] and extensive references there. All the algorithms so far devised compute an interval vector $\mathbf{V}$ guaranteed to contain the set $X^*(\mathbf{A},\mathbf{b})$, but only a few of them ensure optimality of $\mathbf{V}$ in general and such algorithms are very labour-consuming (see, e.g., [4,23]).

The purpose of this work is to construct a class of *PSS-algorithms* to compute effectively optimal or near optimal solutions of ILAS (i.e. of the problem (3)) and of some other similar interval algebraic problems as

well. Section 1 presents necessary notation, reviews some facts of interval analysis and states the problem. Section 2 is devoted to the construction proper of the simplest PSS-algorithm, and Section 3 gives a proof of its convergence to optimal USS estimates. In §4 the basic algorithm is improved to incorporate accuracy control and to save storage. Finally, §5 discusses some possible generalizations of the technique elaborated in the previous sections. In particular, the *outer problem for tied coefficients ILAS*, of practical importance, is considered. This problem statement is non-traditional for interval analysis, but a version of the PSS-algorithm is shown to solve it successfully after introducing some fictitious variables. A good many of the results stated below were first published in abridged form in [27] and in [28].

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1. Notation, conventions and problem statement

Let $I\mathbb{R}$ be the set of all real intervals $[a;b]$ on $\mathbb{R}$, $a \le b$,

$I\mathbb{R}^n$, the set of n-dimensional interval vectors.

In this paper intervals and other interval objects are denoted by boldface letters while non-interval (real) objects are not distinguished in any way. Also, we need the following notation

$\underline{\mathbf{a}}$, $\overline{\mathbf{a}}$ denote lower and upper bounds of the interval $\mathbf{a}$, respectively,

$\operatorname{med}\mathbf{a} = (\overline{\mathbf{a}} + \underline{\mathbf{a}})/2$ is the mean value (median) of the interval $\mathbf{a}$,

$\operatorname{wid}\mathbf{a} = \overline{\mathbf{a}} - \underline{\mathbf{a}}$ the width of the interval $\mathbf{a}$,

$|\mathbf{a}| = \max\{|\overline{\mathbf{a}}|, |\underline{\mathbf{a}}|\}$ – absolute value of interval $\mathbf{a}$,

$$\langle\mathbf{a}\rangle = \begin{cases} \min\{|\overline{\mathbf{a}}|, |\underline{\mathbf{a}}|\}, & \text{if } 0 \notin \mathbf{a}, \\ 0, & \text{if } 0 \in \mathbf{a}, \end{cases}$$

– the mignitude or least distance between points of $\mathbf{a}$ and zero, which is in a sense the opposite of absolute value.

If $\mathbf{a} = (\mathbf{a}_i)_{i=1}^n$ is an interval vector, then all above defined operations shall be understood component-wise, so that $\operatorname{med}\mathbf{a}$, for instance, is real vector $(\operatorname{med}\mathbf{a}_i)_{i=1}^n$.

Throughout the rest of this paper, all arithmetic operations with intervals and interval objects are those of classical interval arithmetic [1,5,14,15].

except for division by a zero-containing interval, where we will make use of the extended Kahan arithmetic [12]. Besides common intervals of $I\mathbb{R}$ its elements are sets of the form $(-\infty;p]\cup[q;+\infty)$, $p\le q$, and $(-\infty;p]$ and $[q;+\infty)$. We allow the equalities $p=-\infty$ and $q=+\infty$, so that the former of these sets includes the latter ones. Results of the division $\mathbf{a}/\mathbf{b}$, $0\notin\mathbf{b}$, and of all other operations on objects from $I\mathbb{R}$ are the same both in classical and in Kahan interval arithmetics. For convenience, we write results concerning the division $\mathbf{a}/\mathbf{b}$, $0\in\mathbf{b}$, in detailed form:

I. $\mathbf{b}=0$.
If $0\in\mathbf{a}$, then $\mathbf{a}/\mathbf{b}=\mathbb{R}$ – the whole real axis, and
if $0\notin\mathbf{a}$, then $\mathbf{a}/\mathbf{b}=\varnothing$.

II. $\mathbf{b}\ne 0$.

1) Let zero be one of endpoints of $\mathbf{b}$.
If $0\notin\mathbf{a}$, then $\mathbf{a}/\mathbf{b}=(-\infty;-\langle\mathbf{a}\rangle/|\mathbf{b}|]$
when $\overline{\mathbf{a}}<0\le\underline{\mathbf{b}}$ or $\overline{\mathbf{b}}\le 0<\underline{\mathbf{a}}$,
and $\mathbf{a}/\mathbf{b}=[\langle\mathbf{a}\rangle/|\mathbf{b}|;+\infty)$
when both $\mathbf{a}$ and $\mathbf{b}$ are nonnegative or nonpositive intervals.
If $0\in\mathbf{a}$, then $\mathbf{a}/\mathbf{b}=\mathbb{R}$.

2) Let $\underline{\mathbf{b}}<0<\overline{\mathbf{b}}$.
If $\mathbf{a}>0$, then $\mathbf{a}/\mathbf{b}=(-\infty;\langle\mathbf{a}\rangle/\underline{\mathbf{b}}]\cup[\langle\mathbf{a}\rangle/\overline{\mathbf{b}};+\infty)$
and
if $\mathbf{a}<0$, then $\mathbf{a}/\underline{\mathbf{b}}=(-\infty;-\langle\mathbf{a}\rangle/\overline{\mathbf{b}}]\cup[-\langle\mathbf{a}\rangle/\underline{\mathbf{b}};+\infty)$.
If $0\in\mathbf{a}$ the set $\mathbf{a}/\mathbf{b}$ is the union $(-\infty;0]\cup[0;+\infty)$, i.e. coincides with the whole of $\mathbb{R}$.

It is worthwhile to note that in the Kahan arithmetic the fundamental property

$$\mathbf{a}*\mathbf{b}=\{\,a*b\mid a\in\mathbf{a},\ b\in\mathbf{b}\}\qquad\text{for }*\in\{+,-,\cdot,/\}$$

holds (which is the basis of the classical interval arithmetic too) as well as inclusion monotonicity.

Let $\mathbf{A}$ be an interval $n\times n$-matrix and $\mathbf{b}\in I\mathbb{R}^n$. As was already noted, the united solution set (USS) of the interval linear algebraic system (2) is

$$\Xi^*(\mathbf{A},\mathbf{b})=\Xi^*=\{\,x\in\mathbb{R}^n\mid(\exists A\in\mathbf{A})\ (\exists b\in\mathbf{b})\ (Ax=b)\},$$

and the problem of concern to us is that of computing the most accurate "outer" component-wise estimates for this set, i.e. the problem of evaluating $\min\{x_k\mid x\in\Xi^*(\mathbf{A},\mathbf{b})\}$ from below and $\max\{x_k\mid x\in\Xi^*(\mathbf{A},\mathbf{b})\}$

from above, $k = 1, 2, \ldots n$. In what follows, our attention will be focused on finding $\min\{x_k \mid x \in X^*(\mathbf{A}, \mathbf{b})\}$ because

$$\max\{x_k \mid x \in X^*(\mathbf{A}, \mathbf{b})\} = -\min\{x_k \mid x \in X^*(\mathbf{A}, -\mathbf{b})\}.$$

Besides, in interval analysis $\mathbf{A}$ is usually assumed to contain non-singular matrices only in order to ensure USS boundedness. We shall consider also that some initial approximation interval vector $\mathbf{V} \supseteq X^*(\mathbf{A}, \mathbf{b})$ is already known. It may be found by any of the algorithms presented in [1,3,5,8,11,13–17,24,25] and its size does not matter in the sequel, though, of course, the choice of more "narrow" initial approximation favors faster convergence of the algorithms developed.

2. Basic algorithm

Let l be a straight line in $\mathbb{R}^n$ with parametric equation

$$\begin{cases} x_1 = r_1 \\ \cdots \\ x_{k-1} = r_{k-1} \\ x_k = y \\ x_{k+1} = r_{k+1} \\ \cdots \\ x_n = r_n \qquad (y \in \mathbb{R} - \text{parameter}), \end{cases} \tag{4}$$

parallel to k-th coordinate axis. Each such line is defined completely by the $(n-1)$-dimensional real vector $r = (r_1, \ldots, r_{k-1}, r_{k+1}, \ldots, r_n)^\mathsf{T}$, and to indicate these line parameters explicitly we will sometimes denote the line as $l(r)$. Let also

$$\Omega(r) = \min\{\, x_k \mid x \in X^*(\mathbf{A}, \mathbf{b}) \cap l(r)\}$$

be the least k-th coordinate value of points from intersection of $l(r)$ with the united solution set of ILAS (2) (if $X^*(\mathbf{A}, \mathbf{b}) \cap l(r) = \varnothing$, then set $\Omega(r) = +\infty$). How are the function values $\Omega(r)$ computed?

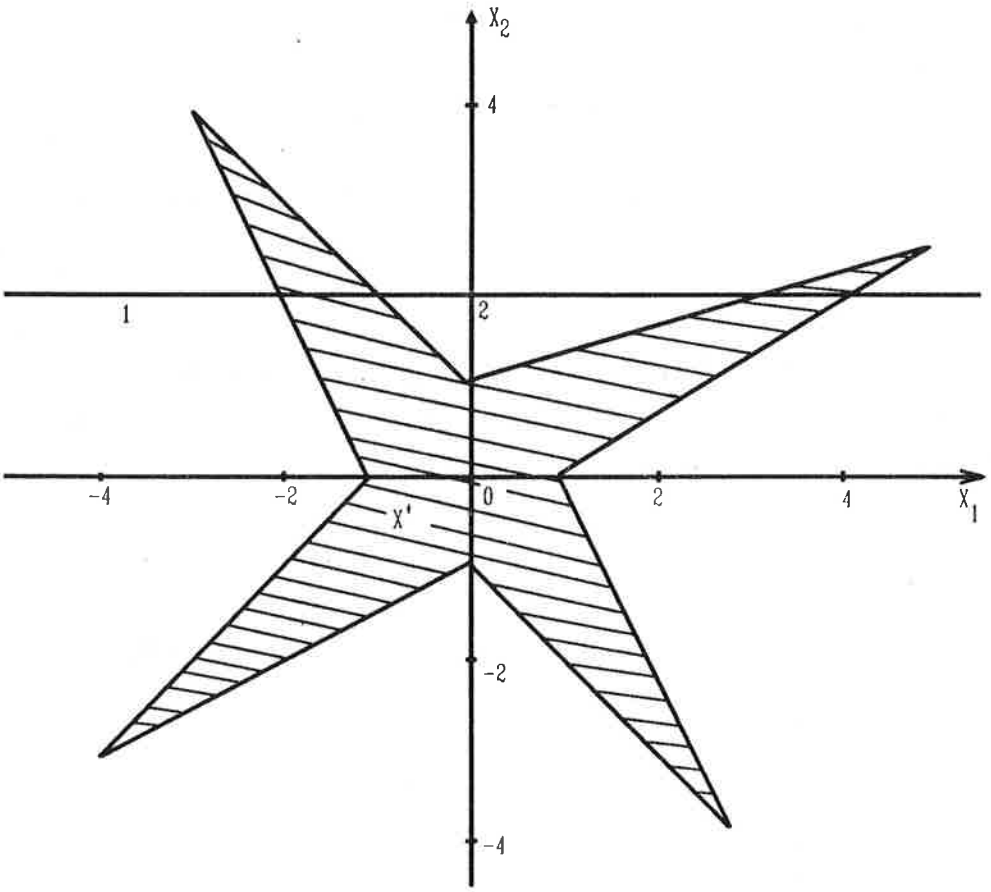

Fig. 1

To answer this question we "substitute" the parametric equation (4) into ILAS (2), which then turns into a system of n linear equations with only one variable y and interval coefficients:

$$\begin{cases} \mathbf{a}_{1k}y + \sum_{\{j=1,j\neq k\}}^{n} \mathbf{a}_{1j}r_j = \mathbf{b}_1, \\ \qquad\qquad \dots \\ \mathbf{a}_{nk}y + \sum_{\{j=1,j\neq k\}}^{n} \mathbf{a}_{nj}r_j = \mathbf{b}_{n1}. \end{cases} \tag{5}$$

or in matrix form

$$\mathbf{A}_k y + \tilde{\mathbf{A}} r = \mathbf{b}\,. \tag{6}$$

where $\mathbf{A}_k$ – k-th column of the matrix $\mathbf{A}$,

$\tilde{\mathbf{A}}$ – interval $n \times (n-1)$-matrix obtained from $\mathbf{A}$ by removing its k-th column.

The essence of this procedure is as follows. When substituting the parametric equation (4) into the point system (1), we get a system of n one-dimensional equations that coincides with (5) in structure, but has real coefficients. Then vary the elements a_{ij} of the matrix and the element b_i of the vector within the prescribed bounds $\mathbf{a}_{ij}$ and $\mathbf{b}_i$, respectively. Clearly the set of all point systems so obtained forms exactly (5)–(6).

It is easily seen that the solution of the i-th equations of this system is the set

$$(\mathbf{b}_i \; - \sum_{\substack{j=1 \\ j \neq k}}^{n} \mathbf{a}_{ij} r_j \;) \; / \; \mathbf{a}_{ik} \tag{7}$$

where "/" is, in general, the Kahan arithmetic division. Having solved separately each of one-dimensional interval linear equations belonging to the system (5), intersect their united solution sets (7). Since within all intervals entering into (5) the corresponding coefficients are varying independently from each other (as in the initial ILAS), the set we have thus gotten gives k-th coordinate values of points from $X^*(\mathbf{A}, \mathbf{b}) \cap l$. Note that it may proved to be empty if the system (5)–(6) is incompatible, or non-connected (as shown in Fig. 1) if some of equations from (5) have solutions $(-\infty; p] \cup [q; +\infty)$, $p < q$. Nevertheless, $X^* \cap l$ is always a bounded set. Indeed, owing to the matrix $\mathbf{A}$'s nonsingularity at least one of $\mathbf{a}_{1k}, \mathbf{a}_{2k}, \ldots, \mathbf{a}_{nk}$ shall not be a zero-containing interval, and the USS of the corresponding equation has to be a bounded set.

For the rest of this paper, the fact of fundamental importance is that of reformulating the outer problem for interval linear algebraic systems as an optimization problem –

$$\begin{aligned} \min\{x_k \mid x \in X^*(\mathbf{A}, \mathbf{b})\} \; &= \; \min\{x_k \mid x \in \bigcup_{l \cap \mathbf{V} \neq \varnothing} (X^*(\mathbf{A}, \mathbf{b}) \cap l\} \; = \\ &= \; \min \{ \; \min\{x_k \mid x \in X^*(\mathbf{A}, \mathbf{b}) \cap l(r) \; \} \mid \\ &\qquad r \in (\mathbf{V}_1, \ldots, \mathbf{V}_{k-1}, \mathbf{V}_{k+1}, \ldots, \mathbf{V}_n) \; \} = \\ &= \; \min\{ \; \Omega(r) \mid r \in (\mathbf{V}_1, \ldots, \mathbf{V}_{k-1}, \mathbf{V}_{k+1}, \ldots, \mathbf{V}_n) \; \} \end{aligned} \tag{8}$$

– i.e. as a problem of minimizing the objective function $\Omega(r)$ on some finite-dimensional compact set. We have already seen how to compute the values $\Omega(r)$, and so one would think that the "outer problem" may

be solved successfully by any of the well-known global optimization algorithms[†]. The objective function $\Omega(r)$, however, has a disagreeable distinction: it is not even continuous in general. Given, for instance, the interval system from [3]

$$\begin{pmatrix} [\ 2;4] & [-2;1] \\ [-1;2] & [\ 2;4] \end{pmatrix} x = \begin{pmatrix} [-2;2] \\ [-2;2] \end{pmatrix}$$

(its united solution set X^* is represented in Fig.1), we have

$$\lim_{r \to -3-0} \Omega(r) = 2 \neq -4 = \lim_{r \to -3+0} \Omega(r)$$

when estimating $\min\{x_1 \mid x \in X^*\}$.

These circumstances crucially limit the range of global optimization algorithms applicable to the problem (8). Currently popular methods from [6] and [20] are obviously unfit for solving (8), because they substantially exploit Lipschitz continuity of the objective function.

Nevertheless, we shall demonstrate that the "outer problem" can be solved by an algorithm of *successively improving estimate* type (in terms of P.S.Pankov's classification [19]). In interval mathematics since the pioneer work of S.Skelboe [30], this this kind of method was developed extensively by R.E.Moore [15], N.S.Asaithambi, Shen Zuhe and R.E.Moore [2], E.Hansen [9], K.Ichida and Y.Fujii [10], H.Ratschek [21] and many others. We need only to construct effective ways to compute a domain minorant for $\Omega(r)$ (its inclusion function's left endpoint).

To put it another way, for any $\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}, \mathbf{r}_{k+1}, \ldots, \mathbf{r}_n)^\top \in I\mathbb{R}^{n-1}$ we have to evaluate

$$\begin{aligned} &\min\{\Omega(r) \mid r \in \mathbf{r}\} = \\ &= \min_{r \in \mathbf{r}} \{x_k \mid x \in X^*(\mathbf{A}, \mathbf{b}) \cap l(r)\} \end{aligned} \tag{9}$$

from below. The simplest way of doing it is as follows. We proceed with the initial interval system (2) just the same as in the case of determining $(X^*(\mathbf{A}, \mathbf{b}) \cap l)$, but now intervals $\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}, \mathbf{r}_{k+1}, \ldots, \mathbf{r}_n$ are

[†]A good survey of Russian works on this subject was done by P.S.Pankov [19].

substituted for $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$ in (2) rather than real numbers $r_1, \ldots, r_{k-1}, r_{k+1}, \ldots, r_n$. Then, as before, we shall compute the united solution set S of the system formed by n one-dimensional interval equations

$$\begin{cases} \mathbf{a}_{1k}y + \sum_{\{j=1, j\neq k\}}^{n} \mathbf{a}_{1j}\mathbf{r}_j = \mathbf{b}_1, \\ \qquad\qquad \ldots \\ \mathbf{a}_{nk}y + \sum_{\{j=1, j\neq k\}}^{n} \mathbf{a}_{nj}\mathbf{r}_j = \mathbf{b}_n, \end{cases} \tag{10}$$

or

$$\mathbf{A}_k y + \tilde{\mathbf{A}}\mathbf{r} = \mathbf{b}\,, \tag{11}$$

in the notation of (6). It is now fairly simple to realize that

$$S \supseteq \bigcup_{r\in\mathbf{r}} \{x_k \mid x \in X^*(\mathbf{A}, \mathbf{b}) \cap l(r)\}$$

inasmuch as

$$\begin{gathered} \{ \text{ USS of the system } \quad \mathbf{A}_k y + \tilde{\mathbf{A}} r = \mathbf{b} \} \subseteq \\ \{ \text{ USS of the system } \quad \mathbf{A}_k y + \tilde{\mathbf{A}}\mathbf{r} = \mathbf{b} \} \end{gathered}$$

for all $r \in \mathbf{r}$. Therefore

$$\Omega(\mathbf{r}) = \min\{ S \cap \mathbf{V}_k \}$$

gives the required low estimate for (9) (cf. with operator Γ in [18]). If the system (10) is incompatible for some $\mathbf{r}$ (that corresponds to $X^*(\mathbf{A}, \mathbf{b}) \cap l(r) = \varnothing$ for all $r \in \mathbf{r}$), then put $\Omega(\mathbf{r}) = +\infty$.

We utilize the notation $\Omega(\mathbf{r})$ to emphasize that the procedure we have implemented is actually a kind of *natural interval extension* of $\Omega(r)$ [15]. We may even consider a function $\Omega : \mathcal{V} \to \mathbb{R}^{\cdot}$ to be defined, with range in the semi-extended real axis $\mathbb{R}^{\cdot} = \mathbb{R} \cup \{+\infty\}$, and with the set $\mathcal{V} = \{\mathbf{r} \in I\mathbb{R}^{n-1} \mid \mathbf{r} \subseteq (\mathbf{V}_1, \ldots, \mathbf{V}_{k-1}, \mathbf{V}_{k+1}, \ldots, \mathbf{V}_n)\}$ as its domain. It is worthwhile to note that $\Omega(\mathbf{r}_1) \geq \Omega(\mathbf{r}_2)$ for $\mathbf{r}_1 \subseteq \mathbf{r}_2$, and evaluating (9) through $\Omega(\mathbf{r})$ is more precise, the thinner the vector $\mathbf{r}$ is, i.e. the smaller $\| \text{ wid } \mathbf{r} \|$ is, provided some natural restrictions on $\mathbf{A}$, $\mathbf{b}$, $\mathbf{V}$, and $\mathbf{r}$ are imposed. This assertion will be discussed thoroughly in §3.

Finally, we are ready to construct our algorithm for determining $\min\{x_k \mid x \in X^*(\mathbf{A},\mathbf{b})\}$. This is an iterative procedure for successively refining this low value estimate based on the well-known "branch and bound method" strategy, similar to what has been done for optimization problems in [2,7,9,10,15,19,21,26,30] and other works. Here, bisections of the initial $(n-1)$-dimensional box

$$(\mathbf{V}_1,\ldots,\mathbf{V}_{k-1},\mathbf{V}_{k+1},\ldots,\mathbf{V}_n)$$

(it contains all vectors r corresponding to lines $l(r)$ that have nonempty intersection with $X^*(\mathbf{A},\mathbf{b})$) to thinner ones $\mathbf{P}$, $\mathbf{P} \subseteq (\mathbf{V}_1,\ldots,\mathbf{V}_{k-1},\mathbf{V}_{k+1},\ldots,\mathbf{V}_n)$, form "branches", while computing $\Omega(\mathbf{P})$ – k-th coordinate estimates of points from $\{X^*(\mathbf{A},\mathbf{b}) \cap l(r) \mid r \in \mathbf{P}\}$ – corresponds to finding "bounds".

The algorithm generates an ordered list Λ consisting of pairs $(\mathbf{P},\Omega(\mathbf{P}))$, $\mathbf{P} \subseteq (\mathbf{V}_1,\ldots,\mathbf{V}_{k-1},\mathbf{V}_{k+1},\ldots,\mathbf{V}_n)$, so that the second members of all pairs increase. The first pair $(\mathbf{Q},\Omega(\mathbf{Q}))$ of the list Λ is of special importance in our consideration. We will call it, as well as the related box $\mathbf{Q}$ and estimate $\Omega(\mathbf{Q})$, the *leading* one. Before starting the algorithm the list Λ contains the only pair $((\mathbf{V}_1,\ldots,\mathbf{V}_{k-1},\mathbf{V}_{k+1},\ldots,\mathbf{V}_n),\underline{V}_k)$. The sequence of steps is then carried out, each divided into several stages:

1. In the leading box $\mathbf{Q}$ choose the largest component $\mathbf{Q}_m$, i.e. the one such that
   $\operatorname{wid}\mathbf{Q}_m = \max \operatorname{wid}\mathbf{Q}_i$. If several components of $\mathbf{Q}$ have maximum width, then m is the number of any one of these.
2. Bisect the leading box $\mathbf{Q}$ in component m to get descendants

$$\mathbf{Q}' = (\mathbf{Q}_1,\ldots,\mathbf{Q}_{m-1},[\underline{\mathbf{Q}}_m;\operatorname{med}\mathbf{Q}_m],\mathbf{Q}_{m+1},\ldots,\mathbf{Q}_n),$$
$$\mathbf{Q}'' = (\mathbf{Q}_1,\ldots,\mathbf{Q}_{m-1},[\operatorname{med}\mathbf{Q}_m;\overline{\mathbf{Q}}_m],\mathbf{Q}_{m+1},\ldots,\mathbf{Q}_n).$$

3. Compute $\Omega(\mathbf{Q}')$ and $\Omega(\mathbf{Q}'')$.
4. Remove the late leading pair $(\mathbf{Q},\Omega(\mathbf{Q}))$ from the list Λ.
5. Insert new pairs $(\mathbf{Q}',\Omega(\mathbf{Q}'))$ and $(\mathbf{Q}'',\Omega(\mathbf{Q}''))$ into Λ in proper order (of increasing second member).

Thus, executing the algorithm yields a non-decreasing (beginning with the second step) leading estimates sequence which is shown in [21] to approximate the required $\min\{x_k \mid x \in X^*(\mathbf{A},\mathbf{b})\}$ from below. In the next section we prove that this sequence converges to the exact $\min\{x_k \mid$

$x \in X^*(\mathbf{A}, \mathbf{b})\}$ value. We shall refer to this class of algorithms to solve the outer problem for interval algebraic equations, based on *adaptive Partitioning of the united Solution Set*, as *PSS-algorithms*. The main idea of the basic method described above will be further developed and updated to result in highly perfected computational procedures.

3. Convergence proof

Lemma. *In the PSS-algorithm the component length sums of the leading boxes tend to zero.*

Proof. We will show that the sequence $\{\sigma_\nu\}$ of component length sums of the leading boxes is majorized by some sequence $\{\Sigma_\nu\} \to 0$. Define

$\sigma(\mathbf{P})$ – the sum of component lengths of a box $\mathbf{P}$,

λ_ν – the set of all boxes $\mathbf{P}$ such that the pair $(\mathbf{P}, \Omega(\mathbf{P}))$ is contained in the list Λ at the beginning of the ν-th algorithm step and then becomes the leading one at a step with some number $\geq \nu$.

It is not hard to see that if

$$\Sigma_\nu = \max_{\mathbf{P} \in \lambda_\nu} \sigma(\mathbf{P})$$

then $\Sigma_\nu \geq 0$, and the sequence $\{\Sigma_\nu\}$ is non-increasing. Indeed, the set $\lambda_{\nu+1}$ contains all boxes from λ_ν except the box $\mathbf{Q}$ that was the leading one at the ν-th step: instead of $\mathbf{Q}$ the set $\lambda_{\nu+1}$ may contain or not contain its descendants $\mathbf{Q}'$ and $\mathbf{Q}''$. Since

$$\sigma(\mathbf{Q}') < \sigma(\mathbf{Q}) \qquad \text{and} \qquad \sigma(\mathbf{Q}'') < \sigma(\mathbf{Q})$$

we conclude

$$\Sigma_\nu = \max_{\mathbf{P} \in \lambda_\nu} \sigma(\mathbf{P}) \geq \max_{\mathbf{P} \in \lambda_{\nu+1}} \sigma(\mathbf{P}) = \Sigma_{\nu+1} > 0.$$

So, what is $\lim \Sigma_\nu$ (which exists by the well-known Weierstrass theorem)?

When $\lim \Sigma_\nu = \delta > 0$ there is positive integer μ such that

$$\frac{2n}{2n-1}\delta > \Sigma_\nu \geq \delta$$

provided $\nu > \mu$ (n denotes dimension) and therefore

$$\frac{2n}{2n-1}\delta \ > \ \sigma(\mathbf{P})$$

for all $\mathbf{P} \in \lambda_\nu$, $\nu > \mu$. Having fixed $\nu > \mu$ consider any box $\mathbf{P} \in \lambda_\nu$. Suppose the largest term in the sum $\sigma(\mathbf{P})$ corresponds to j-th component of $\mathbf{P}$. According to the very definition of λ_ν there exists a positive integer $\mu_{\mathbf{P}}$, $\mu_{\mathbf{P}} > \nu > \mu$ such that $\mathbf{P}$ will become leading box at the $\mu_{\mathbf{P}}$-th step of PSS-algorithm. At that time it will be bisected in its j-th component, and for every descendant $\mathbf{P}'$ of $\mathbf{P}$

$$\sigma(\mathbf{P}') \ \leq \ \sigma(\mathbf{P}) - \sigma(\mathbf{P})/2n \ = \ \frac{2n-1}{2n}\sigma(\mathbf{P}) \ < \delta \quad . \tag{12}$$

is valid. If $M = \max \mu_{\mathbf{P}}$, then inequality (12) holds for all boxes from the set λ_{M+1}. But this contradicts the assumption $\Sigma_\nu \geq \delta \ > \ 0$. Therefore, $\lim \Sigma_\nu = 0$ as required.

This result was also obtained by H.Ratschek [21] and V.I.Senashov [26], but in other ways.

Theorem. *Let the interval linear algebraic system* $\mathbf{A}\,x \ = \mathbf{b}$ *and interval vector* $\mathbf{V} \supseteq X^*(\mathbf{A}, \mathbf{b})$ *be such that*

$$\left\{ \begin{array}{c} \text{for each } i = 1, 2, \ldots, n \text{ the following condition holds :} \\ \text{zero is not an endpoint of } \mathbf{a}_{ik} \\ \text{or} \\ \text{for each } \mathbf{r} \ \subseteq (\mathbf{V}_1, \ldots, \mathbf{V}_{k-1}, \mathbf{V}_{k+1}, \ldots, \mathbf{V}_n) \\ \text{zero is not an endpoint of the interval} \\ (\mathbf{b}_i \ - \ \sum_{\substack{j=1 \\ j \neq k}}^{n} \mathbf{a}_{ij}\mathbf{r}_j). \end{array} \right. \tag{NZ}$$

Then the PSS-algorithm of §2 converges to $\min\{x_k \mid x \in X^*(\mathbf{A}, \mathbf{b})\}$ *from the initial approximation* $\mathbf{V}$.

Proof of the Theorem, in contrast to [2,9,10,15,19,25], is not trivial because the objective function $\Omega(r)$ is in general discontinuous. The most general conditions on the objective function and its domain minorant, sufficient for this type global optimization algorithm to converge, were formulated by Yu.G.Evtushenko and V.A.Rat'kin in [7]. Despite apparent simplicity, their verification, however, may become rather complex

for concrete problems. Our proof of the Theorem boils down to this, actually.

Let $\mathcal{B}$ denote the set of all those $i \in \{1, 2, \ldots, n\}$ for which $0 \notin \mathbf{a}_{ik}$, where the united solution sets of the corresponding equations from (10) are *bounded* intervals $[\underline{y}_i; \overline{y}_i]$. $\mathcal{B}$ is not empty because the interval matrix $\mathbf{A}$ is nonsingular. Also denote $\mathcal{U} = \{1, 2, \ldots, n\} \backslash \mathcal{B}$, and for $i \in \mathbf{CalU}$ let $(-\infty; \underline{y}_i] \cup [\overline{y}_i; +\infty)$ represent the *unbounded* united solution sets of equations from (10) with $0 \in \mathbf{a}_{ik}$, where $\underline{y}_i = -\infty$ or $\overline{y}_i = +\infty$ when the corresponding USS is a ray in $\mathbb{R}$, and $\underline{y}_i = \overline{y}_i = 0$ when the USS coincides with the whole of $\mathbb{R}$.

In interval spaces the standard topology is set by the Hausdorff metric, and all interval arithmetic operations are continuous relative to it [1,14,15]. Hence, $\underline{y}_i$ and $\overline{y}_i$, $i \in \mathcal{B}$, are continuous functions of $(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}, \mathbf{r}_{k+1}, \ldots, \mathbf{r}_n)^\top$ from (10)–(11). But when $0 \in \mathbf{a}_{ik}$ the real numbers $\underline{y}_i$ and $\overline{y}_i$ defining the USS of the one-dimensional equation

$$\mathbf{a}_{ik} y + \sum_{\substack{j=1 \\ j \neq k}}^{n} \mathbf{a}_{ij} \mathbf{r}_j = \mathbf{b}_i,$$

also depend continuously on the interval vector $(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}, \mathbf{r}_{k+1}, \ldots, \mathbf{r}_n)^\top$, if $\underline{\mathbf{a}}_{ik} < 0 < \overline{\mathbf{a}}_{ik}$ or if $(\mathbf{b}_i - \sum_{j \neq k} \mathbf{a}_{ij} \mathbf{r}_j)$ never has zero as one of its endpoints. This follows from formulae of §1 and is ensured by the condition (NZ) of the Theorem. So, we may consider in the sequel the values $\min_{\mathcal{B}} \overline{y}_i$, $\max_{\mathcal{B}} \underline{y}_i$, $\min_{\mathcal{U}} \underline{y}_i$, $\max_{\mathcal{U}} \overline{y}_i$ to be continuous functions of the vector $\mathbf{r}$ from (10)–(11) (as usual $\min \varnothing = +\infty$, $\max \varnothing = -\infty$).

Next we shall prove that the effective domain of the function $\Omega(\mathbf{r})$, i.e. the set $\operatorname{dom} \Omega = \{\mathbf{r} \in \mathcal{V} \mid \Omega(\mathbf{r}) < +\infty\}$, is compact. If the vector $\mathbf{r}$ belongs to $\operatorname{dom} \Omega$, this means that the corresponding system (10)–(11) is compatibile. Then, firstly, the intersection $\cap_{\mathcal{B}} [\underline{y}_i; \overline{y}_i]$ of all bounded solutions of equations from (10)–(11) is nonempty. That is equivalent to $\min_{\mathcal{B}} \overline{y}_i \geq \max_{\mathcal{B}} \underline{y}_i$. Secondly, $\cap_{\mathcal{B}} [\underline{y}_i; \overline{y}_i]$ has nonempty intersection with unbounded solutions $(-\infty; \underline{y}_i] \cup [\overline{y}_i; +\infty)$, $i \in \mathcal{U}$, of one-dimensional equations of the system (10)–(11). The latter is equivalent to

$$(\min_{\mathcal{U}} \underline{y}_i \geq \max_{\mathcal{B}} \underline{y}_i) \vee (\max_{\mathcal{U}} \overline{y}_i \leq \min_{\mathcal{B}} \overline{y}_i)$$

On the whole dom Ω is described by the condition

$$\min\left\{ \min_{\mathcal{B}} \overline{y}_i - \max_{\mathcal{B}} \underline{y}_i,\ \max\{\min_{\mathcal{U}} \underline{y}_i - \max_{\mathcal{B}} \underline{y}_i,\ \min_{\mathcal{B}} \overline{y}_i - \max_{\mathcal{B}} \overline{y}_i\} \right\} \geq 0.$$

Since the function in the right-hand side of this inequality is continuous on $\mathbf{r}$, we have thus proved that dom Ω is closed. Obviously dom Ω is bounded as well, so it is compact.

In general the function $\Omega(\mathbf{r})$ is not even continuous on its effective domain. If $\mathbf{r} \in$ dom Ω, then

$$\Omega(\mathbf{r}) = \begin{cases} \max_{\mathcal{B}} \underline{y}_i & \text{if } \min_{\mathcal{U}} \underline{y}_i \geq \max_{\mathcal{B}} \underline{y}_i, \\ \max\{ \max_{\mathcal{B}} \underline{y}_i,\ \max_{\mathcal{U}} \overline{y}_i\} & \text{if } \min_{\mathcal{U}} \underline{y}_i < \max_{\mathcal{B}} \underline{y}_i \end{cases}$$

(Fig. 2 depicts various situations). Nonetheless, $\Omega(\mathbf{r})$ is lower semi-continuous on $\mathcal{V}$ [31].

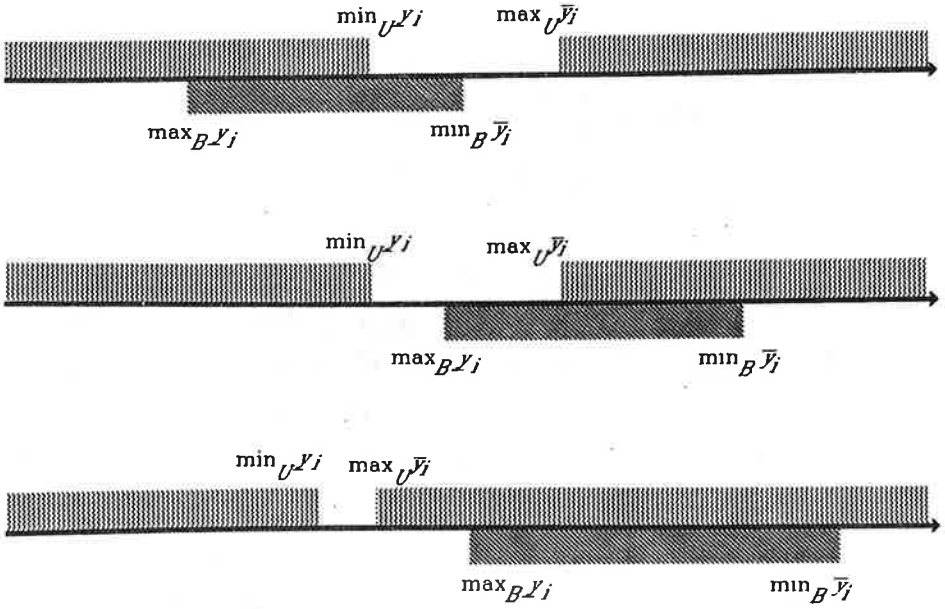

Fig. 2

Indeed, let dom $\Omega = D_0 \cup D_1$, where

$$D_0 = \{\mathbf{r} \in \text{dom } \Omega \subseteq I\mathbb{R}^{n-1} \mid \min_{\mathcal{U}} \underline{y}_i \geq \max_{\mathcal{B}} \underline{y}_i\},$$

$$D_1 = \{\mathbf{r} \in \text{dom } \Omega \subseteq I\mathbb{R}^{n-1} \mid \min_{\mathcal{U}} \underline{y}_i < \max_{\mathcal{B}} \underline{y}_i\}.$$

Then the function $\Omega(\mathbf{r})$ can be defined in the following way:

$$\Omega(\mathbf{r}) = \begin{cases} \max_{\mathcal{B}} \underline{y}_i , & \text{if } \mathbf{r} \in D_0, \\ \max\{\max_{\mathcal{B}} \underline{y}_i , \max_{\mathcal{U}} \overline{y}_i , \} , & \text{if } \mathbf{r} \in D_1 , \\ +\infty , & \text{if } \mathbf{r} \in \mathcal{V}\backslash(D_0 \cup D_1), \end{cases}$$

and $\Omega(\mathbf{r}_0) < \Omega(\mathbf{r}_1) < \Omega(\mathbf{r}_2)$ for any $\mathbf{r}_0 \in D_0$, $\mathbf{r}_1 \in D_1$ and $\mathbf{r}_2 \in \mathcal{V}\backslash\text{dom}\ \Omega$. Moreover, described by non-strict inequality between continuous functions, D_0 is closed, and $\Omega(\mathbf{r})$ is continuous on both D_0 and D_1. So the Lebesgue set $\{\Omega \le c\}$ is closed for any $c \le \sup\{\Omega(\mathbf{r})|\mathbf{r} \in D_0\}$. Suppose now that $c > \sup\{\Omega(\mathbf{r})|\mathbf{r} \in D_0\}$. Then all limit points of $\{\Omega \le c\}$ can belong to $(D_0 \cup D_1)$ only because the complement $\mathcal{V}\backslash(D_0 \cup D_1)$ is open. But the set $\{\Omega \le c\} \cap D_0 = D_0$ is closed in $I\mathbb{R}^{n-1}$ and the set $\{\Omega \le c\} \cap D_1$ is closed in D_1. Hence $\{\Omega \le c\} \cap D_0$ contains all limit points of $\{\Omega \le c\}$ belonging to D_0, and $\{\Omega \le c\} \cap D_1$ contains all limit points of $\{\Omega \le c\}$ belonging to D_1. Since

$$\{\Omega \le c\} = (\{\Omega \le c\} \cap D_0) \cup (\{\Omega \le c\} \cap D_1),$$

we thus get that the Lebesgue set $\{\Omega \le c\}$ is closed for $c > \sup\{\Omega(\mathbf{r})|\mathbf{r} \in D_0\}$ as well. This completes the proof that $\Omega(\mathbf{r})$ is lower semicontinuous [31].

Denote by $\mathcal{W}$ the set of all point vectors from $\mathcal{V}$, i.e. the set $\mathcal{V} \cap \mathbb{R}^{n-1}$. The following reasoning depends heavily on whether we have $\mathcal{W} \cap D_0 \neq \varnothing$ or $\mathcal{W} \cap D_0 = \varnothing$.

If $\mathcal{W} \cap D_0 \neq \varnothing$, then all leading boxes $\mathbf{Q}$ belong to D_0. In fact, for any $p \in \mathcal{W} \cap D_0$ the inequality $\Omega(p) \ge \min\{x_k|x \in X^*(\mathbf{A}, \mathbf{b})\}$ is valid. In case $\mathbf{Q} \in D_1$ we would have $\Omega(\mathbf{Q}) > \Omega(p)$, and thus

$$\Omega(\mathbf{Q}) > \min\{x_k|x \in X^*(\mathbf{A}, \mathbf{b})\},$$

which runs counter to the leading estimate's properties.

Whenever $\mathbf{r} \in D_0$,

$$\min\{\Omega(r)|(r \in \mathbb{R}^{n-1})\&(r \in \mathbf{r})\} = \Omega(r_0) \tag{13}$$

for some real $r_0 \in \mathbf{r}$, $r_0 \in \mathbb{R}^{n-1}$, as long as the lower semicontinuous function $\Omega(\mathbf{r})$ attains its smallest value on the compact set $\{r \in \mathbb{R}^{n-1}|r \in$

$\mathbf{r}\}$ [31]. But $\| r_0 - \mathbf{r} \| \leq \| \operatorname{wid} \mathbf{r} \|$. So, due to uniform continuity of $\Omega(\mathbf{r})$ on D_0, for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 \leq \min\{\Omega(r) | r \in \mathbf{r}\} - \Omega(\mathbf{r}) \leq \epsilon \quad \text{when} \quad \| \operatorname{wid} \mathbf{r} \| \leq \delta \tag{14}$$

Therefore, under condition (NZ), we have rigorously substantiated the assertion of §2 that the accuracy of estimating (9) by means of $\Omega(\mathbf{r})$ is higher the thinner the vector $\mathbf{r}$ is (provided $\mathbf{r} \in D_0$).

Now convergence of the PSS-algorithm of §2 can be easily proved. If $\{\mathbf{Q}^{(\nu)}\}$ is the sequence of leading boxes (as before, ν is a step number), then $\| \operatorname{wid} \mathbf{Q}^{(\nu)} \| \to 0$ by the Lemma. Hence for any $\epsilon > 0$ there is a positive integer N_ϵ such that, analogous to (14), the inequality

$$0 \leq \min\{\Omega(r) | r \in \mathbf{Q}^{(\nu)}\} - \Omega(\mathbf{Q}^{(\nu)}) \leq \epsilon$$

holds for $\nu \geq N_\epsilon$. Also making use of

$$\min\{\Omega(r) | r \in \mathbf{Q}^{(nu)}\} \geq$$

$$\geq \min\{\Omega(r) | r \in (\mathbf{V}_1, \ldots, \mathbf{V}_{k-1}, \mathbf{V}_{k+1}, \ldots, \mathbf{V}_n)\} =$$

$$= \min\{x_k | x \in X^*(\mathbf{A}, \mathbf{b})\} \geq \Omega(\mathbf{Q}^{(\nu)}),$$

we obtain

$$0 \leq \min\{x_k | x \in X^*(\mathbf{A}, \mathbf{b})\} - \Omega(\mathbf{Q}^{(\nu)}) \leq \epsilon \qquad \text{for } \nu \geq N_\epsilon.$$

But this is just the definition of convergence of the simplest PSS-algorithm.

Evidently the above conclusion remains valid if $D_0 = \varnothing$ (then dom $\Omega = D_1$). To make certain of this we shall merely replace D_0 by D_1 in all preceding arguments beginning from (13).

Consider now the case $D_0 \neq \varnothing$ and $\mathcal{W} \cap D_0 = \varnothing$. Let

$$\tau = \min\{ \| \mathbf{r}_0 - r_1 \| \mid (\mathbf{r}_0 \in D_0) \& (r_1 \in \mathcal{W})\}.$$

It is clear that $\tau > 0$, since $\mathcal{W}$ and D_0 are nonintersecting compact sets. Then the set

$$\{\mathbf{r} \in \operatorname{dom} \Omega \mid \min_{r \in \mathcal{W}} \| \mathbf{r} - r \| \leq \tau/2\} \tag{15}$$

also does not intersect D_0, i.e. it is wholly contained in D_1. Obviously $\| \mathbf{r} - r \| \geq \| \text{wid } \mathbf{r} \| /2$, so that

$$\min_r \| \mathbf{r} - r \| \geq \| \text{wid } \mathbf{r} \| /2$$

Hence the set (15) as well as D_1 contains the subset $D_\tau = \{ \mathbf{r} \in \text{dom}\Omega \mid \| \text{wid } \mathbf{r} \| \leq \tau \}$, to which all leading boxes beginning with some number belong. The remaining argument is analogous to that of the previous case: since the function $\Omega(\mathbf{r})$ is uniformly continuous on the compact set D_τ, inequality (14) holds and so on. The theorem is thus completely proved.

4. Improvements

Applying the simplest PSS-algorithm directly to practical problems would evidently be unwise, notwithstanding its above convergence proof. This algorithm can be considerably improved in many ways which are already standard for this kind of method. Usually, such algorithms contain the following modifications (see [2,7,9,10,19] and other works):

after revealing monotonicity of the objective function in some variables one reduces the dimension of boxes from the list Λ;

tracing values of the objective function at some points of boxes along with evaluating over entire boxes enables one to control the precision of the approximation to $\min\{x_k | x \in X^*(\mathbf{A}, \mathbf{b})\}$ and to delete useless pairs (that never become leading pairs) from the list Λ; thanks to the last property growth of the list Λ size is confined to some extent;

based upon local characteristics of the objective function one employs minimization procedures in appropriate boxes which are more effective than bisection;

one constructs a highter quality (more accurate) inclusion function for the objective function.

The latter two improvements are involved ones, and we postpone their careful consideration until a future part of this work. But here we sacrifice efficiency of our algorithms to get generalizations of §5.

As for the first of the above items, noncontinuity of the objective function $\Omega(r)$ seriously complicates determining its monotonicity in any of the variables. The standard way to do this – investigating the sign of

the ranges of the derivative $\partial\Omega(r)/\partial r_i$ – fails now. Moreover, verification of whether $\Omega(r)$ is continuous or not in a box is not at all simple. Taking these facts into account, the author makes ventures to assert that introducing a monotonicity test into the PSS-algorithm does not essentially increase its efficiency, but only weights the algorithm down and makes it more sophisticated. So we shall pursue that idea.

Now, let $\xi(\mathbf{P})$ be a point from $\mathbf{P}$ and let us compute $\Omega(\xi(\mathbf{P}))$ along with the estimate $\Omega(\mathbf{P})$ for boxes $\mathbf{P}$ obtained by the algorithm. It is evident that $\Omega(\xi(\mathbf{P})) \geq \Omega(\mathbf{P})$ and values $\Omega(\xi(\mathbf{P}))$ approximate $\min\{x_k | x \in X^*(\mathbf{A}, \mathbf{b})\}$ from above: if for each step of the algorithm we define

$$\omega = \min \Omega(\xi(\mathbf{P})) \tag{16}$$

for all such boxes $\mathbf{P}$ for which corresponding pairs have ever been in the list Λ up to the current step, then, always,

$$\min\{x_k | x \in X^*(\mathbf{A}, \mathbf{b})\} \leq \omega.$$

On the other hand, given a leading box $\mathbf{Q}$,

$$\Omega(\mathbf{Q}) \leq \min\{x_k | x \in X^*(\mathbf{A}, \mathbf{b})\},$$

and we may now terminate iteration when the quantity $(\omega - \Omega(\mathbf{Q}))$ is sufficiently small.

Therefore a pair $(\mathbf{P}, \Omega(\mathbf{P}))$, satisfying

$$\Omega(\mathbf{P}) > \omega \tag{17}$$

at a some step, never becomes a leading one and deleting it from the list Λ has no influence on the algorithm's performance. This condition is asserted *a priori* for boxes $\mathbf{P}$ with $\Omega(\mathbf{P}) = +\infty$ (it immediately implies $\Omega(\xi(\mathbf{P})) = +\infty$) and corresponding pairs do note even need to be placed into Λ. Altogether, by means of (17) we have to test all newly generated pairs at each step of the algorithm, but completely cleaning the list Λ – running through it and deleting pairs satisfying (17) – makes sense only after the parameter ω changes (i.e. decreases).

The ideal choice for $\xi(\mathbf{P})$ is, of course,

$$\xi(\mathbf{P}) \in \operatorname{Arg} \min\{\Omega(r) | r \in \mathbf{P}\}.$$

In general, however, finding such a desirable $\xi(\mathbf{P})$ is at any rate no easier than solving the initial problem, so we shall take $\xi(\mathbf{P}) = \text{med } \mathbf{P}$ to minimize any possible deviation of $\xi(\mathbf{P})$ from the set $\text{Arg min}\{\Omega(r)|r \in \mathbf{P}\}$.

To summarize, we come to a slightly more perfect version of the PSS-algorithm to compute $\min\{x_k|x \in X^*(\mathbf{A},\mathbf{b})\}$. As before it operates with the list Λ of pairs $(\mathbf{P},\Omega(\mathbf{P}))$, $\mathbf{P} \subseteq (\mathbf{V}_1,\ldots,\mathbf{V}_{k-1},\mathbf{V}_{k+1},\ldots,\mathbf{V}_n)$, ordered in terms of increasing values $\Omega(\mathbf{P})$. Besides, the real parameter ω defined by (16) is connected with the algorithm. At the beginning the list Λ consists only of the pair $((\mathbf{V}_1,\ldots,\mathbf{V}_{k-1},\mathbf{V}_{k+1},\ldots,\mathbf{V}_n),\underline{\mathbf{V}}_k)$, and ω is set to equal $\overline{\mathbf{V}}_k$. One step of executing the algorithm is made up of the following stages :

1. If $(\omega - \Omega(\mathbf{Q})) \leq \epsilon$, then stop computation.
2. In the leading box $\mathbf{Q}$ choose the largest component $\mathbf{Q}_m$.
3. Bisect the leading box in component m to subboxes $\mathbf{Q}'$ and $\mathbf{Q}''$.
4. Remove the previous leading pair $(\mathbf{Q},\Omega(\mathbf{Q}))$ from the list Λ.
5. Compute $\Omega(\mathbf{Q}')$.
6. If $\Omega(\mathbf{Q}') \leq \omega$, then insert the new pair $(\mathbf{Q}',\Omega(\mathbf{Q}'))$ into Λ in the proper order (of increasing second member).
7. Compute $\Omega(\mathbf{Q}'')$.
8. If $\Omega(\mathbf{Q}'') \leq \omega$, then insert the new pair $(\mathbf{Q}'',\Omega(\mathbf{Q}''))$ into Λ in the proper order.
9. Compute

$$\eta' = \begin{cases} \Omega(\text{med } \mathbf{Q}') , & \text{if } (\mathbf{Q}',\Omega(\mathbf{Q}')) \in \Lambda , \\ +\infty , & \text{otherwise} , \end{cases}$$

$$\eta'' = \begin{cases} \Omega(\text{med } \mathbf{Q}'') , & \text{if } (\mathbf{Q}'',\Omega(\mathbf{Q}'')) \in \Lambda , \\ +\infty , & \text{otherwise} , \end{cases}$$

   and $\eta = \min\{\eta',\ \eta''\}$.
10. If $\omega > \eta$, then set $\omega = \eta$ and clean the list Λ : remove from it all pairs $(\mathbf{P},\Omega(\mathbf{P}))$ such that $\Omega(\mathbf{P}) > \omega$.

Here ϵ is the prescribed absolute accuracy. In case ensuring relative accuracy ϵ is required, the termination criterion at stage 1 should be taken as

$$(\omega - \Omega(\mathbf{Q}))/\Omega(\mathbf{Q}) \leq \epsilon$$

or

$$(\omega - \Omega(\mathbf{Q}))/\text{wid } \mathbf{V}_k \leq \epsilon$$

or in some other way in conformity with practical needs.

5. Some generalizations

The version of the PSS-algorithm developed above may be employed to solve interval algebraic systems which are more general than linear equations, i.e. to evaluate the united solution set

$$X_F^* = \{x \in \mathbb{R}^n | (\exists a \in \mathbf{a})(F(x,a) = 0)\}$$

component-wise, for $F(x,a) = (f_1(x,a), f_2(x,a), \ldots, f_n(x,a))$, $a \in I\mathbb{R}^m$ and $f_i(x,a)$ having the form

$$\sum_j a_1^{\alpha_{ij}^1} a_2^{\alpha_{ij}^2} \ldots a_m^{\alpha_{ij}^m} x_1^{\beta_{ij}^1} x_2^{\beta_{ij}^1} \ldots x_n^{\beta_{ij}^n}, i = 1, 2, \ldots, n, \qquad (18)$$

where α_{ij}^l, β_{ij}^l are nonnegative integers. If the USS of the system $F(x, \mathbf{a}) = 0$ is an unbounded set, let X_F^* be an intersection of the whole USS with some interval vector $\mathbf{u}$ given beforehand (similar to [18]).

Suppose we already have an interval vector $\mathbf{V} \supseteq X_F^*$. As is easily seen, further sharpening of k-th component estimate X_F^* to the optimal one can be performed by the PSS-algorithm (leaving the question of convergence open), if the following condition holds :

> substituting real numbers for all variables x_1, ..., x_{k-1}, x_{k+1}, ..., x_n (except x_k) transforms the initial interval system $F(x, \mathbf{a}) = 0$ into a system of one-dimensional linear interval equations with independently varying coefficients (resembling (10)–(11)).

In turn, for the above statement to be valid with $k = 1, 2, \ldots, n$ it is sufficient that

(A) each scalar parameter $a_l^{\alpha_{ij}^m}$ has only one entry in each of $f_i(x,a)$, $i = 1, 2, \ldots, n$,

and

(B) all $f_i(x,a)$ are polylinear functions of the arguments $x_1, x_2, \ldots, x_n$.

Allowing for (A)–(B) leads to a more exact description of a certain class of interval algebraic systems amenable to the PSS-algorithm. Instead of (18) we shall have

$$f_i(x,a) = \sum_j a_j x_1^{\beta_{ij}^1} x_2^{\beta_{ij}^2} \ldots x_n^{\beta_{ij}^n}, \ i = 1, 2, \ldots, n,$$

where $\beta_{ij}^l = 0$ or $\beta_{ij}^l = 1$ for any i, j, l from corresponding sets and each a_j is the only coefficient of one monomial in a single $f_i(x, a)$ (we use the substitution $\tilde{a}_l = a_l^\alpha$ if necessary). If the united solution set is unbounded, employing the PSS-algorithm may require changing slightly the way the function Ω is computed : it shall equal the least k-th coordinate value of points from intersection of the system (10)–(11) USS with the interval vector $\mathbf{u}$.

Sometimes practice requires finding the component-wise united solution set $\tilde{X}^*(\mathbf{A}, \mathbf{b})$ estimates for the interval linear algebraic system $\mathbf{A}x = \mathbf{b}$ whose coefficients are not independent from each other. As in the initial ILAS (2) they are allowed to vary within prescribed intervals $\mathbf{a}_{ij}, \mathbf{b}_i$, but are connected by the additional relations

$$\begin{aligned} &\psi_1(a_{11}, a_{12}, \dots, b_1, \dots) = 0, \\ &\psi_2(a_{11}, a_{12}, \dots, b_1, \dots) = 0, \\ &\dots\dots\dots, \end{aligned} \tag{19}$$

where $\psi_1, \psi_2, \dots$ are some functions. We shall refer to these relations as *ties* between $a_{11}, a_{12}, \dots, b_1, \dots$, and to the corresponding ILAS as a *tied coefficients ILAS*. Traditional interval analysis approaches are unlikely to solve such problems effectively, because classical interval arithmetic operates only with independently varying quantities. Of course, we may ignore ties (19) assuming all coefficients to be independent and then apply any of the well-known methods (e.g., from [1,3–5,8,11,13–17,23,24,25]) to evaluate $\tilde{X}^*(\mathbf{A}, \mathbf{b})$. An interval vector $\mathbf{V}$ will thereby be obtained which contains the USS of the tied coefficients system. $\tilde{X}^*$ and $\mathbf{V}$, however, can differ greatly in size even when $\mathbf{V}$ is optimal for the ILAS without ties. So we ought to consider this vector $\mathbf{V}$ only as an initial approximation for the optimal interval solution of the tied coefficients ILAS.

We shall examine the simplest form of ties (19), namely when some of coefficients a_{ij} are linear combinations of any other parameters

$$a_\gamma = \sum_{k=1}^{t} c_\gamma^k a_k \ , \ c_\gamma^k \in \mathbb{R}, \ \gamma = (i, j) \in \Gamma \subseteq \{1, 2, \dots, n\}^2. \tag{20}$$

Here a_k may be either the initial system's coefficients a_{ij}(after changing notation) or any other, additional parameters. Furthermore, all a_k

are assumed to vary independently within corresponding intervals and none of the a_γ from the left-hand sides of (20) enters any right-hand side of these equalities*. If the ties (20) are already substituted into the real system (1) and all brackets are opened, then some of the a_k may turn out to have several entries in the system so obtained. Assume these are $a_1, a_2, \ldots, a_s$, $s \leq t$. Declaring them as new, extra variables $x_{n+1} = a_1, x_{n+2} = a_2, \ldots, x_{n+s} = a_s$ gives rise to a system of n algebraic equations of degree no more than 2, with $n+s$ variables, in which the independently varying coefficients a_{ij}, $(i,j) \in \{1,2,\ldots,n\}^2 \backslash \Gamma$, $a_{s+1}, a_{s+2}, \ldots, a_t, b_1, b_2, \ldots, b_n$ each enter only once. The corresponding interval system is

$$\begin{cases} \sum_{j\in\mathcal{G}} \sum_{k=1}^{n} c_\gamma^k x_j x_{n+k} + \sum_{j\in\mathcal{G}} \sum_{k=s+1}^{t} c_\gamma^k \mathbf{a}_k x_j + \\ \qquad + \sum_{j\in\mathcal{H}} \mathbf{a}_{ij} x_j = \mathbf{b}_i, \ i \in \mathcal{G}, \\ \sum_{j=1}^{n} \mathbf{a}_{ij} x_j = \mathbf{b}_i, \ i \in \mathcal{H}, \end{cases} \tag{21}$$

where

$$\begin{aligned} \mathcal{G} &= \{i \in \{1,2,\ldots,n\} | (\exists j \in \{1,2,\ldots,n\}\,)(\ (i,j) \in \Gamma\ )\}\,, \\ \mathcal{H} &= \{i \in \{1,2,\ldots,n\} | (\forall j \in \{1,2,\ldots,n\}\,)(\ (i,j) \notin \Gamma\ )\} = \\ &= \{1,2,\ldots,n\} \backslash \mathcal{G}. \end{aligned}$$

It already satisfies the conditions (A)–(B) for applying the PSS-algorithm. On the other hand, $\tilde{X}^*(\mathbf{A},\mathbf{b})$ coincides with the set $\{(x_1, x_2, \ldots, x_n) | x \in Z \subseteq \mathbb{R}^{n+s}\}$ where Z is the intersection of USS of the system (21) with the strip in $\mathbb{R}^{n+s}$ defined by

$$x_{n+1} \in \mathbf{a}_1, \ x_{n+2} \in \mathbf{a}_2, \ldots, \ x_{n+s} \in \mathbf{a}_s.$$

Since an initial approximation for Z is known to be the interval vector $(\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)$, optimal component-wise $\tilde{X}^*(\mathbf{A},\mathbf{b})$ estimates can be found then by applying the PSS-algorithm to the system (21) on the 1-st, the 2-nd, ... , n-th coordinates.

*F.Šik in [28] dealt with a similar problem, but his primary interest was in characterizing the set $\tilde{X}^*(\mathbf{A},\mathbf{b})$. J.Rohn [22] tackled the problem (3) with additional weighed sum constraints imposed on the columns of real matrices from A.

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