Interval Arithmetic – An Elementary Introduction and Successful Applications

by
R. Baker Kearfott

Department of Mathematics
University of Southwestern Louisiana
U.S.L. Box 4-1010
Lafayette, LA 70504-1010 U.S.A.

Abstract

We will first introduce the elements of interval arithmetic. We will then outline areas of applicability in computer science and engineering, giving simple, illustrative examples. We will conclude with references and brief presentations of successful realistic applications.

Outline of Talk

1. Review of Interval Arithmetic
2. Interval Newton Methods
3. A simple example – Computational Existence Verification
4. Global Optimization
5. Successes in Practical Areas

What is Interval Arithmetic?

Interval arithmetic is based on defining the four elementary arithmetic operations on intervals. Let $\boldsymbol{a} = [a_l, a_u]$ and $\boldsymbol{b} = [b_l, b_u]$ be intervals. Then, if $op \in \{+, -, *, /\}$, we define

$$\boldsymbol{a} \ op \ \ \boldsymbol{b} = \{x \ op \ \ y \mid x \in \boldsymbol{a} \text{ and } y \in \boldsymbol{b}\}.$$

For example, $\boldsymbol{a}+\boldsymbol{b} = [a_l+b_l, a_u+b_u]$. In fact, all four operations can be defined in terms of addition, subtraction, multiplication, and division of the endpoints of the intervals, although multiplication and division may require comparison of several results. The result of these operations is an interval except when we compute $\boldsymbol{a}/\boldsymbol{b}$ and $0 \in \boldsymbol{b}$.

Why Interval Arithmetic?

- With *directed roundings*, we can bound round-off error in the computations. We can combine interval arithmetic with tools such as fixed-point iteration theorems to
  - automatically verify rigorous bounds, from approximate solutions.
  - automatic theorem proving.
- Interval arithmetic provides *rigorous bounds on the ranges* of functions. Posession of such bounds can be powerful, especially in *rigorous global optimization*.

An Example of Interval Arithmetic

$$\begin{aligned}[-1,2]\,([3,4]+[-5,6])) &= [-1,2]\,([-2,10]) \\ &= [-10,20],\end{aligned}$$

whereas

$$\begin{aligned}[-1,2][3,4]+[-1,2][-5,6] &= [-4,8]+[-10,12] \\ &= [-14,20]\end{aligned}$$

Here,

$$\begin{aligned}[-14,20] &= \{x \mid x = x_1 + x_2, x_1 \in [-4,8], \\ &\qquad x_2 \in [-10,12]\},\end{aligned}$$

and $[-14,20]$ *contains* the range of $xy+xz$ for $x \in [-1,2]$, $y \in [3,4]$, and $z \in [-5,6]$.

Inclusion Monotonic Interval Extensions of Functions

Definition. If f is a continuous function of a real variable, then an inclusion monotonic interval extension $\boldsymbol{f}$ is a function from the set of intervals to the set of intervals, such that, if $\boldsymbol{x}$ is an interval in the domain of $\boldsymbol{f}$,

$$\{f(x) \mid x \in \boldsymbol{x}\} \subset \boldsymbol{f}(\boldsymbol{x})$$

and such that

$$\boldsymbol{x} \subset \boldsymbol{y} \Rightarrow \boldsymbol{f}(\boldsymbol{x}) \subset \boldsymbol{f}(\boldsymbol{y}).$$

- We may obtain such interval extensions of a polynomial by replacing real operations by corresponding interval operations. For example, if $p(x) = x^2 - 4$, then $\boldsymbol{p}([1,2])$ may be defined by

$$\begin{aligned} \boldsymbol{p}([1,2]) &= ([1,2])^2 - 4 = ([1,4]) - [4,4] \\ &= [-3,0]. \end{aligned}$$

Interval Extensions of Transcendental Functions

- We may use the mean value theorem or Taylor's theorem with remainder formula. For example, suppose $\boldsymbol{x}$ is an interval and $a \in \boldsymbol{x}$. Then, for any $y \in \boldsymbol{x}$, we have

  $$\sin(y) = \sin(a) + (y-a)\cos(a) - (y-a)^2/2\sin(c)$$

  for some c between a and y. If a and y are both within a range where the sine function is non-negative, then we obtain

  $$\sin(y) \in \sin(a) + (\boldsymbol{x}-a)\cos(a) - \frac{(\boldsymbol{x}-a)^2}{2}.$$

  Specifically, if $\boldsymbol{x} = [.1, .3]$ and we use $a = .2$, we would obtain the "value"

  $$\begin{aligned} \mathbf{sin}([.1,.3]) &\subseteq \sin(.2) + [-.1,.1]\cos(.1) \\ &\qquad - \frac{([-.1,.1]^2)}{2} \\ &\subseteq [.0998,.0999] \\ &\qquad + [-.1,.1][.995,.996] + [-.005,0] \\ &= [-.0048,.1995] \end{aligned}$$

- Sharper interval extensions may be obtained in specific cases by using e.g. monotonicity of the original function.

Pitfalls to Naive Interval Arithmetic

- For rigorous bounds on roundoff error, one may be tempted to translate floating-point computer codes by merely replacing “real” with “interval.”
- Due to interval dependencies, such naive development often is unsuccessful.
- Interval arithmetic is successful if it is applied to appropriate tasks and with appropriate algorithms.
  - It provides rigorous results from the computer arithmetic.
  - It can actually result in *faster* algorithms, even if the interval arithmetic is in software and much slower than floating point.
- Researchers continue to enlarge the domain in which interval analysis can make computations rigorous and reliable.

Introduction to Applications

Nonlinear Equations – Interval Newton Methods

Interval analysis can be used either to

- Construct rigorous bounds around an approximate solution, in which an actual solution must lie.
- Exhaustively search a region to find all roots of a nonlinear system.

Verification is easier than exhaustive search. However, both tasks are based on *computational existence / uniqueness theorems.*

Introduction to Applications

A Computational Existence / Uniqueness Metatheorem

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ correspond to the system

$$F(X) = (f_1(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n)) = 0. \quad (1)$$

We write $x = (x_1, x_2, \ldots, x_n)$, and we denote a box (set of interval bounds on the variables x_i) by $\boldsymbol{x}$.

A Computational Existence / Uniqueness Metatheorem

We first transform $F(X) = 0$ to the linear interval system

$$\boldsymbol{F}'(\boldsymbol{x}^{(k)})(\tilde{\boldsymbol{x}}^{(k)} - x^{(k)}) = -F(x^{(k)}) \qquad (2)$$

where $\boldsymbol{F}'(\boldsymbol{x}^{(k)})$ is a suitable interval extension of the Jacobian matrix of F. We then formally solve (2) using interval arithmetic to obtain a box $\tilde{\boldsymbol{x}}^{(k)}$ which satisfies

$$\boldsymbol{F}'(\boldsymbol{x}^{(k)})(\tilde{\boldsymbol{x}}^{(k)} - x^{(k)}) \supset -F(x^{(k)}), \qquad (3)$$

such that $\tilde{\boldsymbol{x}}^{(k)}$ contains all solutions to the original nonlinear system within $\boldsymbol{x}^{(k)}$. We then define the next iterate $\boldsymbol{x}^{(k+1)}$ by

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} \cap \tilde{\boldsymbol{x}}^{(k)}. \qquad (4)$$

Depending on how we obtain $\tilde{\boldsymbol{x}}^{(k)}$, this *interval Newton method* leads to a *root inclusion test*, since

> If $\tilde{\boldsymbol{x}}^{(k)} \subseteq \boldsymbol{x}^{(k)}$, then the nonlinear system of equations has a unique solution in $\boldsymbol{x}^{(k)}$. Conversely, if $\tilde{\boldsymbol{x}}^{(k)} \cap \boldsymbol{x}^{(k)} = \emptyset$ then there are no solutions of the system in $\boldsymbol{x}^{(k)}$.

Computational Existence / Uniqueness

A Simple Example

Ife $n = 1$ and

$$f(x) = x^2 - 4,$$

then the linear interval system becomes

$$2\boldsymbol{x}_i(\tilde{\boldsymbol{x}}_i - x_i) = -f(x_i).$$

The interval Newton iteration equation becomes

$$\tilde{\boldsymbol{x}}_i = \tilde{\boldsymbol{x}} = x_i - \frac{f(x_i)}{2\boldsymbol{x}_i} = x - \frac{f(x)}{2\boldsymbol{x}}.$$

If we choose initial interval and point

$$\boldsymbol{x}^{(k)} = \boldsymbol{x}_i = \boldsymbol{x} = [1, 2.5], \qquad x_i = x = 1.75,$$

then

$$\tilde{\boldsymbol{x}} = 1.75 - \frac{-.9375}{[2,5]} = 1.75 - [.1875, .46875] \quad (5)$$

$$= [1.9375, 2.21875] \subset \boldsymbol{x}, \quad (6)$$

so we may conclude that there is a unique root of f in $\boldsymbol{x}$.

Finding All Roots

- In *univariate* problems, interval Newton methods can be iterated with *extended interval arithmetic* to *always* find *all* roots.
- In *multivariate* problems, interval Newton methods can be combined with *generalized bisection* and *binary search* to *always* find *all* roots.
- In both instances, failure modes are benign:
  - Failure can only occur by exceeding the computer's resources or because of the limited resolution of the floating-point numbers.
  - When failure occurs, the algorithms still print a list of boxes within which all roots must lie.

Finding All Roots

An Example

As before, let $f(x) = x^2 - 4$, but now let $\boldsymbol{x}^{(k)} = \boldsymbol{x}_i = \boldsymbol{x} = [-2, 2]$ and $x_i = x = 0$. The first iteration of the interval Newton method thus becomes

$$\begin{aligned} \tilde{\boldsymbol{x}} &= 0 - \frac{-4}{[-4, 4]} = 0 - ([-\infty, -1] \cup [1, \infty]) \qquad (7) \\ &= ([-\infty, -1] \cup [1, \infty]) \qquad (8) \end{aligned}$$

Thus,

$$\tilde{\boldsymbol{x}} \cap \boldsymbol{x}^{(k)} = [-2, -1] \cup [1, 2]. \qquad (9)$$

- One of the intervals in Equation 9 is put on a list for further processing, and the interval Newton method is iterated on the other.
- The interval Newton method will converge to zero-width intervals for *each* startin interval from Equation 9.
- Hansen has proven that this behavior is true in general.

Global Optimization

With interval methods, we can:

> Find, with certainty, the global minimum of the nonlinear objective function
>
> $$\varphi(X) = \varphi(x_1, x_2, \ldots, x_n) \qquad (10)$$
>
> where bounds $\underline{x}_i$ and $\overline{x}_i$ are known with $\underline{x}_i \leq x_i \leq \overline{x}_i$ for $1 \leq i \leq n$.

To do this, a *branch-and-bound* algorithm with the following general features is used.

- A technique for partitioning a region into subregions is combined with a technique for computing a lower bound $\underline{\varphi}$ and an upper bound $\overline{\varphi}$ of the objective function φ over a region $\boldsymbol{x}$.
- The subboxes are placed in a list in order of increasing $\underline{\varphi}$.
- The list is purged of those boxes for which $\underline{\varphi}$ is greater than $\overline{\varphi}$ for some other box in the list.
- An interval Newton method accelerates the procedure by quickly and rigorously locating critical points.

Global Optimization

An Example

Suppose we are to minimize $f(x) = x^2 - 2x + 1$ (written in that way), and the search region is $\boldsymbol{x} = [.5, 2]$.

- $f((.5 + 2)/2)) = f(1.25) = 0.0625$, so the best estimate of the minimum is 0.0625.
- Split $[.5, 2]$ to $[.5, 1.25]$ and $[1.25, 2]$; compute $\boldsymbol{f}([.5, 1.25]) = [-1.25, 1.5625]$; $f(0.875) = 0.015625 < 0.0625$, so 0.015625 is the new best estimate. Store $[.5, 1.25]$ on a list $\mathcal{L}$.
- $\boldsymbol{f}([1.25, 2]) = [-1.4375, 2.5]$; $f(1.625) = .390625$, so there is no new best estimate. Store $[1.25, 2]$ on $\mathcal{L}$ before $[.5, 1.25]$, since $-1.4375 < -1.25$.

Optimization

(Continued)

- Continue such processing by popping the first item from the list.
  - if a lower bound on a box is bigger than the best estimate, discard the box.
  - Put a box on a final list if its diameter is small.
- Interval Newton methods can be used to accelerate the process.
- Traditional optimization codes are useful to get good best estimates.

An Optimization Example

List of boxes considered

(See blackboard.)

Successes in Practical Areas

Summary

Nonlinear Algebraic Systems

Global Optimization

Linear Algebraic Systems: Error analysis

Sensitivity Analysis: Economic models, etc.

Geometric Computations

Rigorous bounds on solutions of ordinary and partial differential equations are more difficult, but notable successes have occurred recently.

Practical Successes

Nonlinear Algebraic Systems

- **Chemical Engineering Problems** – Carol Schnepper's Ph.D. dissertation (University of Illinois)
- **Continuation Methods** – a foolproof step control for path following – Zhaoyun Xing's dissertation (USL)
- Theory is exhaustively written in a 1990 book by Neumaier.
- We are presently working on a comprehensive **software environment**.

Practical Successes

Global Optimization

- Interval methods are generally very competitive relative to alternatives, both deterministic and stochastic.
- Hansen has a monograph on the subject.
- Moore, Hansen, Leclerc, Jansson, and Knüppel have developed algorithms for parallel architecture.
- Arnold Neumaier and David Gay at AT&T are presently writing software for distribution.
- Our software system will also include such code.

Practical Successes

Sensitivity of Linear Systems

- Fixed-point contraction-mapping like theorems, as above, can be used to obtain guaranteed bounds on solutions.
- Falcó Korn and Christian Ullrich have combined these with the Linpack band matrix routines.
- In theory and practice
  - The interval bound widths are comparable to the LINPACK condition estimator size, but are rigorous.
  - The interval bounds require *much* less computation for very large, sparse systems.

Practical Successes

Economic Models

- Matthews, Broadwater, and Brown have applied interval techniques to estimation of **electric utility expenses** and revenue.
- Uncertainty in items such as interest rates is expressed as intervals.
- Computations are arranged so that interval outputs are sharp and useful.

Practical Successes

Geometric Computation

- Mudur and Koparkar (1984) – Outline techniques for obtaining sharp bounds for geometric computations such as evaluating lengths and areas, curve-curve and surface-surface intersections, testing linearity and planarity.
- Patrikalakis and co-workers (recent)
  - Solution of nonlinear systems related to robot vision and object recognition
  - Computation of singularities and offsets of planar curves.
- Other work is in progress, including applying our work on continuation methods.