



(slide 1)

GlobSol – Present State and Future Developments

R. Baker Kearfott
with
Anthony Holmes

Department of Mathematics
University of Louisiana at Lafayette

INVA2007, February 27, 2007

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering



(slide 2)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Outline

- 1 Introduction**
 - What Kind of Solution is Needed?
 - Local Versus Global Optimization
- 2 Comparing GlobSol and Other Software**
 - A Synopsis of GlobSol
 - Competing Global Optimization Algorithms
 - A Simple Example
 - Results from the Simple Example
 - Observations and Conclusions
- 3 Planned Improvements**
 - Interface and Packaging Improvements
 - On Tessellation and Clustering

The General Global Optimization Problem

Mathematical Description

$$\begin{aligned} & \text{minimize } \varphi(\mathbf{x}) \\ & \text{subject to } c_i(\mathbf{x}) = 0, i = 1, \dots, m_1, \\ & \quad \quad \quad g_i(\mathbf{x}) \leq 0, i = 1, \dots, m_2, \\ & \text{where } \varphi : \mathbb{R}^n \rightarrow \mathbb{R} \text{ and } c_i, g_i : \mathbb{R}^n \rightarrow \mathbb{R}. \end{aligned}$$

- We refer to the region defined by the constraints as \mathbf{D} .
- Often bounds on the search region are given by $\mathbf{x} = ([\underline{x}_1, \bar{x}_1], \dots, [\underline{x}_n, \bar{x}_n])$.
- What kind of answer is useful to the modelers and applications people?
- Where can interval computations contribute?
- How can interval computations, and GlobSol in particular, be made to better fill the modelers' needs?



(slide 4)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Outline

- 1 Introduction**
 - What Kind of Solution is Needed?
 - Local Versus Global Optimization
- 2 Comparing GlobSol and Other Software**
 - A Synopsis of GlobSol
 - Competing Global Optimization Algorithms
 - A Simple Example
 - Results from the Simple Example
 - Observations and Conclusions
- 3 Planned Improvements**
 - Interface and Packaging Improvements
 - On Tessellation and Clustering

Differing Requirements

Examples

- φ is the cost of running a (nominally) \$50,000,000 per month plant:

The plant manager would like the smallest possible operating cost, but would be happy with a 5% lower cost than before.

- φ represents the potential energy of a particular conformation of a molecule:

The globally lowest value for φ gives the most information, but local minima give some information, and finding the global minimum may not be practical.

- A mathematician has reduced a proof to showing that $\varphi \geq 1$ everywhere:

The global minimum must not only be found, but also must be rigorously proven to be so.

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Differing Requirements

Examples

- φ is the cost of running a (nominally) \$50,000,000 per month plant:
The plant manager would like the smallest possible operating cost, but would be happy with a 5% lower cost than before.
- φ represents the potential energy of a particular conformation of a molecule:
The globally lowest value for φ gives the most information, but local minima give some information, and finding the global minimum may not be practical.
- A mathematician has reduced a proof to showing that $\varphi \geq 1$ everywhere:
The global minimum must not only be found, but also must be rigorously proven to be so.

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering



(slide 7)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Differing Requirements

Examples

- φ is the cost of running a (nominally) \$50,000,000 per month plant:
The plant manager would like the smallest possible operating cost, but would be happy with a 5% lower cost than before.
- φ represents the potential energy of a particular conformation of a molecule:
The globally lowest value for φ gives the most information, but local minima give some information, and finding the global minimum may not be practical.
- A mathematician has reduced a proof to showing that $\varphi \geq 1$ everywhere:
The global minimum must not only be found, but also must be rigorously proven to be so.

Differing Requirements

(Continued)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- A portfolio manager has estimated the expected rate of return and a risk measure for various stocks, and has a particular amount to invest among the stocks. The portfolio manager would like to allocate the investment to minimize the total risk, subject to a lower bound on the total rate of return.
- If the global optimizer is not unique, the portfolio manager would like to know that, since he may have other criteria for choosing stocks that weren't included in the original model.

Differing Requirements

(Continued)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- A portfolio manager has estimated the expected rate of return and a risk measure for various stocks, and has a particular amount to invest among the stocks. The portfolio manager would like to allocate the investment to minimize the total risk, subject to a lower bound on the total rate of return.
- If the global optimizer is not unique, the portfolio manager would like to know that, since he may have other criteria for choosing stocks that weren't included in the original model.



(slide 10)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Outline

1 Introduction

What Kind of Solution is Needed?
Local Versus Global Optimization

2 Comparing GlobSol and Other Software

A Synopsis of GlobSol
Competing Global Optimization Algorithms
A Simple Example
Results from the Simple Example
Observations and Conclusions

3 Planned Improvements

Interface and Packaging Improvements
On Tessellation and Clustering

Local Versus Global Optimization

Illustration

(slide 11)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

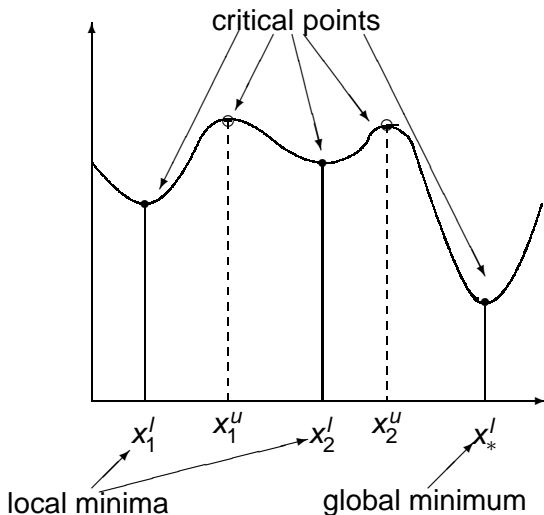
Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering





(slide 12)

Local Versus Global Optimization

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- ***Global Optimization*** is not to be confused with *local optimization*, a much more tractable problem.
- Global optimization algorithms can be deterministic or heuristic.
- Deterministic algorithms can be automatically verified (taking account of roundoff error) or not.
- The goal in global optimization can be:
 - finding the global optimum and (perhaps) a single optimizing point;
 - finding the global optimum and all global optimizing points.



(slide 13)

Local Versus Global Optimization

- *Global Optimization* is not to be confused with *local optimization*, a much more tractable problem.
- Global optimization algorithms can be deterministic or heuristic.
- Deterministic algorithms can be automatically verified (taking account of roundoff error) or not.
- The goal in global optimization can be:
 - finding the global optimum and (perhaps) a single optimizing point;
 - finding the global optimum and all global optimizing points.

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering



Local Versus Global Optimization

(slide 14)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- *Global Optimization* is not to be confused with *local optimization*, a much more tractable problem.
- Global optimization algorithms can be deterministic or heuristic.
- Deterministic algorithms can be automatically verified (taking account of roundoff error) or not.
- The goal in global optimization can be:
 - finding the global optimum and (perhaps) a single optimizing point;
 - finding the global optimum and all global optimizing points.



Local Versus Global Optimization

(slide 15)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- *Global Optimization* is not to be confused with *local optimization*, a much more tractable problem.
- Global optimization algorithms can be deterministic or heuristic.
- Deterministic algorithms can be automatically verified (taking account of roundoff error) or not.
- The goal in global optimization can be:
 - finding the global optimum and (perhaps) a single optimizing point;
 - finding the global optimum and all global optimizing points.



(slide 16)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Outline

- 1 Introduction
What Kind of Solution is Needed?
Local Versus Global Optimization
- 2 Comparing GlobSol and Other Software
A Synopsis of GlobSol
Competing Global Optimization Algorithms
A Simple Example
Results from the Simple Example
Observations and Conclusions
- 3 Planned Improvements
Interface and Packaging Improvements
On Tessellation and Clustering



(slide 17)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

What is GlobSol?

- A Fortran 90 package
 - well-tested.
 - self-contained.
- Solves constrained and unconstrained global optimization problems
- Separate program solves square algebraic systems of equations.
- Utility programs for interval and point evaluation, etc.
- Subroutine / module libraries for interval arithmetic, automatic differentiation, etc.
- Publicly available free of charge:
http://interval.louisiana.edu/GlobSol/download_GlobSol.html

A Brief History of GlobSol

(slide 18)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- Began in 1985 as an individual project of mine.
- The first software package: `INTBIS`, an ACM *Transactions on Mathematical Software* algorithm for finding all solutions to polynomial systems (joint work with Manuel Novoa).
- An interval library for standard functions, `INTLIB`, was developed for `INTBIS` (joint work of several), and also became an ACM TOMS algorithm.
- For object-oriented program structure, the basic algorithm was rewritten in Fortran 90, becoming `INTOPT-90`.

A Brief History of GlobSol



(slide 19)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- Began in 1985 as an individual project of mine.
- The first software package: INTBIS, an ACM *Transactions on Mathematical Software* algorithm for finding all solutions to polynomial systems (joint work with Manuel Novoa).
- An interval library for standard functions, INTLIB, was developed for INTBIS (joint work of several), and also became an ACM TOMS algorithm.
- For object-oriented program structure, the basic algorithm was rewritten in Fortran 90, becoming INTOPT-90.

A Brief History of GlobSol

(slide 20)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- Began in 1985 as an individual project of mine.
- The first software package: INTBIS, an ACM *Transactions on Mathematical Software* algorithm for finding all solutions to polynomial systems (joint work with Manuel Novoa).
- An interval library for standard functions, INTLIB, was developed for INTBIS (joint work of several), and also became an ACM TOMS algorithm.
- For object-oriented program structure, the basic algorithm was rewritten in Fortran 90, becoming INTOPT-90.

A Brief History of GlobSol



(slide 21)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- Began in 1985 as an individual project of mine.
- The first software package: INTBIS, an ACM *Transactions on Mathematical Software* algorithm for finding all solutions to polynomial systems (joint work with Manuel Novoa).
- An interval library for standard functions, INTLIB, was developed for INTBIS (joint work of several), and also became an ACM TOMS algorithm.
- For object-oriented program structure, the basic algorithm was rewritten in Fortran 90, becoming INTOPT-90.



History of GlobSol Continued

More on INTOPT-90

(slide 22)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- Included automatic differentiation.
- Is described in my book *Rigorous Global Search: Continuous Problems* (Kluwer, 1996).
- A spinoff: Module `INTERVAL_ARITHMETIC`, which defines an interval data type in Fortran, using `INTLIB` as supporting library.
- Initially included validated solution of nonlinear equations with interval Newton methods and constraint propagation and some techniques for constrained global optimization.
- Subsequently developed into GlobSol.



History of GlobSol Continued

More on INTOPT-90

(slide 23)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- Included automatic differentiation.
- Is described in my book *Rigorous Global Search: Continuous Problems* (Kluwer, 1996).
- A spinoff: Module `INTERVAL_ARITHMETIC`, which defines an interval data type in Fortran, using `INTLIB` as supporting library.
- Initially included validated solution of nonlinear equations with interval Newton methods and constraint propagation and some techniques for constrained global optimization.
- Subsequently developed into GlobSol.



History of GlobSol Continued

More on INTOPT-90

(slide 24)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- Included automatic differentiation.
- Is described in my book *Rigorous Global Search: Continuous Problems* (Kluwer, 1996).
- A spinoff: Module `INTERVAL_ARITHMETIC`, which defines an interval data type in Fortran, using `INTLIB` as supporting library.
- Initially included validated solution of nonlinear equations with interval Newton methods and constraint propagation and some techniques for constrained global optimization.
- Subsequently developed into GlobSol.



History of GlobSol Continued

More on INTOPT-90

(slide 25)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- Included automatic differentiation.
- Is described in my book *Rigorous Global Search: Continuous Problems* (Kluwer, 1996).
- A spinoff: Module `INTERVAL_ARITHMETIC`, which defines an interval data type in Fortran, using `INTLIB` as supporting library.
- Initially included validated solution of nonlinear equations with interval Newton methods and constraint propagation and some techniques for constrained global optimization.
- Subsequently developed into GlobSol.

History of GlobSol Continued

More on INTOPT-90

(slide 26)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- Included automatic differentiation.
- Is described in my book *Rigorous Global Search: Continuous Problems* (Kluwer, 1996).
- A spinoff: Module `INTERVAL_ARITHMETIC`, which defines an interval data type in Fortran, using `INTLIB` as supporting library.
- Initially included validated solution of nonlinear equations with interval Newton methods and constraint propagation and some techniques for constrained global optimization.
- Subsequently developed into GlobSol.



History of GlobSol Continued

GlobSol Proper

(slide 27)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Starting in 1998 with INTOPT-90, a joint SunSoft Research and Development Contract enabled the development of GlobSol in its present form. Activities included

- formal testing and extensive debugging;
- somewhat more user-friendly interfaces and packaging;
- a more thorough study of constrained problems;
- experience with practical applications.

George Corliss, Chenyi Hu, and various graduate students participated in this contract, while thanks is due to Bill Walster for financial support.



History of GlobSol Continued

GlobSol Proper

(slide 28)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Starting in 1998 with INTOPT-90, a joint SunSoft Research and Development Contract enabled the development of GlobSol in its present form. Activities included

- formal testing and extensive debugging;
- somewhat more user-friendly interfaces and packaging;
- a more thorough study of constrained problems;
- experience with practical applications.

George Corliss, Chenyi Hu, and various graduate students participated in this contract, while thanks is due to Bill Walster for financial support.



History of GlobSol Continued

GlobSol Proper

(slide 29)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Starting in 1998 with INTOPT-90, a joint SunSoft Research and Development Contract enabled the development of GlobSol in its present form. Activities included

- formal testing and extensive debugging;
- somewhat more user-friendly interfaces and packaging;
- a more thorough study of constrained problems;
- experience with practical applications.

George Corliss, Chenyi Hu, and various graduate students participated in this contract, while thanks is due to Bill Walster for financial support.



History of GlobSol Continued

GlobSol Proper

(slide 30)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Starting in 1998 with INTOPT-90, a joint SunSoft Research and Development Contract enabled the development of GlobSol in its present form. Activities included

- formal testing and extensive debugging;
- somewhat more user-friendly interfaces and packaging;
- a more thorough study of constrained problems;
- experience with practical applications.

George Corliss, Chenyi Hu, and various graduate students participated in this contract, while thanks is due to Bill Walster for financial support.

History of GlobSol Continued

GlobSol Proper

(slide 31)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Starting in 1998 with INTOPT-90, a joint SunSoft Research and Development Contract enabled the development of GlobSol in its present form. Activities included

- formal testing and extensive debugging;
- somewhat more user-friendly interfaces and packaging;
- a more thorough study of constrained problems;
- experience with practical applications.

George Corliss, Chenyi Hu, and various graduate students participated in this contract, while thanks is due to Bill Walster for financial support.



(slide 32)

GlobSol

Developments Since the SunSoft Project

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- Linear relaxations, with rigorous underestimates based on the Neumaier / Shcherbina formula have been implemented.
- Symbolic analysis of convex subspaces, along the lines done in work by Epperly and Pistikopoulos, to allow work in smaller-dimensional subspaces, has been implemented.

Although powerful in some contexts, these will only be made publicly available later, pending researching certain licensing and packaging issues.



(slide 33)

GlobSol

Developments Since the SunSoft Project

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- Linear relaxations, with rigorous underestimates based on the Neumaier / Shcherbina formula have been implemented.
- Symbolic analysis of convex subspaces, along the lines done in work by Epperly and Pistikopoulos, to allow work in smaller-dimensional subspaces, has been implemented.

Although powerful in some contexts, these will only be made publicly available later, pending researching certain licensing and packaging issues.

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

GlobSol

Developments Since the SunSoft Project

- Linear relaxations, with rigorous underestimates based on the Neumaier / Shcherbina formula have been implemented.
- Symbolic analysis of convex subspaces, along the lines done in work by Epperly and Pistikopoulos, to allow work in smaller-dimensional subspaces, has been implemented.

Although powerful in some contexts, these will only be made publicly available later, pending researching certain licensing and packaging issues.



(slide 35)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Outline

- 1 Introduction
 - What Kind of Solution is Needed?
 - Local Versus Global Optimization
- 2 Comparing GlobSol and Other Software
 - A Synopsis of GlobSol
 - Competing Global Optimization Algorithms**
 - A Simple Example
 - Results from the Simple Example
 - Observations and Conclusions
- 3 Planned Improvements
 - Interface and Packaging Improvements
 - On Tessellation and Clustering



Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

The Neumaier Benchmark

In “A Comparison of Complete Global Optimization Solvers” (*Math. Prog. B* **103**, pp. 335–356), Neumaier, Shcherbina, Huyer, and Vinkó compare BARON/GAMS, GlobSol, ICOS, LGO/GAMS, LINGO, and OQNLP/GAMS. The analysis shows a clear advantage of BARON, and shows that GlobSol often requires much more time.

- LGO/GAMS, LINGO, and OQNLP/GAMS use heuristics or statistics, and thus are not mathematically rigorous.
- ICOS is rigorous and efficient, but the global optimization capabilities are still being developed (from a constraint satisfaction package).
- BARON would be mathematically rigorous in the absence of roundoff error, but we will see a qualitative difference between solutions given by GAMS / BARON and those given by GlobSol.



Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

The Neumaier Benchmark

In “A Comparison of Complete Global Optimization Solvers” (*Math. Prog. B* **103**, pp. 335–356), Neumaier, Shcherbina, Huyer, and Vinkó compare BARON/GAMS, GlobSol, ICOS, LGO/GAMS, LINGO, and OQNLP/GAMS. The analysis shows a clear advantage of BARON, and shows that GlobSol often requires much more time.

- LGO/GAMS, LINGO, and OQNLP/GAMS use heuristics or statistics, and thus are not mathematically rigorous.
- ICOS is rigorous and efficient, but the global optimization capabilities are still being developed (from a constraint satisfaction package).
- BARON would be mathematically rigorous in the absence of roundoff error, but we will see a qualitative difference between solutions given by GAMS / BARON and those given by GlobSol.



Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

The Neumaier Benchmark

In “A Comparison of Complete Global Optimization Solvers” (*Math. Prog. B* **103**, pp. 335–356), Neumaier, Shcherbina, Huyer, and Vinkó compare BARON/GAMS, GlobSol, ICOS, LGO/GAMS, LINGO, and OQNLP/GAMS. The analysis shows a clear advantage of BARON, and shows that GlobSol often requires much more time.

- LGO/GAMS, LINGO, and OQNLP/GAMS use heuristics or statistics, and thus are not mathematically rigorous.
- ICOS is rigorous and efficient, but the global optimization capabilities are still being developed (from a constraint satisfaction package).
- BARON would be mathematically rigorous in the absence of roundoff error, but we will see a qualitative difference between solutions given by GAMS / BARON and those given by GlobSol.



(slide 39)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Outline

- 1 Introduction
 - What Kind of Solution is Needed?
 - Local Versus Global Optimization
- 2 Comparing GlobSol and Other Software
 - A Synopsis of GlobSol
 - Competing Global Optimization Algorithms
 - A Simple Example**
 - Results from the Simple Example
 - Observations and Conclusions
- 3 Planned Improvements
 - Interface and Packaging Improvements
 - On Tessellation and Clustering



A Linear Financial Problem

(slide 40)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

An investment company needs to decide how to invest \$200,000 in a mix of four stocks, with the following expected rates of return and measures of risk.

STOCKS

	A	B	C	D
Price per share	\$100	\$50	\$80	\$40
Return per share	0.12	0.08	0.06	0.10
Risk measure per \$	0.10	0.07	0.05	0.08

Price per share
Return per share
Risk measure per \$

- The annual rate of return must be at least 9%.
- No one stock can account for more than 50% of the total investment.
- We wish to minimize the risk subject to these conditions.



A Linear Financial Problem

The Corresponding Linear Program

(slide 41)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Minimize $10A + 3.5B + 4C + 3.2D$

Subject to $100A + 50B + 80C + 40D \leq 200,000,$
 $12A + 4B + 4.8C + 4D \geq 18,000,$

$$0 \leq 100A \leq 100000,$$

$$0 \leq 50B \leq 100000,$$

$$0 \leq 80C \leq 100000,$$

$$0 \leq 40D \leq 100000.$$

A Linear Financial Problem

Some Observations

(slide 42)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- This is a simple linear problem, that could appear in an elementary text book.
- This problem was not designed to be special, and there is no reason to think it should be.
- However, we will see that a crucial property of this problem can be overlooked when a given solver is used.
- The property may be of interest in the application area.

A Linear Financial Problem

Some Observations

(slide 43)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- This is a simple linear problem, that could appear in an elementary text book.
- This problem was not designed to be special, and there is no reason to think it should be.
- However, we will see that a crucial property of this problem can be overlooked when a given solver is used.
- The property may be of interest in the application area.

A Linear Financial Problem

Some Observations

(slide 44)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- This is a simple linear problem, that could appear in an elementary text book.
- This problem was not designed to be special, and there is no reason to think it should be.
- However, we will see that a crucial property of this problem can be overlooked when a given solver is used.
- The property may be of interest in the application area.

A Linear Financial Problem

Some Observations

(slide 45)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- This is a simple linear problem, that could appear in an elementary text book.
- This problem was not designed to be special, and there is no reason to think it should be.
- However, we will see that a crucial property of this problem can be overlooked when a given solver is used.
- The property may be of interest in the application area.



(slide 46)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Outline

- 1 Introduction
 - What Kind of Solution is Needed?
 - Local Versus Global Optimization
- 2 Comparing GlobSol and Other Software
 - A Synopsis of GlobSol
 - Competing Global Optimization Algorithms
 - A Simple Example
 - Results from the Simple Example**
 - Observations and Conclusions
- 3 Planned Improvements
 - Interface and Packaging Improvements
 - On Tessellation and Clustering

The Linear Financial Problem

Results from Various Solvers

(slide 47)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

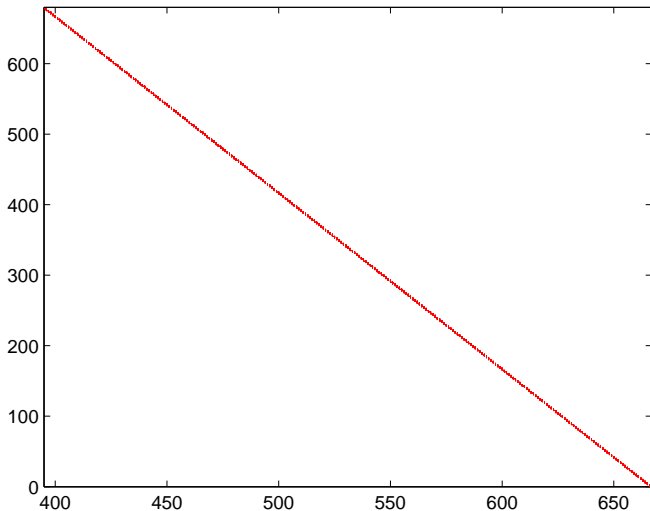
Solver	A	B	C	D	min
MINOS (ampl)	666.6	0	0	2500	14666
CONOPT (gams)	333.3	0	833.3	2500	14666
BARON (neos)	666.6	0	0	2500	14666
GlobSol	See below				14666

- All solvers except GlobSol completed almost immediately with normal termination and no warnings.
- GlobSol did not complete its search after hours, but output a list of boxes that contained both solutions represented by BARON and by CONOPT.

The Linear Financial Problem

GlobSol's Unfinished Boxes – A versus C

x_1 vs x_3 , unresolved boxes



(slide 48)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

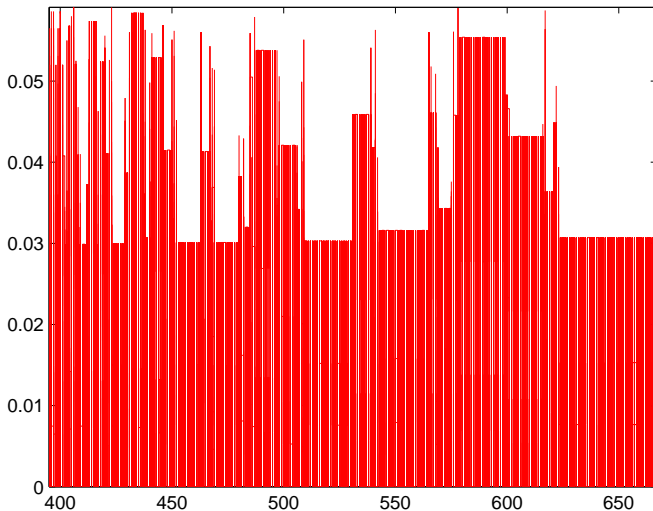
Interface and Packaging Improvements

On Tessellation and Clustering

The Linear Financial Problem

GlobSol's Unfinished Boxes – A versus B

x_1 vs x_2 , unresolved boxes



(slide 49)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

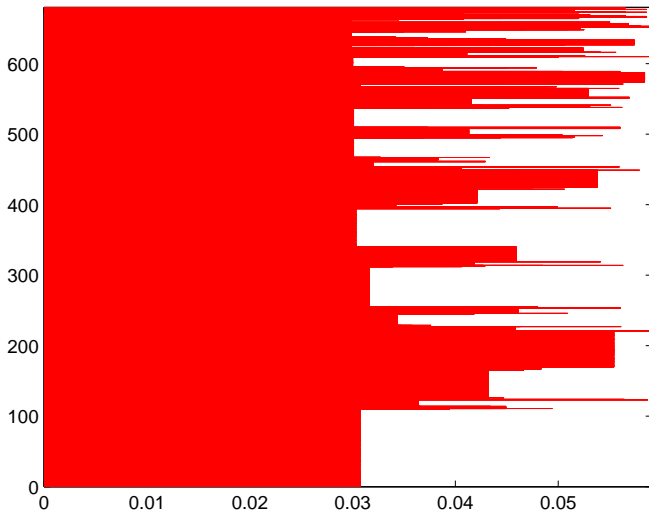
Interface and Packaging Improvements

On Tessellation and Clustering

The Linear Financial Problem

GlobSol's Unfinished Boxes – B versus C

x_2 vs x_3 , unresolved boxes



(slide 50)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering



The Linear Financial Problem

Observations from GlobSol's Output

(slide 51)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- The fourth coordinate of all boxes not yet rejected is pegged at 2500.
- The un-rejected boxes seem to lie along a linear relationship between A and C .
- The values for B in the un-rejected boxes seem clustered around 0.
- There appears to be a “clustering effect.” (Why, and how can the effect be reduced or eliminated?)

The Linear Financial Problem

Additional Elementary Analysis

(slide 52)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- Assuming $A = 0$ and $D = 2500$, we see that the objective contours and the contours of the budget constraint are parallel, and the contour corresponding to the minimum risk corresponds to the line traced by GlobSol; examine:

$$\begin{aligned} &\text{Minimize } 10A + 3.5B + 4C + 3.2D \\ &\text{Subject to } 100A + 50B + 80C + 40D \leq 200,000. \end{aligned}$$

- If we perturb the return and risk estimates for stocks A and C by about 1%, then all four solvers give the same unique solution.
- How should this be interpreted?

A Perturbed Problem

Some Details

(slide 53)

Old / New

	A	B	C	D
Price	\$100	\$50	\$80	\$40
Return	$\frac{0.12}{0.124}$	0.08	$\frac{0.06}{0.061}$	0.10
Risk	$\frac{0.10}{0.101}$	0.07	$\frac{0.05}{0.051}$	0.08

On the perturbed problem, all four solvers gave

A	B	C	D	min
645.161	0	0	2500	14516

which is close to one of the particular solutions to the unperturbed problem.

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

GlobSol's Output

The Perturbed Problem

GlobSol version	# boxes	time
2003	40985	32.5
experimental	163	2.4

- The “2003” version is the publicly available one.
- The “experimental” version includes verified linear relaxations with a publicly available LP solver.
- “# boxes” is the total number of boxes processed.
- “time” is the number of seconds execution time on a 3.2GHz Windows XP machine, compiled without optimization, using the g95 compiler for the 2003 version and the Compaq compiler, version 6, for the experimental version.

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering



(slide 55)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Outline

1 Introduction

What Kind of Solution is Needed?
Local Versus Global Optimization

2 Comparing GlobSol and Other Software

A Synopsis of GlobSol
Competing Global Optimization Algorithms
A Simple Example
Results from the Simple Example
Observations and Conclusions

3 Planned Improvements

Interface and Packaging Improvements
On Tessellation and Clustering

Observations and Conclusions

What kind of answer is most useful?

(slide 56)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example
Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- The singular problem gives a variety of solutions, some of which may be more desirable to the portfolio manager than others (due to items, such as higher possible profit, not included in the original objective).
- The perturbed problem is near the singular problem, but none of the solvers, GlobSol included, finds this.
- This suggests that GlobSol's apparent inefficiency with the exactly singular problem may give more useful information than the apparent efficiency of the other solvers.
- If small intervals were input for the coefficients, the nearness to singularity would be accurately reflected.



(slide 57)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Outline

- 1 Introduction
 - What Kind of Solution is Needed?
 - Local Versus Global Optimization
- 2 Comparing GlobSol and Other Software
 - A Synopsis of GlobSol
 - Competing Global Optimization Algorithms
 - A Simple Example
 - Results from the Simple Example
 - Observations and Conclusions
- 3 Planned Improvements
 - Interface and Packaging Improvements
 - On Tessellation and Clustering



Correctly Rounded I/O

(slide 58)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- Based on David Gay's `gdt oa` package of C functions.
- Uses the Fortran 2003 standard for interoperability with C.
- Allows specification of the number of digits to be displayed.
- Is presently available in my Windows / g95 distribution of GlobSol. (The compiler must support the C interoperability portion of the ISO Fortran 2003 standard.)



Correctly Rounded I/O

Old versus New

Old:

Box no.: 1

Box coordinates:

[0.7071D+00, 0.7071D+00] ...

PHI:

[-0.1500D+01, -0.1500D+01]

New, with 3 digit output specified:

Box no.: 1

Box coordinates:

[.707, .708], [.707, .708]

PHI:

[-1.51, -1.49]

(slide 59)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example
Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering



A Matlab / Intlab interface

(slide 60)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- The answer box lists are output in a format appropriate for import into Matlab.
- “m” files are supplied to plot lists of boxes and cubes. (Illustrations in this presentation were so produced.)
- Direct calling from Matlab and return of results to Matlab is planned.



Other Interface and Packaging Improvements

(slide 61)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- A GAMS / AMPL Interface
 - Possibly implemented using elements of the University of Vienna COCONUT environment.
- Package an LP solver with GlobSol
 - Possibly CLP from the COIN project, using the Fortran 2003 standard for C interoperability.
- More Complete and Up-to-date Documentation



Other Interface and Packaging Improvements

(slide 62)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- A GAMS / AMPL Interface
 - Possibly implemented using elements of the University of Vienna COCONUT environment.
- **Package an LP solver with GlobSol**
 - Possibly CLP from the COIN project, using the Fortran 2003 standard for C interoperability.
- More Complete and Up-to-date Documentation



Other Interface and Packaging Improvements

(slide 63)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- A GAMS / AMPL Interface
 - Possibly implemented using elements of the University of Vienna COCONUT environment.
- Package an LP solver with GlobSol
 - Possibly CLP from the COIN project, using the Fortran 2003 standard for C interoperability.
- **More Complete and Up-to-date Documentation**



(slide 64)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Outline

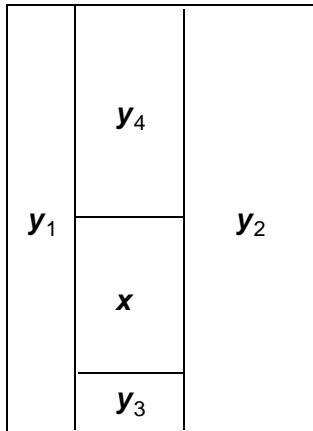
- 1 Introduction
What Kind of Solution is Needed?
Local Versus Global Optimization
- 2 Comparing GlobSol and Other Software
A Synopsis of GlobSol
Competing Global Optimization Algorithms
A Simple Example
Results from the Simple Example
Observations and Conclusions
- 3 Planned Improvements
Interface and Packaging Improvements
On Tessellation and Clustering



Thoughts on Tessellation

GlobSol's Answer Box Removal Process

x is the box to be removed, and y_1 , y_2 , y_3 , and y_4 are the boxes that are produced.



Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

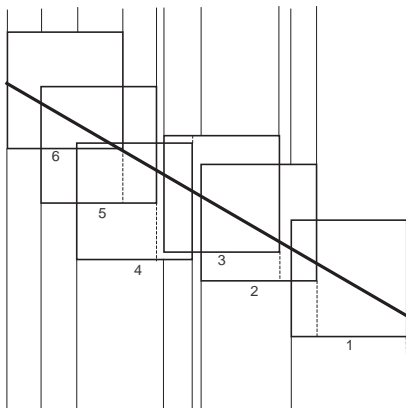
Interface and Packaging Improvements

On Tessellation and Clustering

Tessellations

Problems with Higher-Dimensional Solution Sets

If the singular solutions are found early in the order indicated, then the following tessellation results.



Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

Making GlobSol More Efficient

Eliminating the Clustering Effect

(slide 67)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering

- Change the search order? (Use a criterion other than lowest lower bound to order the list, or use breadth-first search?)
- Change where in the algorithm the complementation process is done?
- Tune minimum box size and other tolerances?

Note: The complementation process works well when there are only isolated solutions.



UNIVERSITY
OF
LOUISIANA
L a f a y e t t e

(slide 68)

Introduction

What Kind of Solution is Needed?

Local Versus Global Optimization

Comparing GlobSol and Other Software

A Synopsis of GlobSol

Competing Global Optimization Algorithms

A Simple Example

Results from the Simple Example

Observations and Conclusions

Planned Improvements

Interface and Packaging Improvements

On Tessellation and Clustering