INTERVAL ANALYSIS: VERIFYING FEASIBILITY

Introduction.

Constrained optimization problems are of the form

$$\begin{array}{lrcl} \text{minimize} & \phi(x) & & \\ \text{subject to} & c_i(x) & = & 0, \quad i=1,\ldots,m, \\ & g_j(x) & \leq & 0, \quad j=1,\ldots q_1 \\ & \underline{x}_{i_k} \leq & x_{i_k}, & k=1,\ldots q_2-\mu, \\ & x_{i_k} \leq & \overline{x}_{i_k}, & k=\mu+1,\ldots q_2, \end{array} \tag{1}$$

where $\phi: \mathbf{R}^n \to \mathbf{R}$, $c_i: \mathbf{R}^n \to \mathbf{R}$, and $g_j: \mathbf{R}^n \to \mathbf{R}$.

In interval branch and bound algorithms for finding global optima for Problem (1), a search box of the form

$$\boldsymbol{x} = \left\{(x_1,x_2,\ldots,x_n)^T \in \mathbf{R}^n \,\middle|\, \underline{x}_i \leq x_i \leq \overline{x}_i, 1 \leq i \leq n\right\}, \tag{2}$$

is generally given, where some of the sides in (2) correspond to bound constraints of problem (1), and the other sides merely define the extent of the search region. If there are no constraints c_i and g_j, then the box $\boldsymbol{x}$ is systematically tessellated into sub-boxes. The branch and bound algorithm, in its most basic form, proceeds as follows: Over each sub-box $\tilde{\boldsymbol{x}}$, $\phi(\check{x})$ is computed for some $\check{x} \in \tilde{\boldsymbol{x}}$, and the range of ϕ over $\tilde{\boldsymbol{x}}$ is bounded (e.g. with a straightforward interval evaluation). (See **Interval analysis : Introduction, interval numbers and basic properties of interval arithmetic**.) If there are no constraints c_i and g_j, then the value $\phi(\check{x})$ represents an upper bound on the minimum of ϕ. The minimum such value $\underline{\phi}$ is kept as the tessellation and search proceed; if any box $\tilde{\boldsymbol{x}}$ has a lower range bound greater than $\underline{\phi}$, it is rejected as not containing a global optimum. See [1], [2], or [3] for details of such algorithms.

The situation is more complicated in the constrained case. In particular, the values $\phi(\check{x})$ cannot be taken as upper bounds on the global optimum unless it is known that $\check{x}$ is feasible. More generally, an upper bound on the range of ϕ over a small box $\check{\boldsymbol{x}}$ can be taken as an upper bound for the global optimum provided it is proven that there exists a feasible point of Problem (1) within $\check{\boldsymbol{x}}$. This article outlines how this can be done.

For the fundamental concepts used throughout this article, see **Interval analysis : Introduction, interval numbers and basic properties of interval arithmetic**.

General feasibility: the Fritz-John conditions.

An interval Newton method (see **Interval analysis: Interval Newton methods**) can sometimes be used to prove existence of a feasible point of Problem (1) that is a critical point of ϕ. In particular, the interval Newton method can sometimes prove existence of a solution to the Lagrange multiplier or *Fritz–John* system within $\check{\boldsymbol{x}}$. For the Fritz–John system, it is convenient to consider the q_2 bound constraints in the same form as the q_1 general inequality constraints, so that there are $q = q_1 + q_2$ general inequality constraints of the from $g_j(x) \leq 0$. With that, the Fritz–John system can be written as

$$F(W) = \begin{pmatrix} u_0\nabla\phi(X) + \sum_{j=1}^{q} u_j \nabla g_j + \sum_{i=1}^{m} v_i \nabla c_i(X) \\ u_1 g_1 \\ \vdots \\ u_q g_q \\ c_1(X) \\ \vdots \\ c_m(X) \\ \left(u_0 + \sum_{j=1}^{q} u_j + \sum_{i=1}^{m} v_i^2\right) - 1 \end{pmatrix} = 0, \tag{3}$$

where $u_j \geq 0$, $j = 1,\ldots,q$, the v_i are unconstrained, and the last equation is one of several

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possible normalization conditions. For details, see [1, §10.5] or [2, §5.2.5].

However, computational problems occur in practice with the system (3). It is more difficult to find a good approximate critical point (for an appropriate small box $\check{\boldsymbol{x}}$) of the entire system (3) than it is to find a point where the inequality and equality constraints are satisfied. Furthermore, if an interval Newton method is applied to (3) over a large box, the corresponding interval Jacobi matrix or slope matrix typically contains singular matrices and hence is useless for existence verification. This is especially true if it is difficult to get good estimates for the *Lagrange multipliers* u_j and v_i. For this reason, the techniques outlined below are useful.

Feasibility of inequality constraints.

Proving feasibility of the inequality constraints is sometimes possible by evaluating the g_j with interval arithmetic: If $\boldsymbol{g}_j(\check{\boldsymbol{x}}) \leq 0$, then every point in $\check{\boldsymbol{x}}$ is feasible with respect to the constraint $g_j(x) \leq 0$; see **Interval analysis: Introduction, interval numbers, and basic properties of interval arithmetic** or [3]. However, if $\check{\boldsymbol{x}}$ corresponds to a point at which g_j is active, then $0 \in \boldsymbol{g}_j(\check{\boldsymbol{x}})$, and no conclusion can be reached from an interval evaluation. In such cases, feasibility can sometimes be proven by treating $g_j(x) = 0$ as one of the equality constraints, then using the techniques below.

Infeasibility.

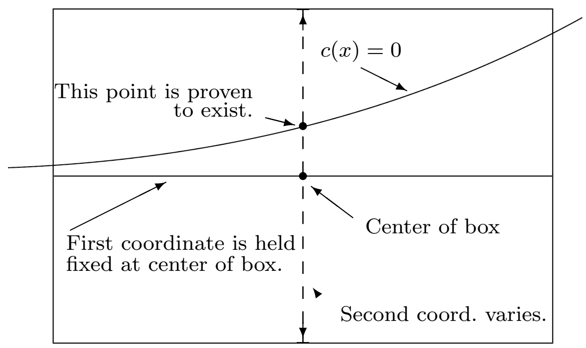
An inequality constraint g_j is proven infeasible over $\check{\boldsymbol{x}}$ if $\boldsymbol{g}_j(\check{\boldsymbol{x}}) > 0$, and an equality constraint c_i is infeasible over $\check{\boldsymbol{x}}$ if either $\boldsymbol{c}_i(\check{\boldsymbol{x}}) > 0$ or $\boldsymbol{c}_i(\check{\boldsymbol{x}}) < 0$. See **Interval analysis: Introduction, interval numbers, and basic properties of interval arithmetic** or [3].

Feasibility of equality constraints.

The equality constraints

$$c(x) = \left(c_1(x), \ldots, c_m(x)\right)^T = 0,$$

$c : \mathbf{R}^n \to \mathbf{R}^m$, $n \geq m$ can be considered an underdetermined system of equations, whereas interval Newton methods generally prove existence / uniqueness for square systems. However, fixing $n - m$ coordinates $\check{x}_i \in \check{\boldsymbol{x}}_i$ allows interval Newton methods to work with $\tilde{c} : \mathbf{R}^m \to \mathbf{R}^m$, to prove existence of a feasible point within $\check{\boldsymbol{x}}$. In principle, indices of the coordinates to be held fixed are chosen to correspond to coordinates in which c is varying least rapidly. For a set of test problems, the most successful way appears to be choosing those coordinates corresponding to the right-most columns after Gaussian elimination with complete pivoting has been applied to the rectangular matrix $c'(\check{x})$ for some $\check{x} \in \check{\boldsymbol{x}}$. The figure below illustrates the process in two dimensions.



Proving that there exists a feasible point of an underdetermined constraint system

Certain complications arise. For example, if bound constraints or inequality constraints are active, then either the point $\check{x}$ must be perturbed or else the bound or inequality constraints must be treated as equality constraints. Handling this case by perturbation is discussed in [2, p. 191 ff].

For the original explanation of the Gaussian elimination-based process, see [1, §12.4]. In [2, §5.2.4], additional background, discussion, and references appear.

References

[1] Hansen, E. R.: *Global Optimization Using Interval Analysis*, Marcel Dekker, Inc., New York, 1992.

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Lagrange multipliers

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